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Ordering of Pivotal Operations on Sparse System of Equations

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1. *Introduction*

A matrix which has many zero elements is called a sparse one. In the computer-aided analysis of fairly large system of equations, of prime importance are sparse matrix storage and solution techniques in terms of reduction in memory space, computation time, and round-off errors. These techniques have recently attracted a considerable amount of interest, and have been extensively developed [1]-[4].

On the subject of solving sparse system of equations, a number of authors have considered the problem of sparsity exploitation associated with the so-called *LU decomposition* [5], intended mainly for reductions in memory space and computation time [6]-[9]. On sparsity-directed techniques in calculating driving-point and transfer immittances for some specified ports of a network from its primitive immittance matrix, the authors have discussed the block-pivoted condensation based on network decomposition [10], [11].

In the present paper we define a pivotal operation, which includes both the pivotal operation in the LU decomposition and the pivotal condensation in the process of calculating driving-point and transfer immittances from a primitive matrix, in point of the growth of nonzero elements. Then we formulate an optimal ordering problem of these pivotal operations and present some approach to the problem with the use of a newly defined 'pivoting graph'.

2. *Optimal Ordering Problem*

Given an $n \times n$ system of linear equations

$$W_0|x\rangle = |b\rangle, \ \det W_0 \neq 0, \tag{1}$$

where $|x\rangle$ and $|b\rangle$ denote column vectors, we henceforth assume without loss of generality that the diagonal elements in W_0 are nonzero. The *LU decomposition* of W_0 involves the breaking down of W_0 into $W_0 = LU$, where L is lower triangular in form with unit as the value of every diagonal element and U is upper triangular. On the other hand, consider the case when, given a primitive

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immittance matrix W_0 and a source vector $|b\rangle=[b_1, b_2, \cdots, b_p, 0, \cdots, 0]^t$ with the superscript t denoting the transposition[1], we are to seek the coefficient matrix $\hat{W}$ of the condensed form $\hat{W}|\hat{x}\rangle=|\hat{b}\rangle$ of (1), where $|\hat{x}\rangle=[x_1, x_2, \cdots, x_p]^t$ and $|\hat{b}\rangle=[b_1, b_2, \cdots, b_p]^t$. In this case, we apply to W_0 the operation, so called the *pivotal condensation*, with respect to the k-th variable $x_k (k\geq p+1)$ one by one.

We first define a sparsity-oriented pivotal operation which includes these two operations in point of the growth of nonzero elements. For this purpose, we define the binary matrix $X(W)=[x_{ij}]$, associated with a given square matrix $W=[w_{ij}]$ $(w_{ii}\neq 0)$ of order m, with each x_{ij} taking the value of unit or zero such that

$$x_{ij}=\begin{cases} 1; & w_{ij}\neq 0, \\ 0; & w_{ij}=0. \end{cases} \tag{2}$$

Definition 1: Let $W^{*k}=[w_{ij}^{*k}]$ denote a matrix of order $m-1$ obtained from W by applying some operation with respect to row k and column k and then deleting row k and column k. If through this operation the following relations hold for the elements x_{ij}^{*k} of $X(W^{*k})$

$$x_{ij}^{*k}=x_{ij}+x_{ik}\cdot x_{kj} \qquad \text{for all } i, j\neq k, \tag{3}$$

where the addition and multiplication are Boolean ones, then we designate this operation as the *s-pivoting* with respect to row k and column k.

Definition 2: Given a square matrix W, define the sets A_k and B_k associated with column $|x_k\rangle$ and row $\langle x_k|$ of $X(W)$, respectively, such that

$$A_k=\{j\,|\,x_{jk}=1, j\neq k\}, \tag{4}$$

$$B_k=\{i\,|\,x_{ki}=1, i\neq k\}. \tag{5}$$

Note that if $X(W)$ is symmetric, then $A_k=B_k$.

Definition 3: For each element i of B_k, define the set L_{ki} as

$$L_{ki}=\{j\,|\,x_{ji}=0, j\varepsilon A_k, i\varepsilon B_k\}=A_k-[\{i\}\cup A_i]. \tag{6}$$

As is readily seen, $x_{ji}=0$ and $x_{ji}^{*k}=1$ for any $j\varepsilon L_{ki}$, that is, during the s-pivoting with respect to row k and column k a new nonzero element is created at the (j, i) position such that $j\varepsilon L_{ki}$. If we denote by $M_k(W)$ the number of the nonzero elements which are newly created by the s-pivoting on W with respect to row k and column k, then $M_k(W)$ is given by

$$M_k(W)=\sum_{i\varepsilon B_k}|L_{ki}|, \tag{7}$$

where $|\cdot|$ denotes the number of elements in the set.

Given a matrix W_0 of order n on which q s-pivoting operations are to be applied[2], we assume that these q rows and columns are located in the leading position of W_0. If this is the case, associated with the symmetric group P of a set $\{1, 2, \cdots, q\}$, let S represent a set of the ordered sequences defined as

1) In the usual network analysis $p=2$.
2) In the LU decomposition $q=n$.

$$S=\left\{s_\nu=(\sigma_1, \sigma_2, \cdots, \sigma_q) \middle| \begin{pmatrix} 1 & 2 & \cdots q \\ \sigma_1 & \sigma_2 & \cdots \sigma_q \end{pmatrix} \varepsilon P \right\}, \quad (8)$$

and for any $s=(k_1, k_2, \cdots, k_q)\varepsilon S$ define $\Delta(s)$ such that

$$\Delta(s)=M_{k_1}(W_0)+M_{k_2}(W_0{}^{*k_1})+\cdots+M_{k_q}(W_0{}^{*k_1k_2\cdots k_{q-1}}), \quad (9)$$

where, for a matrix W of order m and any ordered sequence $\pi=(k_1, k_2, \cdots, k_h)$ ($k_i<m$ for all i), we denote $W^{*\pi}$ as follows,

$$W^{*\pi}=W^{*k_1k_2\cdots k_h}=(\cdots((W^{*k_1})^{*k_2})\cdots)^{*k_h}. \quad (10)$$

Then $\Delta(s)$ represents the total number of nonzero elements which are created all through the q s-pivoting operations, firstly with the k_1th row and column, secondly with the k_2th row and column, $\cdots$, and lastly with the k_qth row and column. Thus, we can formulate the following *optimal ordering problem*:

Seek an optimal order $s=s^*$ such that

$$\Delta(s^*)=\min_{s\varepsilon S} \Delta(s). \quad (11)$$

3. *Pivoting Graph*

Associated with a square matrix W of order m, we define a weighted graph $G=[V(G), E(G)]$ called the *pivoting graph* of W as follows, where $V(G)$ is the set of the nodes of G and $E(G)\subset V(G)\times V(G)$ the set of the edges of G.

Definition 4: The pivoting graph G of W is an oriented graph such that

(i) each $v_i\varepsilon V(G)$ $(i=1, 2, \cdots, m)$ corresponds to column i of W,

(ii) to each $w_{ij}\neq 0$ $(i\neq j)$ of W there corresponds an edge $(v_i, v_j)\varepsilon E(G)$ orienting from v_i to v_j, and

(iii) to each $v_i\varepsilon V(G)$ and to each $(v_i, v_j)\varepsilon E(G)$, the sets A_i and L_{ij} are attached, respectively, as their weights.

This pivoting graph G of W involves the information of which zero elements of W will be replaced with nonzero ones through the s-pivoting with respect to the kth row and column of W. If for $(v_k, v_i)\varepsilon E(G)$ $L_{ki}\neq\phi$[3], then the s-pivoting will substitute each $w_{ji}=0$ with $j\varepsilon L_{ki}$ in column i by $w_{ji}{}^{*k}\neq 0$.

We now consider the procedure to derive the pivoting graph G^{*k} of W^{*k} from G of W.

Lemma 1: *$V(G^{*k})$ and $E(G^{*k})$ of G^{*k} are obtained from G such that*

$$V(G^{*k})=V(G)-\{v_k\}, \quad (12)$$

$$E(G^{*k})=[E(G)-\bigcup_{h\varepsilon A_k}\{(v_h, v_k)\}-\bigcup_{i\varepsilon B_k}\{(v_k, v_i)\}]\cup[\bigcup_{i\varepsilon B_k} H_i], \quad (13)$$

where if $L_{ki}=\phi$ then $H_i=\phi$, otherwise, $H_i=\{(v_j, v_i)\,|\, j\varepsilon L_{ki}\}$.

Lemma 2: *The $A_i{}^{*k}, B_i{}^{*k}$, and $L_{ij}{}^{*k}$ associated with W^{*k} are obtained from A_i and B_i of W as follows,*

$$A_i{}^{*k}=\begin{cases} A_i\cup A_k-\{i, k\} & ;\ i\varepsilon B_k, \\ A_i & ;\ i\notin B_k, \end{cases} \quad (14)$$

3) ϕ denoted an empty set.

$$B_i{}^{*k}=\begin{cases} B_i \cup B_k-\{i,k\}\ ;\ i\varepsilon A_k, \\ B_i \qquad\qquad\quad ;\ i\notin A_k, \end{cases} \tag{15}$$

$$L_{ij}{}^{*k}=A_i{}^{*k}-[\{j\}\cup A_j{}^{*k}]. \tag{16}$$

For a pivoting graph G of W, let $M_k(G)=M_k(W)$. Furthermore, let U_0 be the set of the nodes v_r of G_0 corresponding to rows r and columns r of W_0 with respect to which s-pivoting operations are to be applied. Then we can find an ordered sequence $s_z=(h_1, h_2, \cdots, h_t)$, for each h_i of which is corresponding to v_{h_i} in U_0, such that

$$M_{h_1}(G_0)+M_{h_2}(G_0{}^{*h_1})+\cdots+M_{h_t}(G_0{}^{*h_1h_2\cdots h_{t-1}})=0, \tag{17}$$

where $G_0{}^{*h_1h_2\cdots h_i}$ is a pivoting graph of $W_0{}^{*h_1h_2\cdots h_i}$ for $i=1, 2, \cdots, t-1$, and for any $v_k\varepsilon V(G_0{}^{*s_z})\cap U_0$

$$M_k(G_0{}^{*s_z})\neq 0. \tag{18}$$

For any such s_z the following theorem holds.

Theorem: *Let $S^*=\{s_i{}^*\}\subset S$ denote a set of the sequences $s_i{}^*$ satisfying* (11). *Then there exists a subset $S^\dagger=\{s_j{}^\dagger\}$ of S^* such that any sequence s_z satisfying* (17) *and* (18) *is a leading subsequence of some $s_j{}^\dagger$ in $S^\dagger$, that is,*

$$s_j{}^\dagger=(s_z, s_{jz}{}^\dagger). \tag{19}$$

4. *Near-Optimal Ordering*

Let $U_z=U_0\cap V(G_0{}^{*s_z})$ and $|U_z|=r$. Then to seek an optimal subsequence $s_{jz}{}^\dagger$ of $s_j{}^\dagger$, we might have to evaluate $\Delta(s_z, s_\zeta)$ for all the $r!$ different ordered sequences $s_\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_r)$ with $v_{\zeta_i}\varepsilon U_z$ in the worst case, even if we employed efficient techniques such as the branch-and-bound method. Thus it is of practical use to seek a near-optimal order.

Ogbuobiri *et al.* [8] pointed out the following three schemes of seeking near-optimal sequences.

Scheme 1: Seek a sequence $s^{(1)}=(k_1, k_2, \cdots, k_r)$ such that for the graph $G_z=G_0{}^{*s_z}$.

$$|B_{k_1}|\leq|B_{k_2}|\leq\cdots\leq|B_{k_r}|,\quad v_{k_i}\varepsilon U_0\cap V(G_z). \tag{20}$$

Scheme 2: Seek a sequence $s^{(2)}=(k_1, k_2, \cdots, k_r)$ such that for each graph $G^{(i)}=(G_0{}^{*s_z})^{*s_i}$ with $s_i=(k_1, k_2, \cdots, k_i)$ $(i=0, 1, \cdots, r-1)$

$$|B_{k_{i+1}}|=\min_{v_\lambda\varepsilon U_0\cap V(G^{(i)})}[|B_\lambda|]. \tag{21}$$

Scheme 3: Seek a sequence $s^{(3)}=(k_1, k_2, \cdots, k_r)$ such that for each graph $G^{(i)}=(G_0{}^{*s_z})^{*s_i}$ with $s_i=(k_1, k_2, \cdots, k_i)$ $(i=0, 1, \cdots, r-1)$

$$M_{k_{i+1}}(G^{(i)})=\min_{v_\lambda\varepsilon U_0\cap V(G^{(i)})}[M_\lambda(G^{(i)})]. \tag{22}$$

In what follows we consider another method of seeking a near-optimal order $s_j{}^\#=(s_z, s_{jz}{}^\#)$ for $s_j{}^\dagger=(s_z, s_{jz}{}^\dagger)$.

Step 1) Given a square matrix W_0, put $G\leftarrow G_0$, $U\leftarrow U_0$, $\pi\leftarrow\lambda$ (null sequence), and $\Delta\leftarrow 0$, then proceed to the next step.

Step 2) If $U=\phi$, then stop the prcedure. Otherwise, go to Step 3).

Step 3) Seek a set $V_1=\{v_k|M_k(G)=0,\ v_k\varepsilon U\}$. If $V_1=\phi$, then go to Step 4). Otherwise, associated with each $v_{k_i} \varepsilon V_1$, let $s=(k_1, k_2, \cdots, k_{|V_1|})$ for arbitrary order of the k_i, and put $\pi \leftarrow (\pi, s)$, $G \leftarrow G^{*s}$, and $U \leftarrow U - V_1$. Then return to Step 2).

Step 4) Seek a set V_2 of the nodes $v_k \varepsilon U$ to give

$$\operatorname*{Min}_{v_i \varepsilon U}[\sum_{j \varepsilon B_i} |L_{ij}| + |B_i|]. \tag{23}$$

If $|V_2|>1$, then go to Step 5), or otherwise go to Step 7).

Step 5) Seek a set V_3 of the nodes $v_k \varepsilon V_2$ to give

$$\operatorname*{Max}_{v_j \varepsilon V_2}[\bigcup_{i \varepsilon V_j} L_{ij}]. \tag{24}$$

If $|V_3|>1$, then go to Step 6), or else go to Step 7).

Step 6) Seek a set V_4 of the nodes $v_k \varepsilon V_3$ to give

$$\operatorname*{Min}_{V_i \varepsilon V_3}[\sum_{j \varepsilon B_i} \{\sum_{h \varepsilon L_{ij}} |L_{hj}^{*i}|\}]. \tag{25}$$

Then choose an arbitrary v_k in V_4, and go to Step 7).

Step 7) Let $\pi \leftarrow (\pi, k)$, $G \leftarrow G^{*k}$, $U \leftarrow U - \{v_k\}$, and $\Delta \leftarrow \Delta + M_k(G)$, and return to Step 2).

In this process, the cycles of Step 2) and 3) concern with seeking s_z of $s_j{}^{\#} =(s_z, s_{j_z}{}^{\#})$, and once the procedure enters into Step 4), the subsequent cycles of steps are for seeking $s_{j_z}{}^{\#}$.

[Example] Given a square matrix W_0, associated with which $X(W_0)$ is of the form

$$X(W_0)=\begin{array}{c|ccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & . & . & . & . & . & 1 & 1 & 1 \\ 2 & . & 1 & 1 & . & . & 1 & . & . & . \\ 3 & . & 1 & 1 & . & 1 & 1 & . & . & . \\ 4 & . & . & . & 1 & 1 & . & 1 & 1 & 1 \\ 5 & . & . & 1 & 1 & 1 & . & 1 & 1 & 1 \\ 6 & . & 1 & 1 & . & . & 1 & 1 & . & . \\ 7 & 1 & . & . & 1 & 1 & 1 & 1 & . & . \\ 8 & 1 & . & . & 1 & 1 & . & . & 1 & 1 \\ 9 & 1 & . & . & 1 & 1 & . & . & 1 & 1 \\ \hline \end{array}.$$

Let $U_0=\{v_3, v_4, \cdots, v_9\}$. We first apply Scheme 3 stated before, then we obtain a sequence $s^{(3)}=(3, 6, 4, 8, 9, 5, 7)$ for which $\Delta(s^{(3)})=14$. On the other hand, with the use of procedure stated above, the sequence $s^{\#}=(3, 6, 8, 9, 4, 5, 7)$ with $\Delta(s^{\#})=12$ is obtained, and we see that this $s^{\#}$ is contained in the set S^* of the optimal sequences.

5. *Conclusion*

In employing the procedure discussed here, we may be able to see which zero elements will be replaced with nonzero ones during the course of applying this ordered sequence of s-povoting operations. Thus, sorting techniques with storage allocated only for nonzero elements, such as those discussed in [6] which are to be modified at each growth of nonzero elements, can be simplified by preparing in advance storage allocation for those newly created nonzero elements.

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