Tree of Eigen-value of Two Dimensional Figure and its Application

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1. *Introduction*

Previously, the authors have tried to solve the eigen-value problem expressed by Helmholtz's differential equation using the outer normals of Green function [1], [7], where Helmholtz's differential equation is $\Delta\Phi+\lambda\Phi=0$ in simply connected domain R, $\Phi=0$ on an arbitrary boundary Γ and λ is eigen-value.

In this paper, we have used the finite difference approximation method, Jacobi method and error correction method [2], and have chosen several series of figures (shape of boundary is called as figure), and further have calculated their eigenvalues by digital computer, finally have made the tree of eigen-value [3]. As the result, it is possible to estimate optimum relaxation factor [4].

2. *Method for calculating eigen-value*

2.1 *Lattice space and difference operator*

We put a figure Γ on a regular square net whose cell length is h. Let Γ_h be the approximate boundary which is composed of the nearest links by Γ.

Generally the eigen-value of figure depends on the area of figure, so we have normalized the area M. By taking h^2 as the unit, the area surrounded by Γ_h is represented as follows;

$$M=Sh^2=1, \tag{1}$$

where S nearly equals to the number of lattice points surrounded by Γ_h.

We represent Helmholtz's difference equation as follows;

$$\left[-\frac{1}{h^2}\boldsymbol{A}+\lambda\boldsymbol{E}\right]\boldsymbol{\Phi}=0, \tag{2}$$

where $\boldsymbol{A}$ is a matrix of difference operator, $\boldsymbol{\Phi}$ is a vector of function Φ and $\boldsymbol{E}$ is a unit matrix. Since $\boldsymbol{\Phi}\neq 0$, we have,

$$\left|-\frac{1}{h^2}\boldsymbol{A}+\lambda\boldsymbol{E}\right|=0, \tag{3}$$

Therefore, Helmholtz's equation results in eigen-value problem of matrix $\boldsymbol{A}$, and inverse matrix of $\boldsymbol{A}$ is identical with Green function. Let Λ be eigen-value of $\boldsymbol{A}$, and λ can be calculated from following equation,

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$$\lambda=\frac{1}{h^2}\Lambda=S\Lambda. \tag{4}$$

2.2 *Correction of eigen-value*

Let λ_i be ith analytical eigen-value, $\lambda_{i(h)}$ be ith experimental eigen-value and e_i be relative error, where

$$e_i=\frac{\lambda_i-\lambda_{i(h)}}{\lambda_i}. \tag{5}$$

From experiments we have gotten a relation $e_i \doteqdot a\lambda_{i(h)}$, where $a \doteqdot 0.08/S$, so we have following formula for correction

$$\bar{\lambda}_{i(h)}=\frac{\lambda_{i(h)}}{1-0.08\,\lambda_{i(h)}/S}, \tag{6}$$

where $\bar{\lambda}_{i(h)}$ is corrected eigen-value.

When the boundary Γ is similar to Γ_h, that is, S is sufficiently large, the formula for correction (6) produces good results. In Fig. 1, one example of relative error e_i and $\bar{e}_i$ is shown, where $\bar{e}_i=(\lambda_i-\bar{\lambda}_{i(h)})/\lambda_i$.

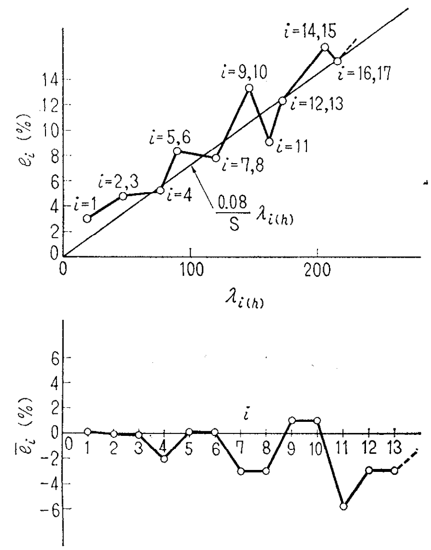

Fig. 1. An example of relative error e_i and $\bar{e}_i$ (Figure: Square, Number of inner point: 81)

2.3 *Explanation of formula for correction*

Equation (6) can be represented in infinite progression as follows;

$$\bar{\lambda}_{i(h)}=\sum_{n=1}^{\infty}\left(\frac{0.08}{S}\right)^{n-1}\lambda^n{}_{i(h)}. \tag{7}$$

On the other hand, Helmholtz's equation can be represented in second and fourth derivatives as follows;

$$\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)\Phi(x, y)=\left\{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-2\frac{\partial^4}{\partial x^2\partial y^2}\right\}\Phi(x, y)$$

$$=\lambda^2\Phi(x, y)-2\frac{\partial^4}{\partial x^2\partial y^2}\Phi(x, y).$$

$$\Delta\Phi(x, y)=\frac{1}{h^2}\left\{\Sigma^*\Phi(x', y')-4\Phi(x, y)-2\sum_{n=2}^{\infty}\frac{h^{2n}}{(2n)!}\left(\frac{\partial^{2n}}{\partial x^{2n}}+\frac{\partial^{2n}}{\partial y^{2n}}\right)\Phi(x, y)\right\}$$
$$=\frac{1}{h^2}\left\{\Sigma^*\Phi(x', y')-4\Phi(x, y)-2\sum_{n=2}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\lambda^n\Phi(x, y)\right.$$
$$\left.+2\frac{h^4}{4!}\frac{\partial^4}{\partial x^2\partial y^2}\Phi(x, y)+\cdots\right\},$$

where $x'=x\pm h$, $y'=y\pm h$ and Σ^* indicates summation of four neighbour point of (x, y).

Let $R(x, y)$ be higher order terms than fourth order terms in right hand side of above equation, we have

$$\Sigma^*\Phi(x', y')-4\Phi(x, y)+R(x, y)$$
$$=-h^2\lambda\Phi(x, y)+2\sum_{n=2}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\lambda^n\Phi(x, y)$$
$$=2\{1-\cos(h\sqrt{\lambda})\}\Phi(x, y) \qquad (\lambda>0).$$

Since $R(x, y)=O(h^2)$, we neglect the term of $R(x, y)$. Let $\lambda_{(h)}$ be eigen-value of Helmholtz's difference equation, we have

$$\Sigma^*\Phi(x', y')-4\Phi(x, y)=2\{1-\cos(h\sqrt{\lambda})\}\Phi(x, y)=h^2\lambda_{(h)}\Phi(x, y). \tag{8}$$

Let $\bar{\lambda}_{(h)}$ be corrected value of $\lambda_{(h)}$, we have a following formula for correction.

$$\bar{\lambda}_{(h)}=\left\{\frac{1}{h}\cos^{-1}\left(1-\frac{h^2\lambda_{(h)}}{2}\right)\right\}^2=S\left\{\cos^{-1}\left(1-\frac{\lambda_{(h)}}{2S}\right)\right\}^2$$
$$=S\sum_{n=1}^{\infty}\frac{(n-1)!}{(2n-1)!!}\cdot\frac{1}{2n}\left(\frac{\lambda_{(h)}}{2S}\right)^n, \tag{9}$$

where $\lambda_{(h)}/2S\leq 2$, $(2n-1)!!=(2n-1)(2n-3)\cdots 1$.

The progression in equation (7) has a simular tendency to equation (9), so the meanings of correction by equation (6) becomes clear. Particularly, in one dimensional region, equation (6) is a formula for complete correction.

Further, the correction by equation (6) is appropriate for case of convex figure. Because W. R. Wasow has shown that $\lim_{h\to 0}\lambda_{(h)}=\lambda(\lambda>\lambda_{(h)})$ when figure is convex and that there are some cases of $\lambda<\lambda_{(h)}$ when figure is concave [8].

3. *Series of figure and examples of tree of eigen-value*

For a systematic calculation of eigen-value, we have chosen several series of convex figures.

(1) Series 1 Equilateral polygons
(2) Series 2 Rectangles
(3) Series 3 Rhombuses
(4) Series 4 Trapeziums
(5) Series 5 Isosceles trapezoids
(6) Series 6 Right-angled triangles
(7) Series 7 Isosceles triangles
(8) Series 8 Ellipsoids

(9) Series 9 Lacked circles

(10) Series 10 Fan shapes

In Fig. 2, one exaple of correlation between eigen-values and geometric informations is shown and the correlation of this sort has ever been considered by G. Polya [5].

Further there are several common parameters to all figures, for examples, a length of periphery [5], ratio of radius of maximum inscribed circle to one of minimum circumscribed circle [3], etc. The correlation between eigen-value and length of periphery L is shown in Fig. 3. We call these correlation graphs as tree of eigen-value of figure.

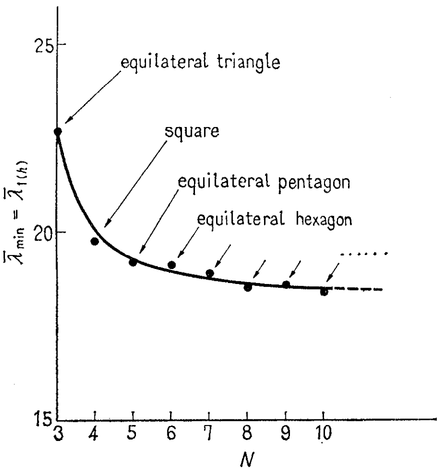

Fig. 2. Minimum eigenvalue of equilaterl polygons (series 1)

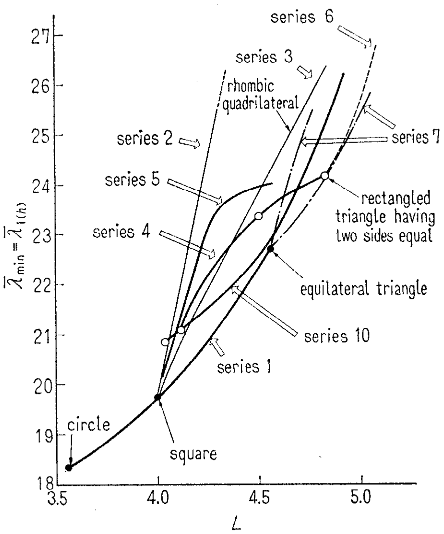

Fig. 3. Minimum eigen-value of various figures (Tree of eigen-value)

4. *Application to estimation of optimum relaxation factor*

4.1 *Definition of optimum relaxation factor*

We have dealt with the boundary value problem by succesive over-relaxation method [4]. Let R be a simply connected domain with a boundary Γ, I be the interior of R and $u(x, y)$ be unknown function, then boundary value problem is represented as follows;

$$\begin{aligned} \Delta u(x, y) &= -f(x, y) \quad \text{in } I \\ u(x, y) &= g(x, y) \quad \text{on } \Gamma, \end{aligned} \tag{10}$$

where $f(x, y)$ and $g(x, y)$ are given.

We have descretized a space and used a difference operator in place of a differential operator. Let (i, j) be a lattice point (x, y), then equation (10) can be represented in matrix form as follows;

$$\boldsymbol{AU}=\boldsymbol{K},$$

and in componentwise,

$\sum_{j=1}^{n} a_{ij}u_j = k_i,\ 1 \leq i \leq n,$ (n is number of inner points).

Let ω be relaxation factor of succesive over-relaxation method and $u_i^{(m)}$ be mth value in succesive process, then above equation is represented as follows;

$$a_{ij}u_i^{(m+1)} = a_{ij}u_i^{(m)} + \omega\left\{-\sum_{j=1}^{i-1} a_{ij}u_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij}u_j^{(m)} + k_i - a_{ii}u_i^{(m)}\right\}. \quad (11)$$

Optimum value ω_{opt} of ω in equation (11) is given as follows [4];

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\mu^2_{\max}}}, \quad (12)$$

$$\mu_{\max} = 1 - \Lambda_{\min}/4, \quad (13)$$

where $\Lambda_{\min}$ is the minimum eigen-value of the matrix $\boldsymbol{A}$.

4.2 *Example of estimation*

If $\mu_{\max}$ is known, ω_{opt} can be calculated from equation (12). Fortunately, $\mu_{\max}$ can be calculated from equation (13), and further $\Lambda_{\min}$ can be calculated from equation (4).

For trial, we have calculated an ω_{opt} for case of rectangular, whose ratio of adjoining sides is 80 : 60. As Γ_h is broken at the four corners comparing with Γ (rectangular) for this case, S is calculated as $80\times 60-2=4798$ and L is calculated as $7\sqrt{3}/3 \fallingdotseq 4.04$. The series 2 of Fig. 3 may be utilized and $\lambda_{\min}$ may be estimated as 20.3.

$$\tilde{\Lambda}_{\min} = \frac{\tilde{\lambda}_{\min}}{S} = \frac{20.3}{4798} \fallingdotseq 0.0042,$$

$$\tilde{\mu}_{\max} = 1 - \tilde{\Lambda}_{\min}/4 \fallingdotseq 0.9989,$$

$$\tilde{\omega}_{\text{opt}} = \frac{2}{1+\sqrt{1-\tilde{\mu}^2_{\max}}} \fallingdotseq \frac{2}{1.05} \fallingdotseq 1.90.$$

For this case, ω_{opt} has been calculated as 1.9124 by J. S. Pearson and J. A. Harrison [6].

5. *Conclusion*

In this paper the authors have suggested the trees of eigen-value of various figures and developped the estimation method of optimum relaxation factor. It must be taken into consideration that this method can be used for calculating ω_{opt} only for case of arbitrary convex figures.

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