Spatial Networks

HIDEMITSU OGAWA*, AKIRA IGARASHI* AND TAIZO IIJIMA*

1. *Introduction*

Features of digital computers which have accomplished wonderful developping are that they reduce all operations to logical and algebraic ones and process them one by one. On the other hand, the human brain processes information not only logically but also emotionally, nondecidably, and in parallel. The problem of pattern recognition belongs to this region. The investigation of the latter functions of information processing becomes very important for realization of true artificial intelligence [1].

There are several different trends in the approach to such problems. In order to find them, let us consider the structure of information sciences. Information sciences consist of the next three parts [2]: (i) the elucidation of information phenomena; (ii) the construction of information theories; (iii) the development of information processing systems.

The theory of the new information processing system, i.e., spatial networks, discussed in this paper aims at (iii) on the basis of (ii). Thoughts which support spatial networks will be described briefly in the following. Since the set dealt with in the problem of the pattern recognition consists of infinite elements, it is impossible to investigate all elements one by one in order to clear its structure. Thus, it may be advisable to adopt the method such that the structure of the original set is estimated with suitable accuracy by finite elements sampled appropriately. In order that the method became effective, properties investigated about one element must hold about its neighboring elements. Hence, we must make good use of not only algebraic but also topological properties of the set for information processing. The information processing systems which can deal with such information are called *the spatial networks.*

In Section 2, we define the space of signals as a topological space, and the function of information processing as an operator of a topological space. In Section 3, we adopt the particular topological space $L_2[-l, l]$ as the space of signals, and introduce the fundamental coordinate system in L_2. The dyadic representation of a function of information processing with respect to this basis is obtained. And in Section 4, we introduce the natural coordinate system and describe the design of spatial networks.

This paper first appeared in Japanese in Joho Shori (the Journal of the Information Processing Society of Japan), Vol. 11, No. 11 (1971), pp. 672-681.
* Electrotechnical Laboratory.

2. *Mathematical Formulation of Information Processing*

2.1 *Space of Signals*

It is easy to understand things not intensively but extensively. So, in order to formulate mathematically the signals having information, we consider the set of these signals and define signals as members of the set. What must we adopt as such a set? We describe it in the following.

Let $\mathfrak{H}$ be the set of all patterns written on a domain. Then, for any f and g belonging to $\mathfrak{H}$, operations of addition and scalar multiplication are defined by equations

$$(f+g)(x)=f(x)+g(x), \tag{1}$$

$$(af)(x)=af(x), \tag{2}$$

respectively, where x is a point moving on the domain. Hence, $\mathfrak{H}$ becomes a real linear space.

Let us consider the case that noise transforms a pattern $f \in \mathfrak{H}$ into a little different one f'. In this case, it is able to be considered that f' is quite different from f. However, since f' is originally the same as f from the viewpoint of pattern recognition, it is natural to regard f' as the same or the near pattern to f. Hence, we can introduce the concept "neighborhood" into $\mathfrak{H}$, and so that $\mathfrak{H}$ becomes a topological space [9].

Let f and g be two distinct patterns, and consider neighborhoods of them. In this case, there exists disjoint neighborhoods of f and g. Hence, $\mathfrak{H}$ becomes a Hausdorff space [9]. If we consider the pattern $f'+g'$ where f' and g' belong to neighborhoods of f and g, respectively, then $f'+g'$ is regarded as a pattern belonging to a neighborhood of $f+g$. Similarly, if we consider the pattern $a'f'$, then $a'f'$ is regarded as a pattern belonging to a neighborhood of af, where a' belongs to a neighborhood of a number a.

Hence, the set of patterns becomes a topological linear space [9].

2.2 *Formulation of a Function of Information Processing*

Let us define *information processing* as a transformation of a point of a topological space into the suitable one of a topological space because we have defined signals having information as points of a topological space in the previous section. Then, an operator of a topological space implies *the function of information processing.*

If we realize this operator with hardwares as faithfully as possible, then the system have the same function as the operator, that is, it can process information not only algebraically but also topologically. The theory of design and analysis for it is called the *theory of spatial networks.*

3. *Representation of a Function of Information Processing*

It has been turned out that the set of patterns is a topological linear space in 2.1. However, it is so general that it is difficult to construct a theory in detail. So, we adopt the complex Hilbert space [8] $L_2[-l, l]$ as the first step. The inner product of L_2 is given by

$$(f, g)\equiv\frac{1}{2l}\int_{-l}^{l} f(x)\overline{g(x)}dx. \tag{3}$$

We adopt the set $\mathfrak{B}$ of all bounded linear operators of L_2 into itself as *the space of operators.*

3.1 *Fundamental Coordinate System*

It is convenient for the construct of an spatial network not by use of optical systems or of microwave circuits but by use of lumped circuits to express the function of information processing in the form of discrete type. It turns out that it is better to adopt l_2 instead of L_2 as the space of signals, because a linear operator of L_2 is represented in the form of an infinite matrix of l_2 [8].

Let $\boldsymbol{A}$ be an operator which transforms $f\in L_2$ into $g\in L_2$:

$$g=\boldsymbol{A}f. \tag{4}$$

Then, it follows that

$$g_m=\sum_{n=-\infty}^{\infty} a_{m,n} f_n (m=0, \pm1, \pm2,\cdots), \tag{5}$$

where

$$f_n\equiv(f, \varphi_n), \ g_n\equiv(g, \varphi_n) \text{ and } a_{m,n}\equiv(\boldsymbol{A}\varphi_n, \varphi_m), \tag{6}$$

and $\{\varphi_n\}$ is an orthonormal basis in L_2.

It turns out from Eq. (6) that the form of the matrix $(a_{m,n})$ depends on the basis $\{\varphi_n\}$ introduced into L_2. Furthermore, since we are free of the choice of the basis it is desirable to use the basis for which matrices have the simplest forms at first. Such a complete orthonormal system is called *the fundamental coordinate system.* The eigenfunctions

$$\varphi_n(x)\equiv exp\left(-i\frac{n\pi}{l}x\right) \quad (n=0, \pm1, \pm2,\cdots) \tag{7}$$

of the translation operator [6] $\boldsymbol{T}(a)$ are adopted as the fundamental coordinate system in this paper.

3.2 *Dyad and Dyadic*

Let φ, ψ, and f be elements of a Hilbert space $\mathfrak{H}$. The operator $<\varphi, \psi>$ defined by the equation

$$<\varphi, \psi>f\equiv(f, \psi)\varphi \tag{8}$$

is called dyad. A linear combination of dyads is called dyadic.

3.3 *Expansion Operator* $\boldsymbol{A}(\lambda)$

Let us consider the expansion operator which transforms $f(x)$ into $f(\lambda x)$ $(0<\lambda<\infty)$ so that we obtain the concept of the dyadic representation of the function of information processing. The expansion operator is well defined for functions $f\in L_2(-\infty, \infty)$. However, since there is a case of $\lambda x\notin[-l, l]$ for $f\in L_2[-l, l]$, it is impossible to define the expansion operator as the operator which transforms $f(x)$ into $f(\lambda x)$ merely.

Let us calculate formally the matrix representation of the expansion operator using the fundamental basis first of all. Then, its elements are as follows:

$$[\varphi_n(\lambda x), \varphi_m(x)] = \frac{\sin(m-\lambda n)\pi}{(m-\lambda n)\pi}. \tag{9}$$

Since this infinite matrix is the bounded linear operator of l_2, there is a bounded linear operator of $L_2[-l, l]$ corresponding to the matrix. Hence, the expansion operator which is denoted by $\boldsymbol{A}(\lambda)$ is defined by the corresponding operator of $L_2[-l, l]$.

Let us consider Eq. (9) again in order to find how process the operator $\boldsymbol{A}(\lambda)$ defined above. The function $\varphi_n(x)$ defined on $[-l, l]$ at first is able to be regarded as the periodic function with period $2l$ defined on $(-\infty, \infty)$. Then, $\varphi_n(\lambda x)$ is defined naturally, i. e., $\varphi_n(\lambda x)$ is the periodic function with period $2l/\lambda$ defined on $(-\infty, \infty)$. Since the left-hand side of Eq. (9) is of the inner product in $L_2[-l, l]$, it turns out finally that $\boldsymbol{A}(\lambda)$ is the operator which transforms $\varphi_n(x)$ onto the part on $[-l, l]$ of $\varphi_n(\lambda x)$ obtained above.

The above description is expressed by the following equations:

$$\begin{aligned} &\boldsymbol{A}(\lambda) \equiv \boldsymbol{\Phi} \boldsymbol{A}'(\lambda) \boldsymbol{\Phi}, \\ &[\boldsymbol{A}'(\lambda) f](x) \equiv f(\lambda x), \end{aligned} \qquad (0<\lambda<\infty) \tag{10}$$

where

$$\boldsymbol{\Phi} \equiv \sum_{n=-\infty}^{\infty} <\varphi_n, \varphi_n> \tag{11}$$

From Eq. (9)-(11) and properties of dyadic, $\boldsymbol{A}(\lambda)$ has the following dyadic representation:

$$\boldsymbol{A}(\lambda) = \sum_{m,n=-\infty}^{\infty} \frac{\sin(m-\lambda n)\pi}{(m-\lambda n)\pi} <\varphi_m, \varphi_n>. \tag{12}$$

3.4 *Dyadic Representation of Functions of Information Processing*

It turns out from the previous section that it is easy for us to deal intuitively with the operator not of $L_2[-l, l]$ but of $L_2P[-l, l]$ which is the set of functions given by extension of functions belonging to $L_2[-l, l]$ to the whole interval $(-\infty, \infty)$ with period $2l$. Then, we first define a bound linear function of information processing as a bounded linear operator $\boldsymbol{A}'$ of $L_2P[-l, l]$ into a topological space. And using the equation

$$\boldsymbol{A} = \boldsymbol{\Phi} \boldsymbol{A}' \boldsymbol{\Phi}, \tag{13}$$

the operator $\boldsymbol{A}'$ is transformed into the bounded linear operator $\boldsymbol{A}$ of a topological space into $L_2P[-l, l]$. Hence, the process of an information processing by an operator $\boldsymbol{A}$ is separated into the following three steps. First, the operator $\boldsymbol{\Phi}$ extracts the part on the interval $[-l, l]$ from a signal on $(-\infty, \infty)$ and extends it to the whole interval $(-\infty, \infty)$ with period $2l$. Secondly, $\boldsymbol{A}'$ carries out the information processing proper to $\boldsymbol{A}$. At the end, $\boldsymbol{\Phi}$ extracts again the part on

$[-l, l]$ from the signal on $(-\infty, \infty)$ and extends it to $(-\infty, \infty)$ with period $2l$. Taking care of our attention only to the interval $[-l, l]$, A is an operator of $L_2[-l, l]$ into itself.

Rewriting Eq. (13) with Eq. (11), we have

$$\boldsymbol{A}=\sum_{m,n=-\infty}^{\infty} a_{m,n} < \varphi_m, \varphi_n >, \tag{14}$$

where

$$a_{m,n} \equiv (\boldsymbol{A}' \varphi_n, \varphi_m) = (\boldsymbol{A} \varphi_n, \varphi_m). \tag{15}$$

Eq. (14) is called *the dyadic representation* of $\boldsymbol{A}$.

By Eqs. (14) and (8), it follows that

$$\boldsymbol{A} \varphi_n = \sum_{m=-\infty}^{\infty} a_{m,n} \varphi_m. \tag{16}$$

It turns out from Eq. (16) that the ratio which the component φ_n of the spatial frequency is transformed by $\boldsymbol{A}$ into the component φ_m is of $a_{m,n}$. Hence, the matrix $\boldsymbol{A} \equiv (a_{m,n})$ is called *the spatial frequency characteristics* of $\boldsymbol{A}$.

4. *Theory of Design of Spatial Networks*

4.1 *Natural Coordinate System*

It is impossible to construct a spatial network directly by the dyadic representation of $\boldsymbol{A}$ obtained in 3.4 since a spatial network has only finite terminals. If we hold fast to the fundamental coordinate system $\{\varphi_n\}$, then it is difficult to obtain f_n by the operation (f, φ_n) when a spatial network accepts an input signal f from the outside.

Let us introduce the new complete orthonormal system $\{\phi_m\}^N_{m=-N}$ into the subspace $\mathfrak{H}_N$ of $L_2[-l, l]$ spanned by $\{\varphi_n\}^N_{n=-N}$ by the equation

$$\phi_m(x) \equiv \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \overline{\varphi_n\left(\frac{2lm}{2N+1}\right)} \varphi_n(x) \tag{17}$$

in order to solve such problems. Then, we have

$$\phi_m(x) = \frac{1}{\sqrt{2N+1}} \frac{\sin \frac{(2N+1)\pi}{2l}\left(x - \frac{2lm}{2N+1}\right)}{\sin \frac{\pi}{2l}\left(x - \frac{2lm}{2N+1}\right)}. \tag{18}$$

For any $f \in \mathfrak{H}_N$, the sampling theorem holds:

$$(f, \phi_n) = \frac{1}{\sqrt{2N+1}} f\left(\frac{2ln}{2N+1}\right). \tag{19}$$

Hence, $\{\phi_n\}^N_{n=-N}$ is called *the natural coordinate system* in $\mathfrak{H}_N$.

4.2 *Theory of Design*

The function of information processing which we can realize by use of spatial networks is not $\boldsymbol{A}$ but

$$\boldsymbol{A}_N \equiv \boldsymbol{E}_N \boldsymbol{A} \boldsymbol{E}_N = \sum_{m,n=-N}^{N} a_{m,n} < \varphi_m, \varphi_n >, \tag{20}$$

because of the restriction of finite number of terminals of a spatial network, where $\boldsymbol{E}_N$ is the projection operator of L_2 onto $\mathfrak{H}_N$.

Rewriting Eq. (20) by use of $\{\psi_n\}$, it follows that

$$\boldsymbol{A}_N=\sum_{m,n=-N}^{N} a^{\wedge}_{m,n}<\psi_m,\ \psi_n>, \tag{21}$$

where

$$a^{\wedge}_{m,n}=\frac{1}{2N+1}\sum_{p,q=-N}^{N} a_{p,q}\,\varphi_p\left(\frac{2lm}{2N+1}\right)\overline{\varphi_q\left(\frac{2ln}{2N+1}\right)}. \tag{22}$$

By Eqs. (21) and (8), it follows that

$$\boldsymbol{A}_N\psi_n=\sum_{m=-N}^{N} a^{\wedge}_{m,n}\,\psi_m. \tag{23}$$

Since $\psi_n\in\mathfrak{H}_N$, it turns out from Eqs. (19) and (23) that we get the value $a^{\wedge}_{m,n}$ at the m-th output terminal if the spatial network accepts the pattern which has the value 1 only at the n-th input terminal and the values 0 otherwise as its input signal. The spatial network which has the function $\boldsymbol{A}_N$ of information processing is able to be constructed by means of connecting the n-th input terminal to the m-th output terminal with the summing amplifiers of which characteristic values are $a^{\wedge}_{m,n}$. Hence, the matrix $(a^{\wedge}_{m,n})$ is called *the connection characteristics* of the spatial network.

The necessary and sufficient condition for A to transform real functions into real functions is that the following equation holds:

$$\overline{a_{-m,-n}}=a_{m,n}(m,\ n=0,\ \pm1,\ \pm2,\cdots). \tag{24}$$

In this condition, $a^{\wedge}_{m,n}$ follows

$$a^{\wedge}_{m,n}=\frac{1}{2N+1}\left[\alpha_{0,0}+2\left(\sum_{p=q=1}^{N}+\sum_{p>q}\right)\left(\alpha_{p,q}\cos\frac{2\pi(mp-nq)}{2N+1}+\beta_{p,q}\sin\frac{2\pi(mp-nq)}{2N+1}\right)\right], \tag{25}$$

where $\alpha_{p,q}$ and $\beta_{p,q}$ are real and imaginary parts of $a_{p,q}$, respectively.

5. *Concluding Remarks*

A fundamental idea of an approach to (ii) among three region of the information science mentioned in Introduction has been described. That is, a signal has been regarded as a point in a topological space. And a function of information processing has been formulated mathematically as an operator of a topological space into a topological space.

On the basis of this thoughts, we have gone over to the development of systems of information processing corresponding to (iii). The theory of linear spatial networks has been developed on the particular topological space $L_2[-l, l]$ adopted as the space of signals. Using the dyadic representation of functions of information processing, it has been turned out for us to be able to discuss in terms of matrix forms in essence. At the same time, since the coordinate system adopted appears explicitely, it has been turned out to be able to recognize clearly every process of information processing one by one.

By introducing the natural coordinate system, the abstract and mathematical theory has been connected with the practical hardwares naturally, so that a new information processing system of a different type from usual digital computers has been obtained.

We cannot assert that this is the optimum model of the brain. However, we are assured that the spatial network is an effective alternative to digital computers, because two kinds of optical character readers have been made already by use of the idea mentioned in this paper with the Iijima's theory of pattern recognition [3] and have been obtained excellent results [4] [5].

Acknowledgement

We wish to express our gratitude to members of Iijima Section for their instructive discussions.

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