On the Calculation Limit of Roots of Algebraic Equations

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In numerical solution of high degree algebraic equations a calculation limit parameter was established in order to avoid vain calculations and endless iterations, and was successfully applied to an ill-conditioned example with 6 close but not equal roots for illustration.

1. *Preface*

Many numerical methods for finding the roots of algebraic equations have hitherto been designed without decisive success, for a vague and too much expectation to get the sufficiently accurate roots with as many significant digits as calculation digits has been entertained.

As Abel gave first the demonstration to the theorem that the algebraic equations of 5th or higher degree can not be algebraically solved, the roots of high degree equations must be sought by iterative methods, where it is indispensable to judge the degree of convergence by any means. When the roots of equations are sought automatically using electronic computers without interruption by human judgment, the difficulty of calculation is exclusively concentrated to judgment of convergence, for which it is necessary to estimate the calculation errors quantitatively.

We happened to discover the fact that the errors in the roots of high degree algebraic equations solved by an ALGOL program using a variable word length computer FACOM 231*** are substantially constant regardless the number of calculation digits. This fact is analysed here and a nearly definite method for solving high degree algebraic equations is offered.

2. *Equations*

Generally the high degree algebraic equations can not be solved algebraically. Hence, in order to seek the zeros of polynomials we must find out by trial and error method the x satisfying

$$f(x)=0. \tag{1}$$

For any numerical method we must determine the roots basically by the condi-

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tion that the left side of eq. (1) is regarded as zero within the range of errors, then the value of $f(x)$ can neither be reduced any more, nor the smaller value of $f(x)$ has any sense, which means the existence of the calculation limit of zero value of $f(x)$.

The range of x making the value of $f(x)$ fall into the range of calculation errors is regarded as the indeterminate range of roots, which is nearly constant for the equation and can not be reduced any more if the number of calculation digits is definite.

Hereinafter an algebraic equation of nth degree with real coefficients and real roots is denoted as

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x+a_0, \tag{2}$$

and calculation is proceeded in decimal L digits.

3. *Errors of polynomial values*

When a solution y is expressed as $y=F(x_1, x_2, \ldots\ldots, x_n)$ in terms of input data $x_1, x_2, \ldots\ldots, x_n$ at every stage of calculation, if x_k has the error Δx_k respectively, then Δy, the error of y, can be estimated as follows;

$$\Delta y=\sum_{k=1}^{n}\frac{\partial F}{\partial x_k}x_k \tag{3}$$

which is the fundamental formula for error estimations valid in any case.

Supposing the errors Δa_k, Δx for a_k, x respectively, and applying eq. (3) to eq. (2) we get formally

$$\Delta f(x)=\sum_{k=0}^{n}\Delta a_kx^k+\Delta f'(x), \tag{4}$$

or in practice we get iteratively

$$\left.\begin{aligned}&\Delta f(x)=\Delta b_0\\&\Delta b_k=\Delta a_k+\Delta x_{k+1}+x\Delta b_{k+1}\\&\Delta b_n=\Delta a_n\\&\quad(k=n-1,\ n\text{-}2\ldots\ldots,\ 1,\ 0).\end{aligned}\right\}. \tag{5}$$

According to eqs. (4) and (5), if the term of Δx is significant in the vicinity of a root, then $f'(x)$ is comparatively large, and so we can make the value of $|\Delta x|$ less, keeping the magnitude of $f(x)$ fixed, therefore we can easily confirm the root. Consequently we need to think about only the case where the term of x is negligible.

The term of Δa_k is very important and useful for investigation of dependence of a root on coefficients, of which we do not discuss here. Now Δa_k comes to be regarded as zero, however, it is convenient to adjust rounded errors by this term. As the accurate estimation of rounded errors is very difficult, we regard the rounde errors as the magnitude of the last digit of the most value of elements taking parts in calculation——especially in addition. That is, we consider as if Δa_k were approximately equal to $a_k * 10^{-L}$ even if $\Delta a_k = 0$. After all, eq. (5) is calculated as follows;

$$\Delta f(x) = \sum_{k=0}^{n} |a_k x^k| * 10^{-L}. \tag{6}$$

4. *The accuracy of polynomial values*

The accurate number of digits of polynomial values is substantially estimated by $\log_{10}|f(x)/\Delta f(x)|$, therefore it is expressed as follows;

the accurate number of digits of $f(x) = L - \alpha$, (7)

where

$$\alpha = \log_{10}\left\{\sum_{k=0}^{n} |a_k x^k| / |\sum_{k=0}^{n} a_k x^k|\right\} \tag{8}$$

and α takes the positive integer value nearest to the right side value. We think $f(x)$ considerably correct when α is small, and we take α for L and think that $f(x)$ has no accuracy even up to one digit when α is larger than L.

This α is determined uniquely if the coefficients a_k and variable x are given, therefore it might be called as *loss digits* considering its relation to calculation digits.

5. *Convergence of roots*

When the roots of high degree algebraic equations are sought by any iterative method, there is no way other than to confirm the roots by testing if $f(x)$ is equal to zero after calculating the value of polynomial $f(x)$.

Therefore if $f(x)$ has the error $\Delta f(x)$, we can not ascertain whether x is a true root all over the range where the condition

$$|f(x)| \leqq |\Delta f(x)|, \tag{9}$$

or

$$|\sum_{k=0}^{n} a_k x_k| \leqq \sum_{k=0}^{n} |a_k x^k| * 10^{-L} \tag{10}$$

is satisfied, then this range is regarded as the root range in the current calculation digits.

The judgment of convergence is done by testing eq. (9) with the errors estimated quantitatively. Many methods hitherto designed have judged convergence by using a criterion that the absolute or relative error of a root is less than an appropriately preassigned constant, but they do not seem good ones, for either absolute or relative error is not directly dependent upon eq. (9). We have met so many cases where the accuracy of a root does not reach a desired value but fall into vain calculation whereas eq. (9) is satisfied.

6. *The accuracy of roots*

The calculation limit of zero value of $f(x)$ is indicated as $M* 10^{-L}$, where

$$M=\sum_{k=0}^{n}|a_k x^k|. \tag{11}$$

If differential coefficient $f'(x)$ is correct at least one digit in Newton-iteration in the vicinity of a root, then the correction shift Δx is obtained as

$$\Delta x=-f(x)/f'(x), \tag{12}$$

therefore

$$\text{the correction limit}=M* 10^{-L}/|f'(x)|, \tag{13}$$

and hence

$$\text{the number of digits of a root}=\log_{10}|x/\Delta x|=L-\alpha, \tag{14}$$

where

$$\alpha=\log_{10}|M/(xf'(x))|=\log_{10}\left\{\sum_{k=0}^{n}|a_k x^k|/|\sum_{k=0}^{n}k a_k x^k|\right\}. \tag{15}$$

This α is the integer nearest to the right side of eq. (15), taking zero if it is smaller than zero, when the accuracy is high, and when it is larger than L the accuracy is no more than one digit.

In the vicinity of a root, α is constant and determined by the quality of the equation, hence it is called *loss digits* in Newton method. Therefore α is a criterion parameter for estimating difficulty of solving high degree equations.

The difficult case in numerical solution of high degree equation is no other than the case to be demanded too high accuracy beyond the calculating limit estimated by eq. (14).

The derivative of a polynomial is generally a polynomial too, so the error and accuracy can be estimated and available for investigation of multiple-roots.

7. *An example*

We take an example

$$f(x)=(x-1.2)(x-1.21)(x-1.22)(x-1.23)(x-1.24)(x-1.25)$$

$$=x^6-7.35x^5+22.5085x^4-36.761025x^3+33.77025274x^2$$
$$-16.544850588x+3.37725036=0,$$

in order to ascertain our theory and show the following results by Newton-iteration with the number of calculating digits $L=20$;

Table 1. Details in the vicinity of roots.

Roots	Approximate values	$\|M/(xf'(x))\|$	$L-\alpha$	N
1.20	1.1999 99999 90850 67448	1.4120×10^{10}	10	6
1.21	1.2100 00000 44370 01783	7.1768×10^{10}	9	5
1.22	1.2199 99998 51908 13213	1.4590×10^{10}	9	4
1.23	1.2299 99997 56493 46369	1.4830×10^{10}	9	5
1.24	1.2399 99999 99579 95175	7.5371×10^{10}	9	5
1.25	1.2499 99999 93459 22409	1.5321×10^{10}	10	7

Table 2. Details in the vicinity of a root 1.23.

Root	Approximate values	$\|M/(xf'(x))\|$
1.23	1.2280 00000 00000 00000	2.0198×10^{11}
	1.2304 24136 42326 19151	1.4529×10^{11}
	1.2300 03965 10253 76505	1.4826×10^{11}
	1.2300 00000 31493 45587	1.4830×10^{11}
	1.2299 99997 56493 46369	1.4830×10^{11}

In Table 1, N is the number of iterations demanded to get the final approximate value of each root, of which the starting value is taken by 0.002 less than the known value.

8. *Conclusion*

Each root of high degree algebraic equations has its characteristic loss digits, which decides the degree of difficulty of solution. Adjacent roots are understood as the difficult problem where more than two roots with large loss digits prevent each other from isolation. We could ascertain our theory on several other difficult examples, of which we can not afford to list up here. Generalization of our theory to other interesting problems such as simultaneous linear equations and eigenvalue problems is naturally possible and in fact brings forth good results to some extent, of which we shall report on another occasion.

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