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Evaluation of Systems by Stochastic Process Models

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1. *Introduction*

It is well understood that a stochastic process is a mathematical model that is usefull in the analysis of a complex system. A stochastic process is based on the concept of state and its transition. In the following, we focus our attention on a finite state, continuous time stochastic process, where a system occupies one of N different states and the time between transitions is a random variable of which distribution function is not confined to particular one but allowed to be of arbitrariry defined.

Our object is to develope a theoretical method to represent a system performance by a single dimensional value that is obtained by combining several characteristics of the system such as capacity, reliability and maintenability, at substantially, non steady state condition, thus allowing easy comparison among many systems and to select one system out of them.

2. *Theory of Economical Evaluation of a System*
——*An Analysis of Stochastic Process with Rewards*

Suppose that a N-state stochastic process produces an amount of value denoted by $p_i(t)\ \Delta t$ when it is in state i between t and $t+\Delta t$, and also produces $r_{ij}(t)$ when it makes a transition from state i to j at t. We call $p_i(t)$ and $r_{ij}(t)$ the "state reward density" and "transition reward" respectively. It must be noted that $p_i(t)$ and $r_{ij}(t)$ are different dimentional quantities and also both are functions of time that means the value of a system depend not only on the probabilistic property but also on the time.

The value may be an economical value but it can be any other physical quantity relevant to the problem.

Thus, as time goes on, the stochastic process produces an amount of value that is obtained by integration of reward density that is composed of a sequence of parts of the several state reward densities added with transition rewards both are governed by the sequence of transitions. The particular sequence of the state reward densities corresponding to one particular sequence of transitions is called

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a "reward density of the system" that is a sample of every possible sequences of transitions in finite time interval.

If we can know the probabilities of occurrence of every possible sequences of transitions, we can get an ensemble average of these sequences of reward densities which is also function of time and we call it an "average reward density function of the system." Thus, by integration of it between $[O, T]$ we can get a quantity that is feasible index of the value of the system, and we call it "system value".

These quantities depend on the initial state of the system and we denote "average reward density function of the system" as $w^i(t)$ and "system value" as $W^i(T)$ indicating their initial states were i.

Besides these, we use an illustrative notation $w_n{}^i(t)$ that denotes a subset out of the ensemble of the reward densities of the system of which number of transitions between $[O, T]$ was n.

Let us trace the transition points and identify them as t_1, t_2, t_3......and denote their inter-transition intervals as T_1, T_2, T_3......, T_n, where $T_j=t_j-t_{j-1}$ $(j=1, 2, 3, \ldots\ldots, n)$. In the following, the "system value" is obtained for several cases.

2. 1. *The Case of 2 States*

We suppose the system has 2 states named state 1 and 2 respectively. When initial state was 1, general form of the "reward density of the system" is

$$w_{2n}{}^1(t)=p_1(t)\left[1-\sum_{i=1}^{n}\{u(t-t_{2i-1})-u(t-t_{2i})\}\right]+p_2(t)\sum_{i=1}^{n}\{u(t-t_{2i-1})-u(t-t_{2i})\},$$

and

$$w^1{}_{2n+1}(t)=p_1(t)\left[1-\sum_{i=1}^{n}\{u(t-t_{2i-1})-u(t-t_{2i})\}-u(t-t_{2n+1})\right]$$

$$+p_2(t)\left[\sum_{i=1}^{n}\{u(t-t_{2i-1})-u(t-t_{2i})\}+u(t-t_{2n+1})\right] \quad , \qquad (2.\ 1)$$

where $u(t)$ is Heaviside's unit step function and $n=0, 1, 2, \ldots\ldots$.

For simplicity, we put $p(t)u(t-t_i)=p(t, t_i)$. Suppose the distribution of the time from the k'th to the $k+1$'th transition is $f_k(t)$, then probability of occurrence of a series of transitions at time t_1, t_2, ...t_n is $\prod_{k=1}^{n} f_k(T_k)\, dT_k$, and if the relation $t_n=\sum_{k=1}^{n} T_k \leq T \leq t_{n+1}=\sum_{k=1}^{n+1} T_k$ is held, ensemble average of the "reward density function of the system" is

$$w^1(t)=w_0{}^1(t)\int_T^\infty f_1(T_1)dT_1+\int_0^T f_1(T_1)w_1{}^1(t)dT_1\int_{T-t_1}^\infty f_2(T_2)dT_2$$

$$+\int_0^T f_1(T_1)dT_1\int_0^{T-t} f_2(T_2)w_2^1(t)dT_2\int_{T-t_2}^{\infty} f_3(T_3)dT_3+\ldots\ldots$$
$$+\int_0^T f_1(T_1)dT_1\int_0^{T-t_1} f_2(T_2)dT_2\int_0^{T-t2}\ldots\ldots\int_0^{T-t_{n-1}} f_n(T_n)w_n^1(t)dT_n$$
$$\times\int_{T-t_n}^{\infty} f_{n+1}(T_{n+1})dT_{n+1} +\ldots\ldots . \tag{2. 2}$$

Substituting Eqs. (2. 1), Eq. (2. 2) yield

$$w^1(t)=p_1(t)-\int_0^T f_1(T_1)\ [p_1(t,\ t_1)-p_2(t,\ t_1)]dT_1+$$
$$+\int_0^T f_1(T_1)dT_1\int_0^{T-t_1} f_2(T_2)\ [p_1(t,\ t_2)-p_2(t,\ t_2)]dT_2+\ldots\ldots$$
$$+(-1)^n\int_0^T f_1(T_1)\,dT_1\int_0^{T-t_1} f_2(T_2)\,dT_2\int_0^{T-t2}\ldots\ldots\int_0^{T-t_{n-1}} f_n(T_n)\ [p_1(t,\ t_n)$$
$$-p_2(t,\ t_n)]dT_n+\ldots\ldots . \tag{2. 3}$$

Now, put $p_k(T-x)=\overline{p_k}(x)$ ($k=1$ or 2),
and we define following convolution functions $h_{km}(t)$ ($k=1$ or 2),

$$h_{k1}(t)=\int_0^t f_n(_i-x)\overline{p_k}(x)dx=f_n{}^*\overline{p_k}\quad ,$$
$$h_{km}(t)=\int_0^t f_{n-m-1}(t-x)h_{m-1}(x)dx=f_{n-m+1}{}^*f_{n-m+2}{}^*\ldots\ldots f_n{}^*\overline{p_k} \tag{2. 4}$$

Then

$$\int_0^t h_{kn}(t)\,dt=\int_0^t f_1(T_1)\,dT_1\int_0^{T-t_1}\ldots\ldots\int_0^{T-t_{n-1}} f_n(T_n)dT_n\int_0^{T-t_n}\overline{p_k}(x)dx \tag{2. 5}$$

can be easily proved and $W^1(T)$ is obtained as follows,

$$W^1(T)=\int_0^T p_1(t)dt+\sum_{i=1}^{\infty}(-1)^i\int_0^T[h_{1i}(t)-h_{2i}(t)]dt. \tag{2. 6}$$

Let Laplace transform of $h_{kn}(t)$, $\overline{p_k}(t)$ and $f_i(t)$ be $H_{kn}(s)$, $P_k(s)$ and $F_i(s)$ respectively. Then we can obtain,

$$W_1(T)=\int_0^T p_1(t)dt+\sum_{i=1}^{\infty}(-1)^i\int_0^T L^{-1}\{[p_1(s)-p_2(s)]\}\prod_{j=1}^{i}F_j(s)dt, \tag{2. 7}$$

where $L^{-1}\{\ \ \}$ means inverse transform operator.

$P_k(s)$ is a Laplace transform of $p_k(T-x)$ in regard to variable x and implicitly include T in it. So, $W^1(T)$ is comparatively complex function of T. However, we suppose temporarily that T included implicitly in $P_k(s)$ is fixed and take Laplace transform of Eq. (2. 7) in regard to T. Then we can obtain

$$W^1(s)=(1/s)P_1(s)+\sum_{i=1}^{\infty}(-1)^i(1/s)[p_1(s)-p_2(s)]\prod_{j=1}^{i}F_j(s). \tag{2. 8}$$

This gives a correct result only when $t=T$. However, $P_1(s)-P_2(s)$ would be independent of T, Eq. (2. 10) is always justified.

In most cases, distribution function is specified to the state as

$f_{2n+1}(t)=f(t)$ and $f_{2n}(t)=g(t)$.

Then we obtain

$$W^1(s)=(1/s)P_1(s)-(1/s)[P_1(s)-P_2(s)]F(s)[1-G(s)]/[1-F(s)G(s)]. \qquad (2.\ 9)$$

In the case initial state was state 2, $P_1(s)$ and $P_2(s)$ are exchanged.

2. 2. *The case of N States*

We suppose the case of N states. The distribution function of the time of trasition from state i to j after then transition to state i from any other state has occurred is designated $f_{ij}(t)$, then $\sum_{j\neq i} f_{ij}(t)=f_i(t)$ means the distribution function of inter-transition time of state i regardless of next state. Suppose the system changes the state in the sequence of i, j, k, ..., 1, m and number of transitions is n between $[O,\ T]$, that is, $T_{n+1}>T>T_n$. Then, as in the preceeding case,

$$\begin{aligned} w_n{}^i(t)=&p_i(t)-p_i(t,\ t_1)+p_j(t,\ t_1)-p_j(t,\ t_2)+\ldots\ldots \\ &+p_l(t,\ t_{n-1})-p_l(t,\ t_n)+p_m(t,\ t_n). \end{aligned} \qquad (2.\ 10)$$

We take an ensemble average and obtain the "average reward density of the system"

$$\begin{aligned} w^i(t)=&p_i(t)\int_T^\infty f_1(T_1)dT_1+\sum_{j\neq i}\int_0^T f_{ij}(T_1)[p_i(t)-p_1(t,\ t_1)+p_j(t,\ t_1)]\,dT_1\int_{T-t_1}^\infty f_j(T_2)dT_2 \\ &+\sum_{j\neq j}\sum_{k\neq j}\int_0^T f_{ij}(T_1)dT_1\int_0^{T-t1} f_{jk}(T_2)[p_i(t)-p_i(t,t_1)+p_j(t,t_1)-p_j(t,\ t_2) \\ &+p_k(t,\ t_2)]dT_2\int_{T-t_2}^\infty f_3(T_3)dT_3+\ldots\ldots \end{aligned} \qquad (2.\ 11)$$

Then Laplace transform of the system value between $[O,\ T]$ is obtained as follows,

$$\begin{aligned} W^i(s)=&P_i(s)/s-(1/s)P_i(s)\sum_{j\neq i}F_{ij}(s)+(1/s)\sum_{j\neq i}F_{ij}(s)P_i(s)-(1/s)\sum_{j\neq i}F_{ij}(s)P_j(s)\sum_{k\neq j}F_{jk}(s) \\ &+(1/s)\sum_{j\neq i}\sum_{k\neq j}F_{ij}(s)F_{jk}(s)P_k(s)-\ldots\ldots \end{aligned} \qquad (2.\ 12)$$

As in the preceeding case, this gives correct result only when $t=T$ unless $p_i(t)$s are independent of T.

We define null functions $F_{ii}(s)\equiv 0$ and use column vector $\boldsymbol{W}(s)$, $\boldsymbol{P}(s)$ and $\boldsymbol{V}(s)$ with components $W^i(s)$, $P_i(s)$ and $P_i(s)F_i(s)$ respectively. Moreover, we define a matrix $\boldsymbol{F}(s)$ with component $F_{ij}(s)$ and a diagonal matrix $\overline{\boldsymbol{F}}(s)$ with component $F_i(s)$, then Eq. (2. 12) becomes

$$\begin{aligned} \boldsymbol{W}(s)=&(1/s)\{\boldsymbol{P}(s)+\boldsymbol{F}(s)\boldsymbol{P}(s)+\boldsymbol{F}(s)^2\boldsymbol{P}(s)+\boldsymbol{F}(s)^3\boldsymbol{P}(s)+\ldots\ldots\} \\ &-(1/s)\{\boldsymbol{V}(s)+\boldsymbol{F}(s)\boldsymbol{V}(s)+\boldsymbol{F}(s)^2\boldsymbol{V}(s)+\boldsymbol{F}(s)^3\boldsymbol{V}(s)+\ldots\ldots\} \\ =&(1/s)[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}[\boldsymbol{P}(s)-\boldsymbol{V}(s)]. \end{aligned} \qquad (2.\ 13)$$

2. 3. *Transition Rewards*

We consider the case when the system produces the reward $r_{ij}(t)$ accompanied with the transition from state i to state j. The transition rewards are derived from the preceeding case introducing auxilary interim state corresponding to each state and making their state times tend to 0. Let us suppose there is interim state $\bar{j}$ always immediately preceeding to state j and the transition from any state to state j is decomposed to two transitions, first to state $\bar{j}$ and then from $\bar{j}$ to j. The stay time in state $\bar{j}$ is supposed to be small constant a and, while in this state, we define the reward density $q_{ij}(t)$ to the system such that it satisfy the relation $aq_{ij}(t)=r_{ij}(t)$ when a tend to 0. By the same process as preceeding case and making a tend to 0, we can get

$$\boldsymbol{W}(s)=(1/s)[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]\boldsymbol{P}(s)+[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}\boldsymbol{R}(s) \tag{2. 14}$$

where $\boldsymbol{R}(s)$ is a column vector with component $\sum_j F_{ij}(s)R_{ij}(s)$.

2. 4. *Markov Process*

When the inter-transition time distributions are all given as exponential, we can get simpler form. Suppose a transition probability from state i to state j is given as $\alpha_{ij}\Delta t$ regardless of its past history, then,

$$f_{ij}(x)=\alpha_{ij}\exp(-\alpha_{ij}x)\prod_{k\neq i,j}\int_x^\infty \alpha_{ik}\exp(-\alpha_{ik}z)dz=\alpha_{ik}\exp\left(-\sum_{k\neq i}\alpha_{ik}\right).$$

We can denote

$$[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}=(1/\Delta)[\Delta_{ij}], \tag{2. 15}$$

where Δ is a determinant of $[\boldsymbol{I}-\boldsymbol{F}(s)]$ and Δ_{ij}'s are the cofactors of $[\boldsymbol{I}-\boldsymbol{F}(s)]$. Let us define a matrix $\boldsymbol{A}$ with $-\alpha_i$ as the diagonal components and α_{ij} as the (i, j) components $(i\neq j)$, then Δ and Δ_{ij} are expressed as follows,

$$\Delta=|s\boldsymbol{I}-\boldsymbol{A}|/\prod_{i=0}^{N}(s+\alpha_i),$$

$$\Delta_{ij}=(s+\alpha_i)\overline{\Delta}_{ij}/\prod_{k=0}^{N}(s+\alpha_k) \tag{2. 16}$$

where $\overline{\Delta}_{ij}$'s are the cofactors of $[s\boldsymbol{I}-\boldsymbol{A}]$. As the diagonal components of $[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]$ are $1-\alpha_i/(s+\alpha_i)=s/(s+\alpha_i)$, we obtain

$$[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}\ [\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]=(s/\Delta)[\Delta_{ij}/(s+\alpha_i)]=s[s\boldsymbol{I}-\boldsymbol{A}]^{-1},$$

and

$$\begin{aligned}\boldsymbol{W}(s)&=(1/s)[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]\{\boldsymbol{P}(s)+[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]^{-1}\boldsymbol{R}(s)\}\\&=[s\boldsymbol{I}-\boldsymbol{A}]^{-1}[\boldsymbol{P}(s)+\boldsymbol{R}(s)],\end{aligned} \tag{2. 17}$$

where $\boldsymbol{R}(s)$ is a column vector with the components $\sum_{i\neq j}\alpha_{ij}R_{ij}(s)$.

In a particular case when $p_i(t)=\text{const.}=p_i$ and $r_{ij}(t)=\text{const.}=r_{ij}$,

$$\boldsymbol{W}(s)=(1/s)[s\boldsymbol{I}-\boldsymbol{A}]^{-1}(\boldsymbol{P}+\boldsymbol{R})$$

is obtained and this coincides with that of R.A. Howard.[1]

2. 5. *Deduction of Several Probabilistic Characteristics*

From the preceeding results, we can deduce several meaningfull characteristics of the system.

2. 5. 1. *Time Fraction of a State at Steady State*

Time fraction of a state, for instance state i, or in other word, the probability ν_i that the system is in state i at arbitrarily chosen time t at steady state, is derived. We substitute $p_i(t)=1$ and $p_j(t)=0$ $(j\neq i)$ into Eq. (2. 13) which,in this case, always gives the correct result. If there exists steady state, ν_i approaches asymtoticaly constant value regardless of initial state, and so we multiply row vector $\boldsymbol{C}_i$ of which components are all 0 except i'th component that is 1 to Eq. (2. 13) from left side and denote it $W_i(s)$, that is,

$$W_i(s)=(1/s^2)\boldsymbol{C}_i[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]\boldsymbol{P}, \tag{2. 18}$$

where $\boldsymbol{P}$ is a column vector with the components 0 except i'th one that is 1. Then, ν_i can be shown as

$$\nu_i=\lim_{T\to\infty} W_i(T)/T=\lim_{T\to\infty} dW_i(T)/dT=\lim_{s\to 0} sL\{dW_i(T)/dT\}=\lim_{s\to 0} s^2 W_i(s). \tag{2. 19}$$

Substituting Eq. (2. 18) into Eq. (2. 19), we obtain

$$\nu_i=\lim_{s\to 0}\boldsymbol{C}_i[\boldsymbol{I}-\boldsymbol{F}(s)]^{-1}[\boldsymbol{I}-\overline{\boldsymbol{F}}(s)]\boldsymbol{P}=\lim_{s\to 0} \Delta_{ii}(1-\boldsymbol{F}_i(s))/|\boldsymbol{I}-\boldsymbol{F}(s)|, \tag{2. 20}$$

where Δ_{ii} is the (i, i) cofactor of $[\boldsymbol{I}-\boldsymbol{F}(s)]$.

This result can be used to know the system availability or the like.

2. 5. 2. *Mean Time to Get at Specifled State First*

Mean time the system first get at any specified state, for instance state k, after starting from state i can also be derived. Let the state k be end state, that is $f_{kj}(x)=0$, $(j\neq k)$ and $p_k(x)=0$ and $p_j(x)=1$ $(j\neq k)$.

Then, mean time τ can be derived as

$$\tau=\lim_{T\to\infty} W(T)=\lim_{s\to 0} s W^i(s)$$

This result can be used to get MTTFF (Mean Time To First Failure) or the like.

2. 5. 3. *The Probability That the System Is in a Specified State at Given Time*

Let the probability that the system is in a state k at time t after starting from state i be $b_k^i(t)$ and its Laplace transform be $B_k^i(s)$. As the system value is expressed as following way

$$\sum_i b_k^i(t)p_j(t)=dW^i(t)/dt,$$

we can get $B_k^i(s)$ substituting $p_k(t)=1$ and $p_j(t)=0$, $(j\neq k)$, into Eq. (2. 13) and calculating $sW^i(s)$.

2. 6. *A Simulation by the Analog Computation Method*

The calculation of the system reward is troublesome and it is hoped to get the reward more easily for practical purpose. Here, we propose a simulation by the analog computation method. As an illustrating example, we use the result of the 2 states. In Eq. (2. 9), first term of right side is a simple integration term and few problems. The second term is analogous to a response function in the feed back control system illustrated in Fig. 1 with the input $\overline{p_1}(t)-\overline{p_2}(t)$, and the output of this system at $t=T$ starting from $t=0$ is what we want. An example of this method is shown in the next section.

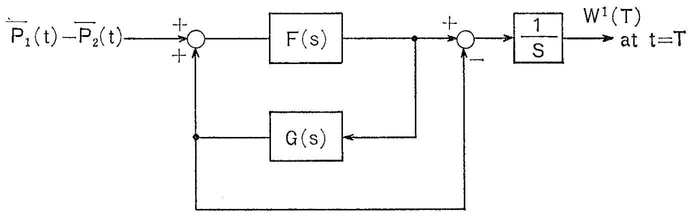


Fig. 1. An Example of Analog Simulation Block Diagram.

3. *Some Applications to the System Design*

The preceeding method is applied to the system design of real time information processing system. A real time system is not always in the steady state because its input rate varies with time and this fact is one of the reasons why the system becomes too large because a system is sometimes designed to adapt to the peak input rate. However, a system designer must take into account the reliability and maintenability as well. These factors of the system design are all combined to one dimensional "system value" to allow easy evaluation of the system.

The system value, here, is thought to depend on the number of input requests that could not be processed immediately because of (1) overflow to the system and (2) the system's failure.

Contribution of each lost input to the system value is supposed to be C_1 for case (1) and C_2 for case (2) (These are negative "value" to the system).

As an example, we suppose the case of 2 states, that is, the system is operating (state 1) and the system has failured and is under repaire (state 2).

state reward densities are approximately $p_1(t)=C_1$ max $(\lambda(t)-\mu,\ o)$ and $p_2(t)=C_2\lambda(t)$ respectively where $\lambda(t)$ is input rate and μ is system's capacity $(=1/\text{throughput})$. As the space is restricted, we show only a few examples.

Fig. 2. shows the case of single system where the system is supposed to fail by chance, that is, in accordance with negative exponential distribution with MTBF (Mean Time Between Failure)$1/\alpha$. Then, $F(s)=\alpha/(s+\alpha)$ is substituted in Eq. (2. 9). Two cases of distributions of $g(x)$ are shown.

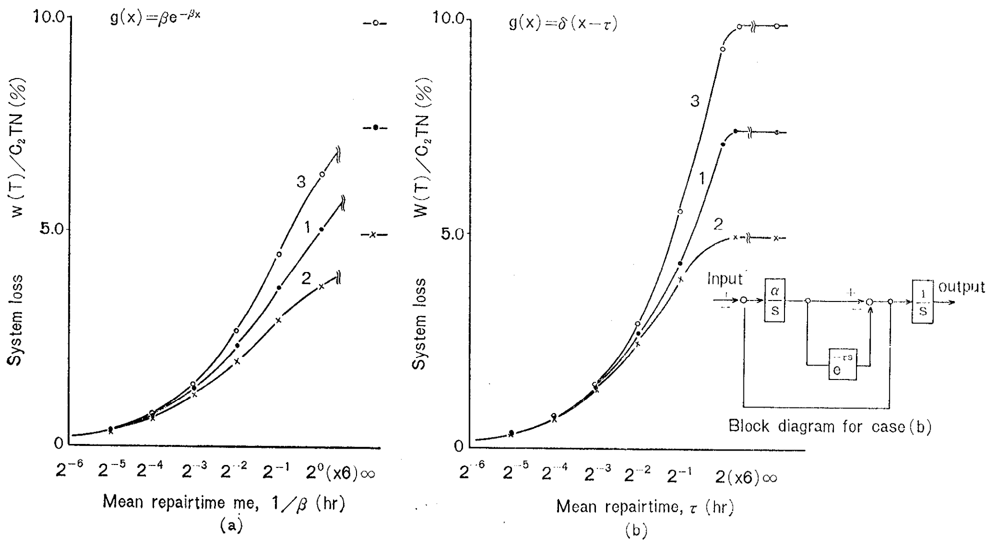
Parallel redudant system with one repaire man also supposed to fail by chance is represented by 3 states; state 0 (both subsystem operating), state 1 (one subsystem operating and the other under repair), and state 2 (one subsystem under repair and the other waiting). We can obtain

$$
\begin{aligned}
&F_{01}(s)=2\alpha/(s+2\alpha),\\
&F_{10}(s)=G(s+\alpha),\\
&F_{12}(s)=\alpha[1-G(s+\alpha)]/(s+\alpha),\\
&F_{21}(s)=[s/(s-\alpha)][G(\alpha)-G(s)]/[1-G(\alpha)],
\end{aligned}
$$

while all other components are 0. $G(s)$ is Laplace transform of repair time distribution.

Substituting them into Eq. (2, 13), we can get the system value.

Time fraction of state 2 and MTTFF when system's initial state was state 0 are derived, as



$$
\lambda(t)=\begin{cases} N, \ldots\ldots\ldots\ldots\ldots\ldots \text{Cace 1} \\ 2N(T-t)/T \ldots \text{Case 2} \\ 2Nt/T \ \ldots\ldots\ldots \text{Case 3} \end{cases}
\qquad f(x)=\alpha\, e^{-\alpha x} \quad \alpha=1/51.2 \text{ hr.}
$$

Fig. 2. System loss vs. mean repaire time (Single system).

$$\nu_2 = 2[m\alpha + G(\alpha) - 1]/[2m\alpha + G(\alpha)],$$
$$\text{MTTFF} = (3/2\alpha)[1 - (2/3)G(\alpha)]/[1 - G(\alpha)],$$

where $m = -G'(0)$ means mean repair time.

In the case of parallel redundant system with two repair men, there need infinite number of states to represent the system except when repair time is also exponentially distributed because every state besides both operating depend on the past history and have different distributions of time. This infinite state system is out of our scope and more study is necessary.

Besides these simple examples, many other system configurations were treated. The problem of analytical comparison between parallel redundant and stand-by system with the file recovery time in mind and multiple processor system supervised by one control unit are few examples of them though not shown here.

4. *Conclusion*

We tried to develope the method to evaluate a system with time variational reward density using a stochastic process. If the system is described as a Markov process with the fixed rewards, simpler method is already obtained. We investigated the problem from the another point of view and obtained the results for the cases not confined to a simple Markov process and with the time variational rewards. As the real system can not always be represented by the Markov process, this extension seems to be necessary.

Meanwhile, there arose a problem of infinite state process that can not be treated by our method and more study is necessary.

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