Sensitvity of Method for Testing Randomness Utilizing Random Walk Process

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1. *Introduction*

Previously the method for testing randomness utilizing random work process has been suggested by the author ([1]). In this method the following procedures are needed:

(1) Normalized random walk simulation ([2]) is carried out in a simply connected domain M with a boundary Γ set up in a space of lattice lines using random numbers to be tested. However, the boundary Γ is made up of n lattice points $(Q_1, Q_2, \cdots\cdots, Q_n)$.

(2) Frequency $\tilde{U}(P\to Q_i)$ $(i=1, 2, \cdots\cdots, n)$ that a random walker starting from a point P arrives at a point Q_i on Γ is determined experimentally, however the starting point P is set in the domain M. ($\tilde{U}(P\to Q_i)$ is called experimental value.)

(3) Normalized random walk simulation is ideally carried out in the domain M with the boundary Γ using complete random numbers, and probability $U(P\to Q_i)$ is determined theoretically. ($U(P\to Q_i)$ is called theoretical value.)

(4) χ^2 test is used between theoretical values and experimental values.

This method is the one for testing randomness of random numbers used in the simulation in general. Further, the method has a function of the so-called frequency test. For this purpose, it is necessary to know partial differential coefficient of probability function $U(P\to Q_i)$ numerically. But in general, it is difficult to calculate the coefficient analytically.

In this paper, several examples of partial differential coefficient are shown. They are concerned with a boundary Γ of simple shape, and are obtained by the numerical calculation named mass division method ([3]) (explosive methed[4]).

The partial differential coefficient of probability function corresponds to the sensitivity of frequency test by means of this method.

2. *Relation between probability $U(P\to Q_i)$ and deviation of probability whitch random numbers are generated*

For simple expression, random walk simulation is to be carried out in two-diamensional space.

A random walker being at one point of the lattice may move to any one of

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the four neighboring points of the lattice point, with equal probability (1/4) in case of two-dimensions, according to the instruction which is the form of either a random number or a random signal such as $+X$, $-X$, $+Y$ and $-Y$, which defines one direction of four neighbores.

Let α, β, γ, and δ represent the probabilities which correspond to the respective ones ones of generating random number $+X$, $-X$, $+Y$ and $-Y$, and the function $U(P\rightarrow Q_i)$ have the variable α, β, γ and δ. Then the function may be written as $U(P\rightarrow Q_i;\ \alpha, \beta, \gamma, \delta)$. For the case of complete random number (ideally equi-distributed random number), the variable α, β, γ and δ should be all 1/4 theoretically, but really, there are deviations $d\alpha$, $d\alpha$, $d\gamma$ and $d\delta$ respectively from 1/4 in generating random numbers which are to be tested. Therefore, a function $\tilde{U}(P\rightarrow Q_i;\ \alpha, \beta, \gamma, \delta)$ is generally rewritten as

$$\tilde{U}\left(P\rightarrow Q_i;\ \frac{1}{4}+d\alpha, \frac{1}{4}+d\beta, \frac{1}{4}+d\gamma, \frac{1}{4}+d\delta\right)$$

When $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ are sufficiently small, the function $\tilde{U}$ can be expanded as follows using Taylor's theorem:

$$\tilde{U}\left(P\rightarrow Q_i;\ \frac{1}{4}+d\alpha, \frac{1}{4}+d\beta, \frac{1}{4}+d\gamma, \frac{1}{4}+d\delta\right)-U\left(P\rightarrow Q_i;\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$
$$=d\alpha\frac{\partial U}{\partial \alpha}+d\beta\frac{\partial U}{\partial \beta}+d\gamma\frac{\partial U}{\partial \gamma}+d\delta\frac{\partial U}{\partial \delta}+\frac{1}{2!}\left(d\alpha\frac{\partial}{\partial \alpha}+\cdots\cdots+d\delta\frac{\partial}{\partial \delta}\right)^2 U\cdots\cdots$$
$$(i=1, 2, \cdots\cdots, n). \qquad (1)$$

The left hand side of Eq. (1) is difference between experimental value $\tilde{U}$ and theoretical value U. On the other hand, if the deviations are sufficiently small, the terms with greater or equal second order of deviation in the right hand side of Eq. (1) may be neglected, and the terms with first order $d\alpha\frac{\partial U}{\partial \alpha}+\cdots\cdots$ $+d\delta\frac{\partial U}{\partial \delta}$ remain. If the first order of derivatives $\frac{\partial U}{\partial \alpha}, \frac{\partial U}{\partial \beta}, \frac{\partial U}{\partial \gamma}, \frac{\partial U}{\partial \delta}$ are obtained numerically, the deviations $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ may be determined solving the simultaneous equations with four unknows by means of least squares. When the value of partial differential coefficient $\left(\frac{\partial U}{\partial \alpha}\right)_{P,\ Q_i:\ \text{fixed}}, \cdots\cdots$ is more than unity, it is regarded that this method is highly sensitive.

3. *Numerical calculation of the partial differential coefficient*

If only a starting point P, a boundary Γ and the lattice points Q_i $(i=1, 2, \cdots\cdots, n)$ are given, the values of $\left(\frac{\partial U}{\partial \alpha}\right)_{P,Q_i:\ \text{fixed}}, \cdots\cdots$ are caluculated using the mass division method.

First, values of coefficient for case of simplest shape as shown in Fig. 1 (a) are caluculated.

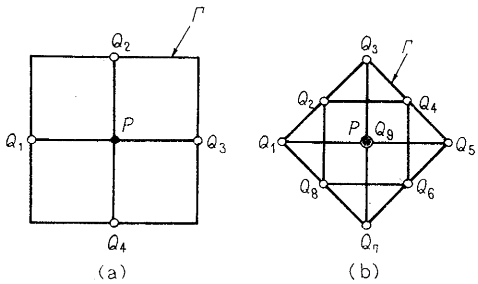

Fig. 1. An example of high symmetrical boundary Γ.

$$U(P\to Q_i;\ \text{all } 1/4)=\frac{1}{4} \qquad (i=1,2,3,4)$$

$$U(P\to Q_1;\ 1/4+d\alpha, 1/4+d\beta, \cdots\cdots)=\frac{1}{4}+d\gamma$$

....

$$U(P\to Q_4;\ 1/4+d\alpha, 1/4+d\beta, \cdots\cdots)=\frac{1}{4}+d\delta$$

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\delta}=0, \qquad \frac{\partial U}{\partial\gamma}=1 \qquad \text{for } i=1$$

....

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\gamma}=0, \qquad \frac{\partial U}{\partial\delta}=1 \qquad \text{for } i=4.$$

Next, values for case of shape as shown in Fig. 1 (b) are calculated. As the starting point P is the boundary point Q_9 at the same time, probability $U(P\to Q_9)$ is meant a recurrent probability.

$$U(P\to Q_i;\ \text{all } 1/4)=\frac{1}{16} \qquad (i=1,3,5,7)$$

$$U(P\to Q_i;\ \text{all } 1/4)=\frac{1}{8} \qquad (i=2,4,6,8)$$

$$U(P\to Q_9;\ \text{all } 1/4)=\frac{1}{4}$$

$$U(P\to Q_1;\ 1/4+d\alpha, \cdots\cdots)=\left(\frac{1}{4}+d\gamma\right)^2$$

....

$$U(P\to Q_9;\ 1/4+d\alpha, \cdots\cdots)=2\left[\left(\frac{1}{4}+d\alpha\right)\left(\frac{1}{4}+d\gamma\right)+\left(\frac{1}{4}+d\beta\right)\left(\frac{1}{4}+d\delta\right)\right]$$

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\delta}=0, \qquad \frac{\partial U}{\partial\gamma}=\frac{1}{2} \qquad \text{for } i=1$$

....

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\gamma}=\frac{\partial U}{\partial\delta}=\frac{1}{2} \qquad \text{for } i=9$$

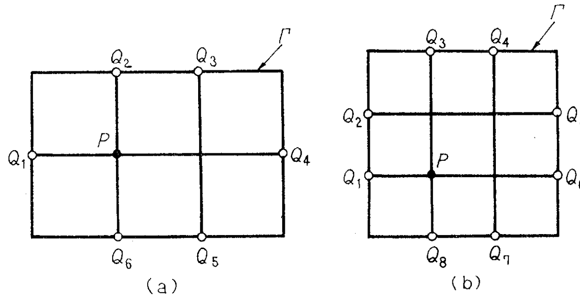

Fig. 2. An example of low symmetrical bounding Γ.

Further, the following results are obtained for case of shape as shown in Fig. 2 (a).

$$U(P\to Q_i;\ \text{all } 1/4)=\frac{4}{15} \qquad (i=1,2,6)$$

$$U(P\to Q_j;\ \text{all } 1/4)=\frac{1}{15} \qquad (j=3,4,5)$$

$$U(P\to Q_1;\ 1/4+d\alpha,\ \cdots\cdots)=\left(\frac{1}{4}+d\gamma\right)+\left(\frac{1}{4}+d\gamma\right)^2\left(\frac{1}{4}+d\alpha\right)+\cdots\cdots$$

$$\doteqdot\frac{4}{15}+\frac{16}{225}d\alpha+\frac{256}{225}d\gamma$$

....

$$\frac{\partial U}{\partial\alpha}=\frac{16}{225},\ \frac{\partial U}{\partial\gamma}=\frac{256}{225}>1,\ \frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\delta}=0, \qquad \text{for } i=1$$

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\gamma}=\frac{16}{225},\ \frac{\partial U}{\partial\beta}=\frac{240}{225}>1,\ \frac{\partial U}{\partial\delta}=0, \qquad \text{for } i=2$$

$$\frac{\partial U}{\partial\alpha}=\frac{64}{225},\ \frac{\partial U}{\partial\beta}=\frac{60}{225},\ \frac{\partial U}{\partial\gamma}=\frac{4}{225},\ \frac{\partial U}{\partial\delta}=0, \qquad \text{for } i=3$$

$$\frac{\partial U}{\partial\alpha}=\frac{124}{225},\ \frac{\partial U}{\partial\beta}=\frac{\partial U}{\partial\delta}=0,\ \frac{\partial U}{\partial\gamma}=\frac{4}{225}, \qquad \text{for } i=4$$

$$\frac{\partial U}{\partial\alpha}=\frac{64}{225},\ \frac{\partial U}{\partial\beta}=0,\ \frac{\partial U}{\partial\gamma}=\frac{4}{225},\ \frac{\partial U}{\partial\delta}=\frac{60}{225}, \qquad \text{for } i=5$$

$$\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\gamma}=\frac{16}{225},\ \frac{\partial U}{\partial\beta}=0,\ \frac{\partial U}{\partial\delta}=\frac{240}{225}>1, \qquad \text{for } i=6$$

There are several coefficients which are more than unity.

Last, in Table 1 and Table 2, the values for case of Fig. 2 (b) and Fig. 3 are shown respectively.

4. *An example of experiment*

This method is applied to test randomness of artificial random numbers (Kitagawa's Table [3]). In Table 3, results of usual frequency test are shown. In Table 4～Table 7, results of random walk simulation are shown. However, the boundaries as shown in Fig. 1 (a), (b) and Fig. 2 (a), (b) were used.

Table 1. Partial differential coefficients of function U (shown in fig. 2 (b)).

Point of arrival	$\frac{\partial U}{\partial \alpha}$	$\frac{\partial U}{\partial \beta}$	$\frac{\partial U}{\partial \gamma}$	$\frac{\partial U}{\partial \delta}$
Q_1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{272}{225}>1$	$\frac{1}{9}$
Q_2	$\frac{11}{144}$	$\frac{53}{144}$	$\frac{59}{144}$	$\frac{5}{144}$
Q_3	$\frac{5}{144}$	$\frac{59}{144}$	$\frac{53}{144}$	$\frac{11}{144}$
Q_4	$\frac{7}{36}$	$\frac{13}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
Q_5	$\frac{13}{36}$	$\frac{7}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
Q_6	$\frac{59}{144}$	$\frac{5}{144}$	$\frac{11}{144}$	$\frac{53}{144}$
Q_7	$\frac{53}{144}$	$\frac{11}{144}$	$\frac{5}{144}$	$\frac{59}{144}$
Q_8	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{272}{225}>1$

Table 2. Partial differential coefficients of function U (shown in Fig. 3).

Point of arrival	$\frac{\partial U}{\partial \alpha}$	$\frac{\partial U}{\partial \beta}$	$\frac{\partial U}{\partial \gamma}$	$\frac{\partial U}{\partial \delta}$
Q_1	$\frac{19}{144}$	$\frac{19}{144}$	$\frac{119}{144}$	$\frac{54}{144}$
Q_2	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{27}{32}$	$\frac{5}{16}$
Q_3	$\frac{19}{144}$	$\frac{54}{144}$	$\frac{119}{144}$	$\frac{19}{144}$
Q_4	$\frac{19}{144}$	$\frac{119}{144}$	$\frac{54}{144}$	$\frac{19}{144}$
Q_5	$\frac{5}{16}$	$\frac{27}{32}$	$\frac{5}{16}$	$\frac{3}{16}$
Q_6	$\frac{54}{144}$	$\frac{119}{114}$	$\frac{19}{144}$	$\frac{19}{144}$
Q_7	$\frac{119}{144}$	$\frac{54}{144}$	$\frac{19}{144}$	$\frac{19}{144}$
Q_8	$\frac{27}{32}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$
Q_9	$\frac{119}{144}$	$\frac{19}{144}$	$\frac{19}{144}$	$\frac{54}{144}$
Q_{10}	$\frac{54}{144}$	$\frac{19}{144}$	$\frac{19}{144}$	$\frac{119}{144}$
Q_{11}	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{27}{32}$
Q_{12}	$\frac{19}{144}$	$\frac{19}{144}$	$\frac{54}{144}$	$\frac{119}{144}$

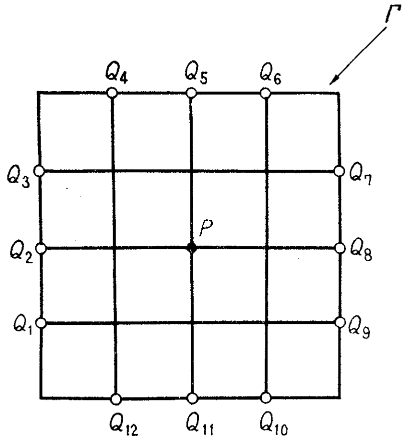

Fig. 3. An example of high symmetrical boundary Γ.

Table 3. Results of frequency test

Random number	Frequency	Deviation
$+X$	0.268	+0.018
$-X$	0.254	+0.004
$+Y$	0.233	−0.017
$-Y$	0.245	−0.005
Total	1.000	0.000

Table 4. Results of random walk simulation using Fig. 1 (a).

Point of arrival	U	$\tilde{U}$	$\tilde{U}-U$
Q_1	0.250	0.233	−0.017
Q_2	0.250	0.254	+0.004
Q_3	0.250	0.268	+0.018
Q_4	0.250	0.245	−0.005
Total	1.000	1.000	0.000

1,000 random trips are carried out.

Table 5. Results of random walk simulation using Fig. 1 (b).

Point of arrival	U	$\tilde{U}$	$\tilde{U}-U$
Q_1	0.063	0.056	−0.007
Q_2	0.125	0.118	−0.007
Q_3	0.063	0.068	0.005
Q_4	0.125	0.120	−0.005
Q_5	0.063	0.074	0.011
Q_6	0.125	0.128	0.003
Q_7	0.063	0.066	0.003
Q_8	0.125	0.096	−0.029
Q_9	0.250	0.274	0.024
Total	1.002	1.000	−0.002

500 random trips are carried out.

Table 6. ResSlts of random walk simulation using Fig. 2 (a).

Point of arrival	U	$\tilde{U}$	$\tilde{U}-U$
Q_1	0.267	0.246	−0.021
Q_2	0.267	0.276	0.009
Q_3	0.067	0.068	0.001
Q_4	0.067	0.080	0.013
Q_5	0.067	0.066	−0.001
Q_6	0.267	0.262	−0.005
Total	1.002	0.998	−0.004

740 random trips are carried out.

Table 7. Results of random walk simulation using Fig. 2 (b).

Point of arrival	U	$\tilde{U}$	$\tilde{U}-U$
Q_1	0.292	0.273	−0.019
Q_2	0.083	0.081	−0.002
Q_3	0.083	0.091	0.008
Q_4	0.042	0.032	−0.010
Q_5	0.042	0.059	0.017
Q_6	0.083	0.093	0.010
Q_7	0.083	0.087	0.004
Q_8	0.292	0.283	−0.009
Total	1.000	0.999	−0.001

494 random trips are carried out.

Table 8. Estimates of deviation of probability.

Used Γ	$d\alpha$	$d\beta$	$d\gamma$	$d\delta$
Fig. 1 (a)	0.018	0.004	−0.017	−0.005
Fig. 1 (b)	0.018	0.004	−0.017	−0.005
Fig. 2 (a)	0.022	−0.001	−0 014	−0.007
Fig. 2 (b)	0.009	0.012	−0.018	−0.003

In Table 8, estimates of deviation of probability which random numbers are generated are shown, and they are obtained solving the simultaneous equations by means of least squares.

5. *Conclusion*

It seems that there is a discrepancy between mathematical definition of randomness and interpretation of randomness in its tests. In order to eliminate to this discrepancy, the partial derivatives of probability function will be useful.

References

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