On the Separation of All Roots of a Polynomial by Determining the Number of Roots in the Unit Circle

TATSUO TORII* AND TOSHIHIKO YOSHIOKA**

1. *Introduction*

In various iterative methods for solving a polynomial equation there have been required, for any equation, both the convergence and the reduction of the computing time. But none of them satisfies these requirements[1].

Recently with the development of digital computers, it has been taken consideration on the global convergence in a iterative process. Lehmer's method is remarkable one such that it reduces the difficulty of the convergence by contracting some region of complex plane where roots exist[2].

In this paper we find all the roots of a polynomial $f(z)$ without decreasing its degree by making clear two points:

(1) The number of roots of $f(z)$ in the unit circle Γ.

(2) All the arguments of roots on Γ, if $f(z)$ has at least one root on Γ.

2. *Number of Roots in the Unit Circle*

We begin to describe some definitions and lemmas without proofs which you may find in our Japanese paper.

We denote nth degree polynomial with complex coefficients by

$$f(z)=a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n \qquad (1)$$

for which we define a conjugate polynomial such that

$$\left.\begin{aligned}\{f(z)\}^*&\equiv f^*(z)\\ &\equiv \bar{a}_nz^n+\bar{a}_{n-1}z^{n-1}+\ldots+\bar{a}_1z+\bar{a}_0\\ &=z^n\overline{f(1/\bar{z})}.\end{aligned}\right\} \qquad (2)$$

Besides, if each root of $f(z)$ is equal to that of $f^*(z)$, we call $f(z)=0$ the reciprocal equation.

Lemma 1. Let $P(z)$ and $Q(z)$ be two polynomials such that

$$|P(z)|<|Q(z)| \qquad \text{for} \qquad |z|=1.$$

Then, $Q(z)$ and $P(z)+Q(z)$ have the same number of roots in the unit circle (Rouché's theorem).

This paper first appeared in Japanese in Joho Shori (the Journal of the Information Processing Society of Japan), Vol. 8, No. 1 (1967), pp. 9-15.

* Faculty of engineering, Osaka University, Osaka.
** Faculty of Engineering, Yamaguchi University, Ube.

Lemma 2. Eliminating the highest power of nth degree polynomial $f(z)$ by

$$g(z)=\overline{f(0)}f(z)-\overline{f^*(0)}f^*(z),$$

we get the followings:

(1) If $g(0)\neq 0$, the highest common factor of $f(z)$ and $f^*(z)$ is equal to that of $g(z)$ and $g^*(z)$. Hereafter we will neglect its constant common factor.

(2) The highest common factor $f(z)$ and $f^*(z)$ is contained in that of $g(z)$ and $g^*(z)$ for any value of $g(0)$.

(3) If $g(0)>0$, $f(z)$ and $g(z)$ have the same number of roots in the unit circle Γ. If $g(0)<0$, $f^*(z)$ and $g(z)$ have the same number of roots in that region[2].

(4) If $g(0)=0$ and if the roots of $g(z)$ do not exist on Γ, then all $g(z)$, $f(z)$ and $f^*(z)$ have the same number of roots in Γ. Moreover, if $g(z)=0$ identically, $f(z)$ and $f'^*(z)$ have the same number of roots in Γ, where $f'^*(z)\equiv\left\{\frac{d}{dz}f(z)\right\}^*$.

Following Lehmer's algorithm, we construct a polynomial sequence

$$f_1(z)=\overline{f_0(0)}f_0(z)-\overline{f_0^*(0)}f_0^*(z)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{k+1}(z)=\overline{f_k(0)}f_k(z)-\overline{f_k^*(0)}f_k^*(z)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{h+1}(z)\equiv f_{h+1}=\overline{f_h(0)}f_h(z)-\overline{f_h^*(0)}f_h^*(z)$$

from the given nth degree polynomial $f_0(z)\equiv f(z)$.

He stated a necessary and sufficient condition that polynomial $f(z)$ has at least one root in Γ, using the sequence

$$f(z)\equiv f_0(z),\ f_1(z),\ \ldots\ldots,\ f_h(z),\ f_{h+1}(z)\equiv f_{h+1} \qquad (3)$$

from which, however, we can find the number of roots in Γ.

Let n_k be the degree of $f_k(z)$. Then

$$n=n_0>n_1>\ldots\ldots>n_h>n_{h+1}=0. \qquad (4)$$

Theorem 1. In order that $f_k(z)$ and $f_k^*(z)$ $(k=0,1,\ldots\ldots,h)$ are prime to each other, it is necessary and sufficient that

$$f_{h+1}\neq 0.$$

Besides, if $f_1(0)f_2(0)\ldots\ldots f_h(0)\neq 0$ and if $f_{h+1}=0$, then the highest common factor of $f(z)$ and $f^*(z)$ is $f_h(z)$.

Proof. Sufficiency. From a relation

$$f_{h+1}=\overline{f_h(0)}f_h(z)-\overline{f_h^*(0)}f_h(z)$$

it is obvious that $f_h(z)$ and $f_h^*(z)$ are coprime if $f_{h+1}\neq 0$. Next, when $f_h(z)$ and $f_h^*(z)$ are coprime, we can find, from lemma 2, (2), the same relation between $f_{h-1}(z)$ and $f_{h-1}^*(z)$ for any value of $f_h(0)$. Similarly, we can know $f_k(z)$ and $f_k^*(z)$ are coprime.

Necessity. If $f_0(z)$ and $f_0^*(z)$ are coprime and if $f_1(0)\neq 0$, then $f_1(z)$ and $f_1^*(z)$ have evidently no common factor, applying lemma 2, (1) to the relation

$$f_1(z)=\overline{f_0(0)}f_0(z)-\overline{f_0^*(0)}f_0^*(z).$$

Similarly, $f_k(z)$ and $f_k^*(z)$ $(k=2,3, \ldots\ldots, h)$ are coprime, if $f_2(0)f_3(0)\ldots\ldots f_h(0)\neq 0$.

Therefore, by the definition of f_{h+1}, it must be non zero constant.

Now, we will prove the later half of this theorem. Applying repeatedly lemma 2, (1) to $f_k(z)$ and $f_k^*(z)$ $(k=1,2, \ldots\ldots, h)$ under the given condition $f_1(0)f_2(0)\ldots\ldots f_h(0)\neq 0$, we find that all $f_k(z)$ and $f_k^*(z)$ have the same highest common factor. Since $f_{h+1}=0$, $f_h(z)$ is equal to $f_h^*(z)$, identically, if we neglect their constant common factors. This completes the proof.

Theorem 2. Let n_k be the degree of $f_k(z)$ and μ_k be the number of roots of $f_k(z)$ in $|z|<1$.

If $f_{h+1}\neq 0$ in the sequence (3), then we get a recurrence formula

$$\mu_k=1(-f_{k+1}(0))n_k+sgn(f_{k+1}(0))\mu_{k+1}$$
$$\mu_{h+1}=0, \ k=h, \ h-1, \ \ldots\ldots, \ 1, \ 0$$

where $1(x)$ is unit function and $sgn(x)$ is sign function.

Proof. From theorem 1 $f_k(z)$ and $f_k^*(z)$ $(k=0,1, \ldots\ldots, h)$ are prime to each other when $f_{h+1}\neq 0$. So that, $f_k(z)$ has no root on the unit circle Γ.

Supposing $f_{k+1}(0)\neq 0$ in a relation

$$f_{k+1}(z)=\overline{f_k(0)}f_k(z)-\overline{f_k^*(0)}f_k^*(z)$$

we get

$$\mu_{k+1}=\mu_k, \ f_{k+1}(0)>0$$
$$\mu_{k+1}=n_k-\mu_k, \ f_{k+1}(0)<0$$

from lemma 2, (3).

If $f_{k+1}(0)=0$, then

$$\mu_{k+1}=\mu_k=n_k-\mu_k$$

from lemma 2, (4).

We can rewrite the above three relations into one formula

$$\mu_k=1(-f_{k+1}(0))n_k+sgn(f_{k+1}(0))\mu_{k+1}.$$

When $k=h+1$, it is evident $\mu_{h+1}=0$ for $f_{h+1}\neq 0$.

Theorem 3. If $f_1(0)f_2(0)\ldots\ldots f_h(0)\neq 0$ and if $f_{h+1}=0$, then

$$\mu_k=1(-f_{k+1}(0))(n_k-n_h+2\mu_h)+sgn(f_{k+1}(0))\mu_{k+1}$$
$$k=h, \ h-1, \ \ldots\ldots, \ 1, \ 0.$$

Besides, $f_h(z)$ and $f_h'^*(z)$ have the same number of roots inside the unit circle.

Theorem 4. A necessary and sufficient condition that a polynomial $p(z)$ has all the roots on the unit circle Γ is that $p(z)=0$ is a reciprocal equation and that $p'(z)$ has no root outside Γ. When $p'(z)$ has all the roots inside Γ, roots of $p(z)$ are all distinct.

3. *Argument of the Root*

We consider only about the roots of $f(z)$ on the unit circle Γ, because the roots of $f(z)$ on the circumference of radius r can be transformed on Γ by replacing

$f(z)$ with $f(rz)$. Futhermore, the roots of $f(z)$ on Γ are those of $p(z)$ which is the highest common factor of $f(z)$ and $f^*(z)$. So, we may consider only about the roots of $p(z)$ on Γ. From theorem 1, the common factor $p(z)$ is determined by

$$p(z)=f_h(z)$$

when $f_1(0)f_2(0)\ldots\ldots f_h(0)\neq 0$ and $f_{h+1}=0$. Moreover, $p(z)=0$ is a reciprocal equation.

Lemma 3. If a polynomial $p(z)$ always satisfies a relation

$$p^*(z)=cp(z)$$

c; complex number independent of z

we get the followings:

(1) $p(z)=0$ is a reciprocal equation and $|c|=1$.

(2) By writing

$$p_0(z)\equiv c^{\frac{1}{2}}p(z),\ \ 0\leq\arg c^{\frac{1}{2}}\leq\pi$$

we have

$$p_0^*(z)=p_0(z).$$

Now, let us call $p_0(z)=0$ a normalized reciprocal equation, and rewrite

$$p_0(z)=a_0z^m+a_1z^{m-1}+\ldots\ldots+a_m,\ \ a_0\neq 0 \qquad (5)$$

Lemma 4. If we take a suitable polynomial $h(z)$ whose degree is k $\left(m\geq k\geq\left[\frac{m}{2}\right]\right)$, then we can write $p_0(z)$ using $h(z)$ such as

$$p_0(z)=z^{m-k}h(z)+h^*(z).$$

Especially, denoting the minimum value of k by l $\left(=\left[\frac{m}{2}\right]\right)$, $h(z)$ can be written

$$h(z)=\begin{cases}a_0z^l+a_1z^{l-1}+\ldots\ldots+a_{l-1}z+a_l, & m=2l+1 \qquad (6a)\\ a_0z^l+a_1z^{l-1}+\ldots\ldots+a_{l-1}z+\frac{1}{2}a_l, & m=2l \qquad (6b)\end{cases}$$

where $[x]$ is a largest integer not greater than x.

Theorem 5. If we denote mth degree normalized reciprocal equation by

$$p_0(z)=a_0z^m+a_1z^{m-1}+\ldots\ldots+a_m=0$$

then we get the followings:

(1) Let m be the odd number. Then, $p_0(z)$ has at least one root on unit circle Γ.

(2) Let m be even number, i, e,. $m=2l$. Define a lth degree polynomial $h(z)$ by

$$h(z)=a_0z^l+a_1z^{l-1}+\ldots\ldots+a_{l-1}z+\frac{1}{2}a_l$$

Then, all the roots of $p_0(z)$ on Γ agree with those of a equation

$$\mathrm{R_e}\{h(z)\}=0,\ \ |z|=1.$$

Here $\mathrm{R_e}\{z\}$ is real part of z.

We prove only the later half of the theorem. If m is even, we can write

$$p_0(z)=z^lh(z)+h^*(z)$$

by the lemma 4.

Since $h^*(z)=z^l\overline{h(z)}$ on $|z|=1$, it follows that

$$p_0(z)=z^l\{h(z)+\overline{h(z)}\}=2z^l\mathrm{R_e}\{h(z)\}.$$

Therefore, $p_0(z)$ and $\mathrm{R_e}\{h(z)\}$ have the common roots on $|z|=1$.

4. *Computational Method*

First, we determine the modulus of roots. This algorithm is based on theorem 1 and theorem 2.

Let us construct the sequence (3) from given polynomial $f(z)$. If $f_{h+1} \neq 0$, the number μ of roots of $f(z)$ in the unit circle Γ is easily obtained by theorem 2. Taking an arbitrary positive number r, we can find $\mu(r)$ which is the number of roots of $f(z)$ in a circle of radius r by transforming $f(z)$ to $f(rz)$.

Given only $\mu(r)$, we can determine the modulus of roots with a preassigned accuracy by the bisection method which may be simplest.

Secondarily, we assume that $f(z)$ has at least one root on Γ. The highest common factor of $f(z)$ and $f^*(z)$ is given by $f_h(z)$ from theorem 1. Futhermore, $f_h(z)=0$ is a reciprocal equation.

Let $f_h(z)$ normalize to $p_0(z)$ and m be its degree. When $m=1$, we can easily solve $p_0(z)=0$. If $m>1$ is odd number, we change $p_0(z)$ to even degree polynomial by multiplying $(z+1)$ to $p_0(z)$. So, we may concern with only even degree reciprocal equation $p_0(z)=0$. To solve it on Γ, we may determine the roots of equation

$$R_e\{h(z)\}=0, \ |z|=1$$

from theorem 5. Here, we note the degree of $h(z)$ is $l=\frac{m}{2}$.

Let us consider the polynomial with complex coefficients

$$h(z)=(\alpha_l+i\beta_l)z^l+(\alpha_{l-1}+i\beta_{l-1})z^{l-1}+\dots\dots+(\alpha_0+i\beta_0)$$

where α'$_s$ and β'$_s$ are real numbers.

If we transform $z=e^{i\theta}$ in $R_e\{h(z)\}$, then we get lth degree trigonometric polynomial

$$R_e\{h(e^{i\theta})\}=\sum_{k=0}^{l}\alpha_k\cos k\theta-\sum_{k=1}^{l}\beta_k\sin k\theta \qquad (7)$$

Hence, we may determine the real roots of this equation. But we omit the algorithm.

In table 1 we show numerical examples which are computed by floating point number with 15 digits mantissa.

Table 1.

Example 1. $f(z)=z^6+(0.125+0.25i)z^5+2iz^4-(0.5-8.25i)z^3-(2-i)z^2-16z-2-4i=0.$

No.	Roots	Numerical solutions	
1	$2i$	0.00000 00000 0000	$+1.99999\ 99999\ 9982i$
2	$\pm\sqrt{3}\ \ -i$	1.73205 08075 6872	$-0.99999\ 99999\ 99906i$
3		−1.73205 08075 6872	$-0.99999\ 99999\ 99949i$
4	$\pm1\ \ \mp i$	0.99999 99999 99943	$-0.99999\ 99999\ 99943i$
5		−0.99999 99999 99943	$+0.99999\ 99999\ 99943i$
6	$-0.125\ -0.25i$	−0.12499 99999 99997	$-0.25000\ 00000\ 00023i$

Example 2. $f(z)=z^9+(0.2+0.1i)z^8+10iz^7-(1-2i)z^6-35z^5-(7+3.5i)z^4-50iz^3+(5-10i)z^2+24z+4.8+2.4i=0.$

No.	Roots	Numerical solutions	
1	$\pm\sqrt{2}\quad\mp\sqrt{2}\,i$	1.41421 35623 7402	−1.41421 35623 7402i
2		−1.41421 35623 7402	+1.41421 35623 7402i
3	$\pm\sqrt{\frac{3}{2}}\quad\mp\sqrt{\frac{3}{2}}\,i$	1.22474 48713 9170	−1.22474 48713 9170i
4		−1.22474 48713 9170	+1.22474 48713 9170i
5	$\pm 1\quad\mp i$	1.00000 00000 0038	−1.00000 00000 0038i
6		−1.00000 00000 0038	+1.00000 00000 0038i
7	$\pm\frac{1}{\sqrt{2}}\quad\mp\frac{1}{\sqrt{2}}\,i$	0.70710 67811 86535	−0.70710 67811 86535i
8		−0.70710 67811 86535	+0.70710 67811 86535i
9	$-0.2\quad-0.1i$	−0.20000 00000 00077	−0.09999 99999 99988i

Acknowledgement. The authors wish to thank Prof. K. Joh for the valuable advices.

References

[1] Ralston, A. AND H.S. Wilf, Mathematical Methods for Digital Computers. Wiley (1960), 233-241.

[2] D.H. Lehmer, A Machine Methods for Solving Polynomial Equations. *Jour. Assoc. Comput. 8* March (1961), 151-162.