A Theory and Design of Free-Formed Surface

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1. *Introduction*

This paper deals with mathematical expressions of free-formed surfaces and space curves which have suitable characteristics for the engineering design process. There are many engineering products which use free-formed surfaces, but their designs and productions are not easy, because a good design must satisfy not only functional requirements, but also aesthetic ones, and mold making process of free-formed surfaces needs special human skill. If a computer is to be introduced to expedite its design and manufacturing, it must be able to construct and display the shapes by accepting designer's ideas, and after the design is fixed it must supply the data for automatic manufacturing and other processes. For this purpose, the computer must be provided with suitable input- output devices, special hardware to generate continuous vectors, software system which can treat the list of graphical data and appropriate mathematical expressions of surfaces and curves. We have already reported several studies on hard-and software of computer graphics and this paper gives another mathematical expressions for free-formed surfaces and curves, curve generating hardware algorithm and some design examples.

2. *Equation of surface*

A potiron $\widehat{P_0P_1}$ of a three dimentional curve is denoted by a vector from $\boldsymbol{R}(t)$ as shown in fig. 1, where its origin is at point P_0 and the vector head moves along the curve from P_0 to P_1 as the parameter t varies from 0 to 1. These are four curves $\boldsymbol{R}_1(t)$, $\boldsymbol{R}_2(t)$, $\boldsymbol{R}_3(t)$ and $\boldsymbol{R}_4(t)$, connected and closed in the space as shown in fig. 2. The vector equation of the surface bounded by these closed curves is expressed as follows.

$$\boldsymbol{S}(u,\ v)=\boldsymbol{R}_1(u)+\{\boldsymbol{R}_3(u)-\boldsymbol{R}_1(u)\}F(u,\ v)+\boldsymbol{R}_2(v)+\{\boldsymbol{R}_4(v)-\boldsymbol{R}_2(v)\}$$
$$\times G(u,\ v)+\boldsymbol{Q}F(u,\ v)\cdot G(u,\ v)+\boldsymbol{S}_0(u,\ v),\quad 0\leq(u,\ v)\leq 1 \qquad (1)$$

where the origin coincides with that of curve $\boldsymbol{R}_1(t)$, u and v are parameters of the surface, functions $F(u,\ v)$ and $G(u,\ v)$ must satisfy the next conditions,

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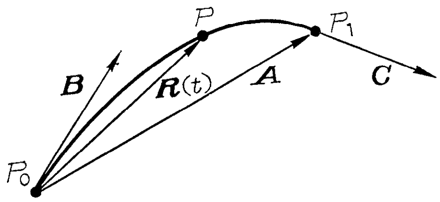


Fig. 1. Curve.

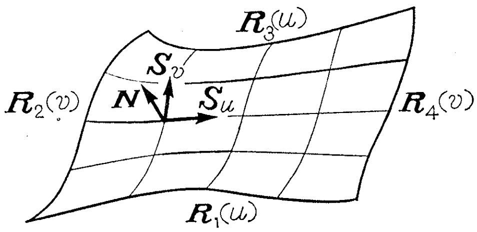


Fig. 2. Surface.

$$\left.\begin{array}{l} F(u,\ 0)=G(0,\ v)=0,\ F(u,\ 1)=G(1,\ v)=1 \\ F(0,\ v)=f_2(v),\ G(u,\ 0)=f_1(u) \\ F(1,\ v)=f_4(v) \\ G(u,\ 1)=f_3(u), \end{array}\right\} \qquad (2)$$

where the conditions of $f_i(t)$ s for any i are

$$f_i(0)=0,\ f_i(1)=1. \qquad (3)$$

The function $\boldsymbol{S}(u,\ v)$ is equal to zero for u and/or v are 0 or 1,

$$\boldsymbol{S}(u,\ v)=0, \quad u,\ v=0,\ 1. \qquad (4)$$

The constant vector $\boldsymbol{Q}$ is, denoting $\boldsymbol{R}_i(1)=\boldsymbol{R}_{i1}$,

$$\boldsymbol{Q}\equiv\tfrac{1}{2}(\boldsymbol{R}_{11}+\boldsymbol{R}_{21}-\boldsymbol{R}_{31}-\boldsymbol{R}_{41})=(\boldsymbol{R}_{11}-\boldsymbol{R}_{31})=(\boldsymbol{R}_{21}-\boldsymbol{R}_{41}). \qquad (5)$$

This relation is derived from the closed curve characteristic, and

$$\boldsymbol{R}_i(0)=0. \qquad (6)$$

In eq. (1), when u and v are put to 0 or 1, then

$$\left.\begin{array}{l} \boldsymbol{S}(u,\ 0)=\boldsymbol{R}_1(u).\ \boldsymbol{S}(0.\ v)=\boldsymbol{R}_2(v) \\ \boldsymbol{S}(u,\ 1)=\boldsymbol{R}_{21}+\boldsymbol{R}_3(u) \\ \boldsymbol{S}(1,\ v)=\boldsymbol{R}_{11}+\boldsymbol{R}_4(v). \end{array}\right\} \qquad (7)$$

These equations show that the four boundaries of the surface are the given curves. Eq. (1) expresses the general surface, and next we introduce a surface whose normal vector at the boundary is the function of the boundary curve, the normal vectors and their derivatives at the end points of that boundary curve. For simplicity, we put

$$\left.\begin{array}{l} F(u,\ v)=f(v),\ G(u,v)=f(u) \\ \boldsymbol{S}_0(u,\ v)=\boldsymbol{T}_{00}\ g(u)g(v)+\boldsymbol{T}_{10}h(u)g(v) \\ \qquad +\boldsymbol{T}_{01}g(u)h(v)+\boldsymbol{T}_{11}h(u)h(v), \end{array}\right\} \qquad (8)$$

where $$T_{ij}=\frac{\partial^2 S}{\partial u \partial v}(i,\ j),\quad i,\ j=1,\ 0 \tag{9}$$

$f(t)$, $g(t)$ and $h(t)$ must satisfy the next conditions.

$$\left.\begin{array}{l} f(0)=g(0)=g(1)=h(0)=h(1)=\dot{f}(0)=\dot{f}(1)=0 \\ \dot{g}(1)=\dot{h}(0)=0,\ f(1)=\dot{g}(0)=\dot{h}(1)=1 \end{array}\right\} \tag{10}$$

Then the tangent vector along u curve at the boundary $u=0$ or 1 of the surface are

$$\left.\begin{array}{l} \boldsymbol{S}_u(0.\ v)=\dot{\boldsymbol{R}}_1(0)+\{\dot{\boldsymbol{R}}_3(0)-\dot{\boldsymbol{R}}_1(0)\}f(v)+\boldsymbol{T}_{00}g(v)+\boldsymbol{T}_{01}h(v) \\ \boldsymbol{S}_u(1,\ v)=\dot{\boldsymbol{R}}(1)+\{\dot{\boldsymbol{R}}_3(1)-\dot{\boldsymbol{R}}_1(1)\}f(v)+\boldsymbol{T}_{10}g(v)+\boldsymbol{T}_{11}h(v). \end{array}\right\} \tag{11}$$

The tangent vectors along the v curve are

$$\boldsymbol{S}_v(0\ v)=\dot{\boldsymbol{R}}_2(v),\ \boldsymbol{S}_v(1,\ v)=\dot{\boldsymbol{R}}_4(v). \tag{12}$$

As the normal vector is the vector product of $\boldsymbol{S}_u$ and $\boldsymbol{S}_v$, the normal is determined by only the values of the boundary concerned. Then it is possible to connect smoothly the surfaces A and B without affecting the respective opposite boundary curves. (fig. 3) If the values pertaining to each surfac are descriminated by the superfixes A and B, then the conditions for smooth connection are

$$\left.\begin{array}{l} \boldsymbol{R}_4^A(v)=\boldsymbol{R}_2^B(v) \\ \boldsymbol{R}_2^A(v)\cdot\{\boldsymbol{S}_u^A(1,\ v)\times\boldsymbol{S}_u^B(0,\ v)\}=0. \end{array}\right\} \tag{13}$$

From this, for the simplest case the conditions are

$$\left.\begin{array}{c} \boldsymbol{R}_4^A(v)=\boldsymbol{R}_2^B(v),\ \dot{\boldsymbol{R}}_1^A(1)=\dot{\boldsymbol{R}}_1^A(1),\ \dot{\boldsymbol{R}}_3^A(1)=\dot{\boldsymbol{R}}_3^B(0) \\ \boldsymbol{T}_{10}^A=\boldsymbol{T}_{00}^B,\ \boldsymbol{T}_{11}^A=\boldsymbol{T}_{01}^B. \end{array}\right\} \tag{14}$$

Of course there are other cases which satisfy the eqs. (13), but for simplicity they are not discussed here. When the connection occurs on some portion of a boundary $\boldsymbol{R}_4^A(v^A)$ between $v_1^A \leqq v^A \leqq v_2^A$, as shown in fig. (4), $\boldsymbol{R}_2^B(0)$, $\boldsymbol{R}_2^B(1)$, $\boldsymbol{T}_{00}^B$, $\boldsymbol{T}_{01}^B$ are calculated from values of $\boldsymbol{S}_u^A(1,\ v_1^A)$ and $\boldsymbol{S}_u(1,\ v_2^A)$. Then $\boldsymbol{R}_2^B(v^B)$ and

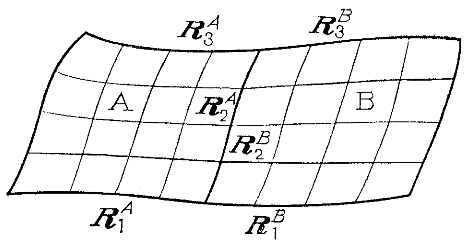


Fig. 3. Connection of two surfaces.

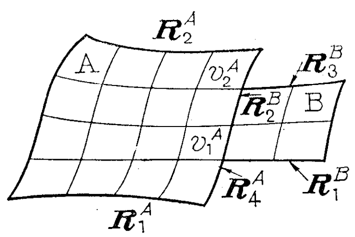


Fig. 4. Connection of two surfaces.

$S_u^B(0,\ v^B)$ of the surface B can be made equal to those of the surface A, where the variables v^A, v^B have the relation.

$$v^B=(v^A-v_1^A)/(v_2^A-v_1^A). \tag{15}$$

3. *Equation of curve*

Though the surface boundary curves may be expressed in any form, it is desirable to express them in algeblaic equations for easy manipulation and storage. A vector equation $\boldsymbol{R}(t)$ is assumed to be expressed by its chord vector $\boldsymbol{A}$ and two tangent vectors $\boldsymbol{B}$, $\boldsymbol{C}$ at the both ends of the chord. Then

$$\boldsymbol{R}(t)=\boldsymbol{A}f(t)+\boldsymbol{B}g(t)+\boldsymbol{C}h(t), \tag{16}$$

where $f(t)$, $g(t)$, and $h(t)$ are same as eqs. (10) and their simplest polynomial forms are,

$$\begin{aligned} &f(t)=t^2(3-2t),\ g(t)=t(t-1^2),\ h(t)=t^2(t-1),\\ &f(t)=t-g(t)-h(t). \end{aligned} \tag{17}$$

Now let x-axis coincident with vector $\boldsymbol{A}$, z-axis is in the direction $\boldsymbol{A}\times\boldsymbol{B}$ and y-axis is perpendicular to x and z-axes. then the components of the vector $\boldsymbol{R}(t)$ are

$$\left.\begin{aligned} x&=R_x(t)=Af(t)+B_xg(t)+C_xh(t)\\ y&=R_y(t)=\qquad\quad B_yg(t)+C_yh(t)\\ z&=R_z(t)=\qquad\qquad\qquad C_zh(t). \end{aligned}\right\} \tag{18}$$

The characteristics of the curves are indicated as follows,

(I). A intersection point with x-axis in x-y plane is determined by the parameter t whose value is

$$t=B_y/(B_y+C_y), \tag{19}$$

if sign of B_y and C_y are different, there is no intersection.

(II). Number of points where $x=0$ is at most two, that of $y=0$ is one or two that of $z=0$ is two where $t=0$ and $t=2/3$.

(III). A curve can have a loop in $x-y$ plane. The value of the parameter t at the entrance of the loop is given by roots of the next equation.

$$t^2-(p-q+1)t+(p-q)^2-q=0, \tag{20}$$

where $\quad p=-AC_y/\{(A-B_x)C_y+(A-C_x)B_y\},\ q=-pB_y/C_y.$

(IV). A component of $\boldsymbol{R}(t)$ to a line whose direction vector is λ, can be a linear function of t, then the next relation holds.

$$\lambda\cdot\boldsymbol{A}=\lambda\cdot\boldsymbol{B}=\lambda\cdot\boldsymbol{C}. \tag{21}$$

(V). When a chord vector $\boldsymbol{A}$, directions of tangent vectors and coordinate $(x,\ y\ z)$ of one point on the curve are given, value of t on that point is obtained from a root of the next equation.

$$Af(t)+\{y-z(\cos\gamma_y/\cos\gamma_z)\}(\cos\beta_x/\cos\beta_y)$$
$$+z(\cos\gamma_x/\cos\gamma_z)-x=0. \tag{22}$$

Where β_x, β_y, γ_x, γ_y, γ_z are direction angeles of the tangent vectors B and C, whose magnitudes can be calculated by using eq. (18) with the obtained value of t.

(VI). In case of plane curve, coordinate (x, y) and derivative (dy/dx) of a point are necessary to determine the magnitude of B and C. The value of t of the point is given by a root of the next equation,

$$a_3t^3+a_2t^2+a_1t+a_0=0 \tag{23}$$

Where, putting

$$dy/dx=\tan\theta,\ x=r\cos\theta',\ y=r\sin\theta',\ \beta_x=\beta,\ \gamma_x=\gamma,$$

then

$$\left.\begin{aligned}
a_3&=\tan\theta\cdot\sin(\beta+\gamma)+2\sin\gamma\cdot\sin\beta\\
a_2&=-3\tan\theta(3\sin\gamma\cdot\cos\beta-4\sin\beta\cdot\cos\gamma)-3\sin\gamma\cdot\sin\beta\\
a_1&=3r\{-\sin(\theta'-\gamma)\cdot\sin(\beta+\theta)+\sin(\theta'-\beta)\cdot\sin(\gamma+\theta)\}/\cos\theta\\
a_0&=r\{\sin(\theta'-\gamma)\cdot\sin(\beta+\theta)-2\sin(\theta'-\beta)\cdot\sin(\gamma+\theta)\}/\cos\theta.
\end{aligned}\right\} \tag{24}$$

(VII). When the tangent vector magnitudes B, C are fixed from other conditions, and a curve is required to pass a certain point and to satisfy more conditions, a vector

$$\boldsymbol{D}(t)=t^2(t-1)^2\sum_i \boldsymbol{d}_i t^i \tag{25}$$

is added to eq. (16) and vectors $\boldsymbol{d}_i$'s in a equation are determined from the given conditions.

(VIII) When a portion of a curve $\boldsymbol{R}(t)$ is coincident with another curve, the latter can be expressed by parameters of the former. Symbols are descriminated by the superscripts L and M, then

$$\left.\begin{aligned}
&\boldsymbol{R}^M(t)=\boldsymbol{A}^Mf(t^M)+\boldsymbol{B}^Mg(t)+\boldsymbol{C}^Mh(t^M),\ 0\leq t^M\leq 1\\
&\boldsymbol{R}^L(t^L)=\boldsymbol{A}^Lf(t^L)+\boldsymbol{B}^Lg(t^L)+\boldsymbol{C}^Lh(t),\ t_1{}^L\leq t^L\leq t_2{}^L\\
&t^M=(t^L-t_1{}^L)/(t_2{}^L-t_1{}^L),
\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}
\boldsymbol{A}^M&=\boldsymbol{R}^L(t_2{}^L)-\boldsymbol{R}^L(t_1{}^L)\\
\boldsymbol{B}^M&=(t_2{}^L-t_1{}^L)\dot{\boldsymbol{R}}^L(t_1{}^L)\\
\boldsymbol{C}^M&=(t_2{}^L-t_1{}^L)\dot{\boldsymbol{R}}(t_2{}^L).
\end{aligned}\right\} \tag{27}$$

(IX) When a curve, whose tangent and curvature are continuous, is required to pass n points P_i, $(i=0,1,\ldots\ldots,n-1)$, the tangent vectors at each point must satisfy the next equations, if a curve between the consecutive points P_{i-1}, P_i is descrided by a corresponding equation $\boldsymbol{R}_{i-1}(t)$.

$$\left.\begin{array}{l}\boldsymbol{C}_{i-1}=K_i\boldsymbol{B}_i \\ \boldsymbol{B}_{i-1}+2(K_i+K_i^2)\boldsymbol{B}_i+K_iK_{i+1}\boldsymbol{B}_{i+1}=3(\boldsymbol{A}_{i-1}+K_{i1}\boldsymbol{A}_i),\end{array}\right\} \quad (28)$$

where $\quad K_i=1$ or $K_i=|A_{i-1}|/|A_i|\cdot$

When two additional vectors $\boldsymbol{B}$'s at the given points are assumed, all $\boldsymbol{B}_i$'s can be obtained by solving the recurrence eqs. (28). If $K_i=1$, the equations are simple, $K_i \fallingdotseq 1$, the calculated curve is seemed more natural.

4. *Curve generating operation*

As it takes too much time to pilot continuous curves by calculating equations (1) and (16) with digital computation, the author uses the incremental technique to generate continuous curves for high speed display and plotting. A component of curve or surface vector can usually be expressed by a cubic equation of t as

$$x=at+bt^2+ct^3, \quad (29)$$

so the generation of increment Δx of eq. (29) is described.

There are four registers A, B, C and R of $3n+1$ bit length as shown in fig. (5). When a pulse which represent Δx generates, the next inter-register operations occur, $A+B\to R$, $B+A\to A$ and $C+B\to B$. After i'th Δt pulse, the contents of the registers are denoted by A_i, B_i, and R_i, then from the relation

$$R_i=R_{i-1}+A_{i-1},\ A_i=A_{i-1}+B_{i-1},\ B_i=B_{i-1}+C_i, \quad (30)$$

R_i can be obtained as

$$R_i=ai\cdot 2^{-n}+b(i\cdot 2^{-n})^2+c(i\cdot 2^{-n})^3. \quad (31)$$

with the initial contents of registers,

$$\left.\begin{array}{l}R_0=0,\ A_0=a\cdot 2^{-n}+b\cdot 2^{-2n}+c\cdot{}^{-3n} \\ B_0=2\cdot 2^{-2n}+6c\cdot 2^{-3n},\ C_0=6c\cdot 2^{-3n}.\end{array}\right\} \quad (32)$$

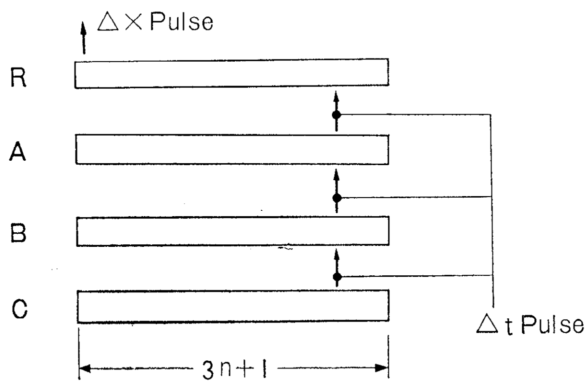


Fig. 5. Generation of $\triangle x$ pulse.

Comparing eq. (29) with eq. (31), a overflow or underflow pluse from the highest position of R register represents a positive or negative increment Δx. This pulse is supplied to a pulse motor of the plotter or the position resister of display system. As t changes from 0 to 1, t's are counted by 2^n, the upper parts of C register and lower part of R register may be omitted without producing error in Δx. Fig. (6) shows curves thus generated, with $\boldsymbol{A}$ taken as x-axis, changing $\boldsymbol{B}$ and $\boldsymbol{C}$ in the plane. The eq. (16) has the power to generate a fairly large class of the curves.

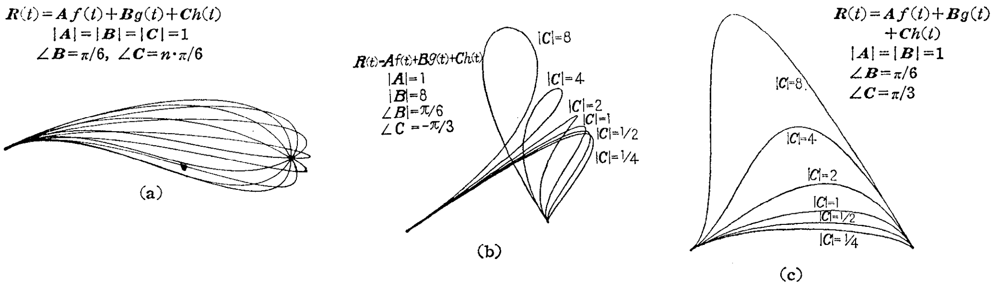


Fig. 6. Example of $R(t)$.

5. *Designing process of free-formed surface*

A designer draws what he has in his mind on the cathod ray tube or the mechanical graphic 1/0 device. Its characteristic lines are specified approximately. Then these curves are converted to an analytical form of eq. (16) with the methods stated in section 3. The computer display shows the curves which are seen from arbitrary directions and the designer corrects, modifies them until he thinks satified, then he fixes the boundaries of patch surfaces which are generated by the computer with the equation (1). When solid models are given, the measurement of the shape are made with the aid of the computer, then a fairing process follows, showing the result to the designer who will make the final decision. The fairing of a curve is simulated by the computer with a elastic line passing through the allowable small area around the measured points, under the given constraints, making the stored elastic energy minimum.

Fig. (7) and fig. (8) are the examples of the surface synthesized by the present methods. Fig. (7) is the perspective views of a surface with the fixed external boundaries and their normals, and the variable deflections and the normal directions of the central position. This is composed of four surfaces. Fig. (8) is a surface synthesized from six patch surfaces and is represented in an isometric and a perspective projection.

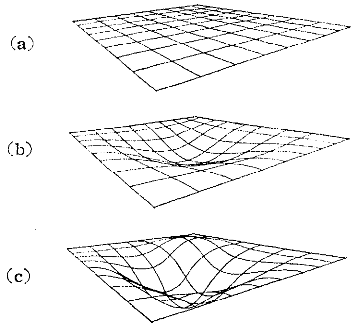
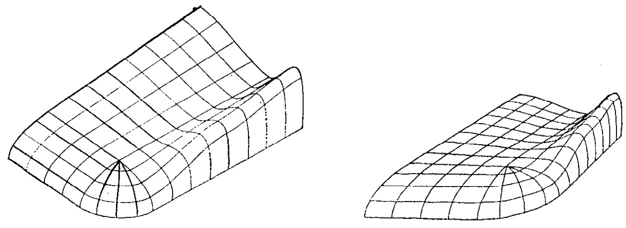


Fig. 7. Synthesized surtace, perspective projection.



(a) Isometric projection. (b) perspective projection.

Fig. 8. Synthesized surface.

Rejerences

[1] HOSAKA, M. AND T. TANI, On the new DDA. *Proc. 3rd Cof. Jap. Soc. Info. Proc.* (*JSIP*) December (1962), 25-26.

[2] HOSAKA, M., The digital resolver. *Proc. 2nd Conf. of Jap. Soc. Instr. Contr. Eng.* October (1963 , 356-370.

[3] HOSAKA, M. AND H. MURATA, Generation of functions and graphics by DDA. *Proc. 3rd Conf. of Jap. Soc. Instr. Contr, Eng.*, September (1964), 479-493.

[4] HOSAKA, M. AND M. ENDO, On generation, storage and processing of graphical data, *Joho Shori* (*Info. Processing*) *6*, 3, May (1965), 129-139.

[5] HOSAKA, M., M. ENDO AND K. KAJIWARA, Graphical 1/0 devices and applications. *Proc. 7th Conf. of JSIP*, December (1965), 15-16.

[6] HOSAKA, M., Storage and processing of curves and surfaces, *ibid*, 13-14.

[7] HOSAKA, M., K. KAJIWARA AND K. YAMADA, Divices of on-line graphical display and control. *Proc. 8th Conf. of JSIP*, December (1966), 21-22.

[8] HOSAKA, M. AND M. ENDO, Design automation applied to mechanical eng. *ibid.* 19-20.