On the Computation of the General Eigenproblem

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The study of the eigenproblem has been very important in pure mathematics while the computation of the eigenvalues and eigenvectors of a matrix is necessary in the various fields of applied mathematics and mechanics. The method in this paper depends upon the properties of similarity transformations and companion matrices.

The main part of the computational procedure of similarity transformations is the same as that of Danilevskii's or Givens' method. Our method is one of the most effective methods for the numerical computation of the eigenproblem of general matrices. Jacobi's method is very effective for the computation of eigenvalues of symmetric matrices but it is no longer applicable to the non-symmetric case. Our method is rather effective than Jacobi's method even for the symmetric case.

Power method due to von Mises is a powerful method for the eigenproblem but is powerless for degenerate matrices the eigenvectors of which do not generate the whole space.

1. *Similarity Transformations and a Canonical Form*

Let $A, B, \cdots$ be square matrices of the order n in the real field and $x, y, \cdots$ be n-dimensional vectors and λ be a scalar. The λ which satisfies the equation

$$Ax=\lambda x \tag{1.1}$$

is a root of the characteristic polynomial

$$\begin{aligned} f(\lambda)=|\lambda I-A| &= \lambda^n+a_1\lambda^{n-1}+\cdots+a_n \\ &=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \end{aligned} \tag{1.2}$$

where the λ_is are the roots of the characteristic polynomial and called eigenvalues. By a non-singular transformation $x=Sy$, the equation (1.1) becomes

$$S^{-1}ASy=\lambda y \tag{1.3}$$

where S is a non-singular matrix and S^{-1} is its inverse.

Two matrices A and B are called similar if there exists a non-singular

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matrix S such that $B=S^{-1}AS$, and we write $A\sim B$.

Similarity is reflexive, symmetric and transitive. The characteristic polynomial of a matrix is an invariant of the similarity transformation. In fact,

$$|\lambda I-S^{-1}AS|=|S^{-1}(\lambda I-A)S|=|\lambda I-A|.$$

Thus the trace $tr(A)$, the determinant $|A|$ and the rank of a matrix are also invariants of a similarity transformation.

A matrix C of the form

$$C=\begin{pmatrix} 0 & \cdots & 0 & -c_n \\ 1 & \cdots & 0 & -c_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -c_1 \end{pmatrix} \tag{1.4}$$

is called the companion matrix of a polynomial $C(\lambda)=\lambda^n+c_1\lambda^{n-1}+\cdots+c_n$. The characteristic polynomial of the companion matrix C is $C(\lambda)$.

$$C(\lambda)=|\lambda I-C|$$

The eigenvector associated to an eigenvalue λ of a companion matrix C is easily given by solving the homogeneous linear equations $Cx=\lambda x$. Its components are:

$$\left.\begin{aligned} &x_n=1, \\ &x_{n-1}=\lambda+c_1, \\ &x_{n-2}=\lambda^2+c_1\lambda+c_2, \\ &\cdots\cdots \\ &x_1=\lambda^{n-1}+c_1\lambda^{n-2}+\cdots+c_{n-1}. \end{aligned}\right\} \tag{1.5}$$

The eigenvalues of A are the same as those of C and the eigenvector u of A is equal to Sv where v is an eigenvector of C. Thus our problem is to find a matrix S such that $C=S^{-1}AS$ and to solve the algebraic equation $C(\lambda)=0$.

It is clear from the equations (1.5) that there is only one eigenvector associated to a multiple root of the characteristic equation of a companion matrix. In this case the number of eigenvectors of the companion matrix is less than the dimension number n. The matrix A which is similar to such degenerate companion matrix is also degenerate. Our method is applicable to such a degenerate case while 'Power method' is powerless.

The numerical method is following. Let $A=(a_{ij})$, $i, j=1, 2, \cdots, n$ and $a_{21}\neq 0$. The Gaussian procedure with the (2, 1) pivotal element which eliminates the elements of the first column of A is excuted by the mutiplication of S_1^{-1} on the left of A. Where S_1^{-1} is the inverse matrix of S_1 and the columns of S_1 are those of the unit matrix I except the second column which is the first column of A. The postmultiplication of S_1 on any matrix M keeps the first column of M invariant.

Thus the first column of $A^{(2)}=S_1^{-1}A^{(1)}S_1$ is a unit vector whose second element is 1, where $A^{(1)}=A$.

The first column of $A^{(2)}$ is equal to the first column of the companion matrix C by the similarity transformation. If the next pivotal element $a_{32}^{(2)}$ of $A^{(2)}$ does not vanish, we have the second similarity transformation $A^{(3)}=S_1^{-2}A^{(2)}S_2$, where the third column of the transformation matrix S_2 is the second column of $A^{(2)}$ and the other columns of S_2 are the same as those of the unit matrix I.

The first two columns of $A^{(3)}$ are the same as those of the companion matrix C by the successive similarity transformations of A; $A^{(3)}=S_2^{-1}(S_1^{-1}AS_1)S_2$. After $n-1$ steps of these similarity transformations, we have a companion matrix $A^{(n)}=C=S^{-1}AS$, where $S=S_1S_2\cdots S_{n-1}$ unless the successive pivotal elements do not vanish.

The transformation matrix S is real because A is real while one of the eigenvalues of A may be imaginary. The imaginary eigenvector of a companion matrix associated to the imaginary eigenvalue is easily computed from (1.5). The imaginary eigenvector of A which is similar to the companion matrix is computed by multiplying S on the left of the real and imaginary parts of the imaginary eigenvector of the companion matrix respectively.

Example 1 (referred from [1], [4]).

$$A=\begin{pmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{pmatrix},\quad S=\begin{pmatrix} 1 & 6 & 44 & 296 \\ 0 & 4 & 48 & 400 \\ 0 & 4 & 32 & 224 \\ 0 & 4 & 48 & 416 \end{pmatrix},\quad C=\begin{pmatrix} 0 & 0 & 0 & -16 \\ 1 & 0 & 0 & 48 \\ 0 & 1 & 0 & -44 \\ 0 & 0 & 1 & 12 \end{pmatrix}.$$

$$C(\lambda)=|\lambda I-C|=|\lambda I-A|=\lambda^4-12\lambda^3+44\lambda^2-48\lambda+16=(\lambda^2-6\lambda+4)^2,$$

$$\lambda_1=\lambda_2=3+\sqrt{5},\qquad \lambda_3=\lambda_4=3-\sqrt{5}.$$

Eigenvectors of C:

$$\begin{pmatrix} -12+4\sqrt{5} \\ 22-6\sqrt{5} \\ -9+\sqrt{5} \\ 1 \end{pmatrix} \text{ for } 3+\sqrt{5} \qquad \begin{pmatrix} -12-4\sqrt{5} \\ 22+6\sqrt{5} \\ -9-\sqrt{5} \\ 1 \end{pmatrix} \text{ for } 3-\sqrt{5}$$

Eigenvectors of A:

$$\begin{pmatrix} 20+12\sqrt{5} \\ 56+24\sqrt{5} \\ 24+8\sqrt{5} \\ 72+24\sqrt{5} \end{pmatrix} \text{ for } 3+\sqrt{5} \qquad \begin{pmatrix} 20-12\sqrt{5} \\ 56-24\sqrt{5} \\ 24-8\sqrt{5} \\ 72-24\sqrt{5} \end{pmatrix} \text{ for } 3-\sqrt{5}$$

2. *The Case Where the Pivotal Element Vanishes*

If the pivotal element $a_{i+1,i}^{(i)}$ of $A^{(i)}$ vanishes, the matrix S_i is singular and then there are no similarity transformations. There are two cases in this case.

(1) There is at least one non-zero element $a_{ji}^{(i)}$ under $a_{i+1,i}^{(i)}$, $(i+1<j\leqq n)$.

(2) All elements under $a_{i+1,i}^{(i)}$ vanish.

In the first case, the pivotal element becomes non-zero after 'positioning for size' by a similarity transformation $P_{i+1,j}^{-1} A^{(i)} P_{i+1,j}$ where $P_{i+1,j}$ $(=P_{i+1,j}^{-1})$ is a permutation matrix which is given by interchanging the $i+1$-th column and j-th column of the unit matrix I.

In the second case, $A^{(i)}$ is of the form (2.1).

$$A^{(i)} = \left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & a_{1i}^{(i)} & a_{1,i+1}^{(i)} & \cdots & a_{1n}^{(i)} \\ 1 & \cdots & 0 & a_{2i}^{(i)} & \vdots & & \vdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & a_{ii}^{(i)} & a_{i,i+1}^{(i)} & \cdots & a_{in}^{(i)} \\ \hline & & 0 & & & A_2^{(i)} & \end{array}\right) \tag{2.1}$$

The characteristic polynomial of $A^{(i)}$ has a factor of the degree i such that

$$\lambda^i - a_{ii}^{(i)} \lambda^{i-1} - \cdots a_{2i}^{(i)} \lambda - a_{1i}^{(i)} . \tag{2.2}$$

The eigenvector of $A^{(i)}$ associated to the root λ of this polynomial is computed from the next equations.

$$\left.\begin{array}{l} x_n = x_{n-1} = \cdots = x_{i+1} = 0 , \\ x_i = 1 , \\ x_{i-1} = \lambda - a_{ii}^{(i)} , \\ \cdots\cdots \\ x_1 = \lambda^{i-1} - a_{ii}^{(i)} \lambda^{i-2} - \cdots - a_{2i}^{(i)} . \end{array}\right\} \tag{2.3}$$

Thus our canonical form is of the form

$$B = \begin{pmatrix} \boxed{C_{n_1}} & \boxed{b_2} & & \boxed{b_k} \\ & \boxed{C_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{C_{nk}} \end{pmatrix} , \tag{2.4}$$

where C_{n_i} is a companion matrix of the order n_i, $\sum_{i=1}^{k} n_i = n$ and b_i is a vector. Matrices of this form are a special class of Hessenberg's matrices (see [10], [17]). An eigenvalue of this matrix is one of those of the minor C_{n_i}. If all elements of b_is vanish, B is decomposed into the direct sums of companion matrices:

$$B = C_{n_1} \dotplus C_{n_2} \dotplus \cdots \dotplus C_{nk}$$

(see [2], [9], [18], [23], [26], [28], [30], [38], [41]).

3. *Decomposition into Direct Sums*

In any field, an arbitrary matrix is decomposed to the direct sums of companion matrices by a similarity transformation.

This theorem has been used as a starting point in the development of the entier theory of matrices. A. A. Bennett discussed the computational aspect of the methods of Lattès, Kowalewski and Dickson. But it seems to be difficult to develop Benntt's method.

W. Krull proved that an arbitrary matrix A is similar to one of the form

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$

where A_1 is the companion matrix of the minimum equation of A. The above matrix is shown to be similar to the direct sum $A_1 \dotplus A_2$. The process is continued with A_2 until the above theorem is shown (see [24]).

The appearance of the minimum equation makes it difficult to apply this procedure to the actual computation. A practical method of the decomposition to direct sums is the following.

Let I_n be the unit matrix of the order n.

$$\begin{pmatrix} I_k & -S \\ O & I_l \end{pmatrix}\begin{pmatrix} A_1 & B \\ O & A_2 \end{pmatrix}\begin{pmatrix} I_k & S \\ O & I_l \end{pmatrix}=\begin{pmatrix} A_1 & A_1S-SA_2+B \\ O & A_2 \end{pmatrix} \tag{3.1}$$

If the matric equation

$$A_1S-SA_2+B=0 \tag{3.2}$$

holds, the right side of (3.1) is decomposed to the direct sum $A_1 \dotplus A_2$. The matric equation (3.2) is a system of $k \times l$ linear equations with $k \times l$ unknown elements of the matrix S. When A_2 is a scalar, that is, a matrix of the order 1, the equation (3.2) becomes

$$(aI_{n-1}-A_1)S=B \tag{3.3}$$

and this is solvable unless a is an eigenvalue of the minor A_1.

The computation of the eigenvectors of a matrix which is decomposed into direct sums of minors is very simple while the computation of the transformation matrix S of (3.2) may not be easy in general.

The transformation of a companion matrix to Jordan's canonical form is very simple when its eigenvectors generate the whole space (see [1], [39]). In this case, the transformation matrix is composed after the eigenvectors of the original matrix are known.

4. *Algorithm and Computaion*

```
procedure Begleit (A, N, IND, S);
value N; array A, S; integer N; integer array IND;
begin integer I, J, K, L, M; real W, T, MAX;
  for I:=1 step 1 until N do IND [I]:=0;
  for I:=1 step 1 until N do for J:=1 step 1 until N do
      S[I, J]:=if I=J then 1 else 0;
START: for L:=2 step 1 until N do
  begin K:=L-1; M:=L; MAX:=abs(A[L, K]);
POSITIONING: for I:=L+1 step 1 until N do
    begin W:=abs(A[I, K]); if MAX<W then
        beign MAX:=W; M:= I end end;
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if M=L then go to SIMILAR;
for I:=1 step 1 until N do
      begin W:=A[I, L]; A[I, L]:=A[I, M]; A[I, M]:=W;
      W:=S[I, L]; S[I, L]:=S[I, M]; S[I, M]:=W end;
for J:=1 step 1 until N do
    begin W:=A[L, J]; A[L, J]:=A[M, J]; A[M, J]:=W end;
SIMILAR: if A[L, K]=0 then begin IND[K]:=K; go to ADV end;
  for I:=1 step 1 until N do
    begin W:=T:=0; for J:=1 step 1 until N do
        begin W:=A[I, J]×A[J, K]+W; T:=S[I, J]×A[J, K]+T
        end;
        A[I, L]:=W; S[I, L]:=T
      end;
  for J:=K step 1 until N do A[L, J]:=A[L, J]/A[L, K];
  for I:=1 step 1 until N do if I≠L then
    begin for J:=K step 1 until N do
          A[I, J]:=A[I, J]−A[L, J]×A[I, K]
    end; ADV:
end START
end Begleit
```

'Positioning for size' is applied in order to get good results of computations (see [42]).

The computation of this procedure was tried on several matrices of the orders 10, 16 and 20 with NEAC 2203 and TOSBAC 3121. The accuracies of these computations were 8 digits in the 9 with NEAC 2203 and 9 in the 10 with TOSBAC 3121. The same procedure written in FORTRAN language has been tried on some test matrices of high order.

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