Hybrid Isograph

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1. *Introduction*

The history of analog computing systems for solving algebraic equations known as Isograph dates back as early as late 19th century [1]~[14], and mechanical, electromechanical, and electronic Isographs have been deviced for analysis and synthesis of linear systems. Letting the equation to be solved be $w=f(z)=\sum_{k=0}^{n} a_k z^k=0$, Isographs developed in the past can be classified into the following two categories:

(a) w-plane type

Electrical or mechanical signals that represent the value of $w(z)$ are generated and the values of z that satisfy the condition $w(z)=0$ are sought for as the roots of the equation on an Isograph of this type.

Non-oscillatory signals are used on some machines for finding only real roots, and sinusoids are used for finding complex as well as real roots on other machines.

On an Isograph of this type, one sets a value of z in some way (say by setting dials) and observes the value of $w(z)$ corresponding to that value of z to see if the condition $w(z)=0$ is satisfied. Thus one has to stray with frusrations over the entire complex number plane or the real axis examining possible values of the variable z by himself, helped by his knowledge of the peculiarity of the equation at the best.

(b) z-plane type

Two types of z-plane Isograph have been proposed in the past.

The principle of the one type lies in the property of electromagnetic field distribution on a two dimensional medium.

One feeds electric currents in a proper manner at points on the medium and detects zero potential at locations corresponding to the roots [5] on the medium that represent the complex number plane. The value of $w(z)$ is generated on the other type z-plane Isograph in more or less the same way as on a w-plane type Isograph but the value is not displayed.

Instead the value of $w(z)$ is continually generated as the value of the variable

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z is changed systematically scanning over a portion of the z-plane and it is continually monitored.

In order to indicate the roots, a signal is generated every time when $w(z)$ tends to zero to drive the indicator to indicate a mark for the root on a location corresponding to the value of z at that moment.

Marshall proposed a second type z-plane Isograph in the section titled as EXTENSION in his paper published in 1950, and Löfgren another in 1953.[7], [8]

An experimental model of the second type z-plane Isograph was built by one of the authors around 1959 and various electronic tricks for root indication including Marshall's scheme and Löfgren's scheme, were tried and the authors believe Marshall's scheme is quite practical. In the present paper, the authors propose a z-plane Isograph of the type mentioned by Marshall with additional capability of automatic variable transformation, and the principles of the proposed system are confirmed by digital simulation.

2. *Simple Isograph*

Describing the equation to be solved in polar form with z multiplied to the both side in order to avoid the transmission of the DC components, causing an extra root, $z=0$, one obtains

$$\sum_{k=0}^{n} a_k r^{k+1} \cos(k+1)\theta + j \sum_{k=0}^{n} a_k r^{k+1} \sin(k+1)\theta = 0 \tag{1}$$

where a_k's are assumed to be real.

For displaying the distribution of the roots of equation (1) on the CRT, one generates two analog quantities that represent r and θ, to synthesize the real and imaginary part of the equation. The analog variables are changed so that (r, θ) covers portions of the number plane to be displayed, and the CRT spot

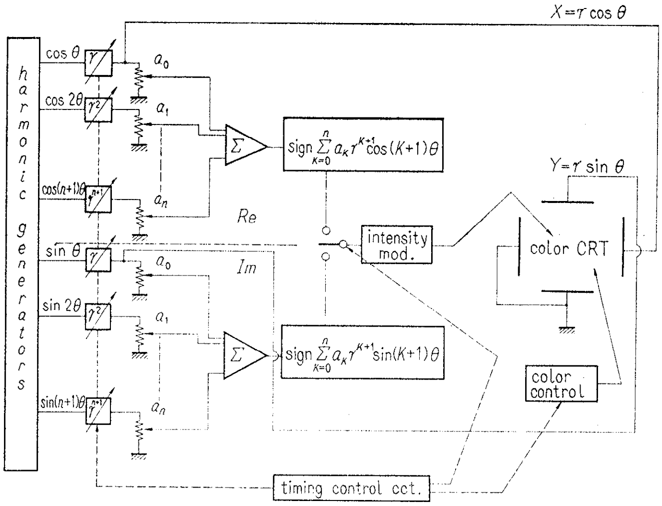

Fig. 1. A hardware configuration for the simple Isograph.

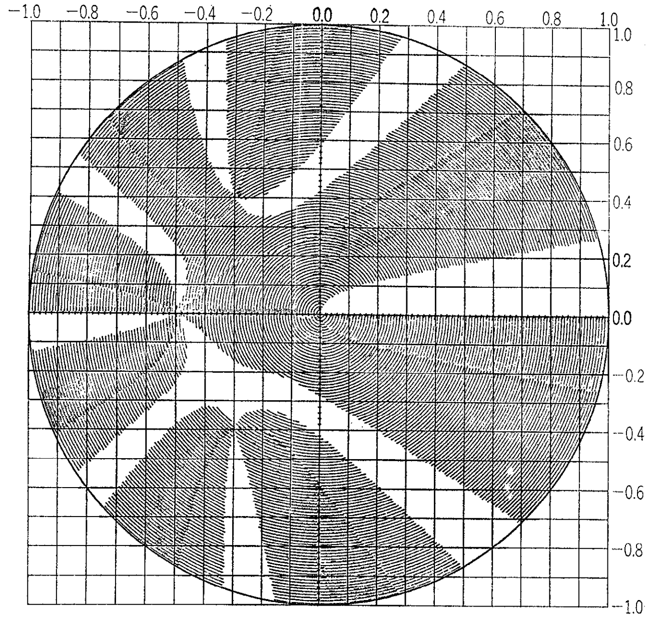

Fig. 2. Solution of $(z+0.5)^3(z+0.3\pm j0.4)^2=0$ on the simple Isograph simulated by digital plotter.

is deflected to continuously follow the (r, θ) with an intensity modulation representing the sign of the real and imaginary part of equation (1).

Fig. 1 shows a hardware configuration for the simple Isograph and Fig. 2 is a digital simulation of an equation displayed on the CRT screen. It has been shown that the boundaries of the brightened areas on the CRT divide 2π at a m-ply root into 4m equal angles. [12], [14]

3. *Moving Magnifier Type Isograph*

When roots are closely located it is desirable to shift the origin of the display to give a magnified view of the distribution of the roots. A transformation $z'=p(z-q)$ gives this effect, where q is an arbitrary complex constant, and p is an arbitrary real constant that is set by the user of the Isograph.

Denoting q as $q=\gamma_0\varepsilon^{j\theta_0}$ one obtains $z=z'p^{-1}+r_0\varepsilon^{j\theta_0}$. Substituting this into equation (1) and multiplying z' again, the real and imaginary components of the equation are

$$\sum_{k=0}^{n} a_k \sum_{\rho=0}^{k} ({}_kC_\rho p^{-\rho}\gamma_0^{k-\rho})\gamma'^{\rho+1} \begin{matrix}\cos\\ \sin\end{matrix}[(\rho+1)\theta'+(k-\rho)\theta_0]=0 \qquad (2)$$

Coefficients ${}_kC_\rho p^{-\rho}\gamma_0^{k-\rho}$ can be represented by potentiometers for p, and r_0 of limited range, therefore, it is possible to organize an Isograph so that the transformation $z'=p(z-q)$ is automatically carried out when p, and q are set by dial, using phase shifters representing $(k-\rho)\theta_0$.

Fig. 3 shows a digital simulation of an equation on this type of Isograph.

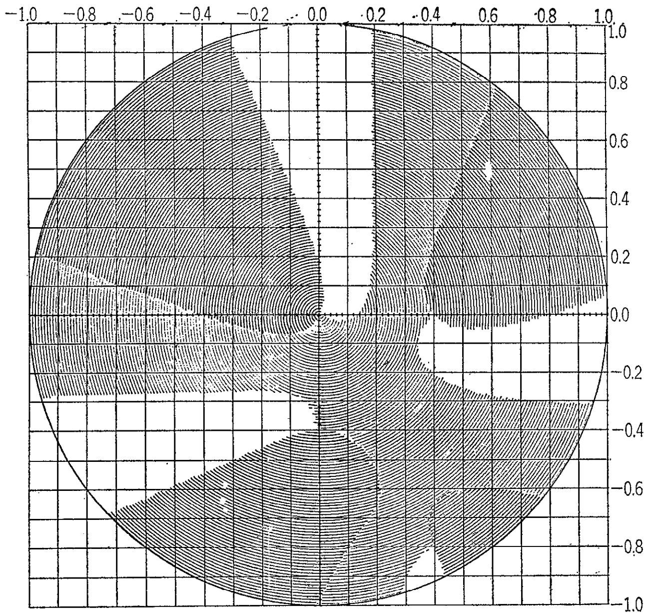

Fig. 3. Solution of $(z-0.5)(z-0.52\pm j0.02)=0$ on the moving magnifier type Isograph with $p=20, q=0.5+j0.02$ simulated by digital plotter.

4. *Bird's Eye Type Isograph*

When it is desired to display the distribution of the roots of the equation over the entire number plane, a transformation $z'=(j-z)/(j+z)$ can be used to circumvent the essential limitation of the physical size of the device used for the display. The upper half and the real axis of the z-plane are mapped onto the inside and the periphery of the unit circle on the z'-plane by the transformation. Since the distribution of the roots of an algebraic equation with real coefficient is symmetric about the real axis on the z-plane, the physical limitation is essentially circumvented by operating the Isograph on the z'-plane for displaying the periphery and inside of the unit circle. Applying the transformation to equation (1) and after some algebraic manipulation [14] one obtains polar expressions for real and imaginary parts:

$$\sum_{k=0}^{n} a_k \sum_{\rho=1}^{n+1} b_{k\rho} \gamma'^{\rho} \begin{matrix}\cos\\ \sin\end{matrix}\left(\rho\theta' + \frac{k\pi}{2}\right)=0 \tag{3}$$

where the constants $b_{k\rho}$ depend only upon the degree of the equation and the transformation, and are independent of the coefficients of the equation to be solved. This is the reason why the transformation can be automated. Fig. 4 is a digital simulation of a solution of an equation on this type of isograph. This transformation can be automated without phase shifters.

5. *Isograph with Bird's Eye and Moving Magnifier*

A combination of the transformation $z'=(j-z)/(j+z)$, and $z''=p(z'-q')$ can

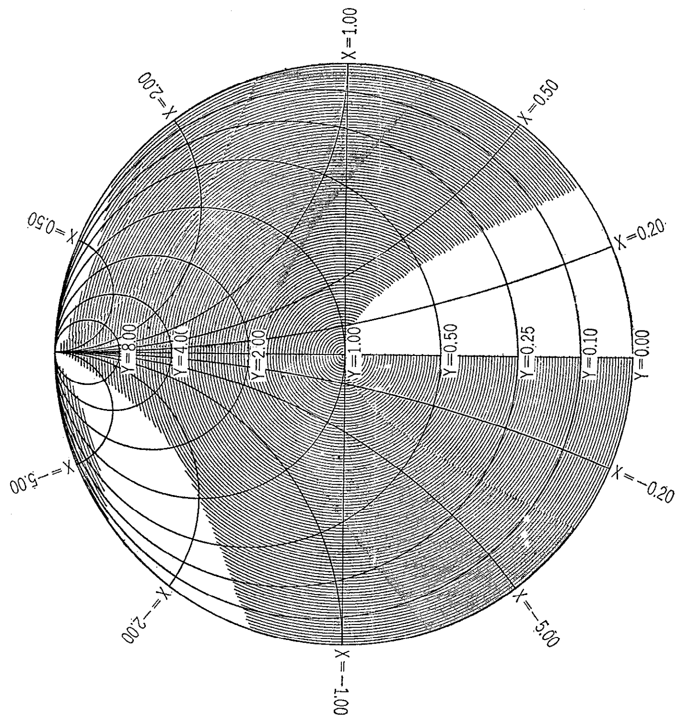

Fig. 4. Solution of $(z+50)(z-20\pm j20)=0$ on the bird's eye view type Isograph simulated by digital plotter.
The transformation $z'=(j-z)/(j+z)$ maps the upper half and the real axis of the z plane onto the unit circle and its periphery around the origin on the z' plane.

be automated as well. Application of the two transformations gives polar expressions for the real and imaginary parts:

$$\sum_{k=0}^{n} a_k \sum_{\rho=1}^{n+1} b_{\rho k} \sum_{s=0}^{\rho-1} {}_{\rho-1}C_s p^{-s} \gamma_0'^{(\rho-s-1)} \gamma''^{s+1} \times \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\left[(s+1)\theta''+(\rho-s-1)\theta_0'+\frac{k\pi}{2}\right]=0 \qquad (4)$$

for which phase shifters are needed again.

6. *A Hybrid System*

Solution of algebraic equation is needed for analysis and synthesis of linear systems. Physical constants of a linear system and the coefficients of the characteristic equation are related through functional relations in general as:

$$a_k=F_k(p_1, p_2, \cdots, p_s) \quad k=0, 1, \cdots, n \qquad (5)$$

where $p_1, p_2, \cdots, p_s$ are the physical constants of the system. For designing a linear system, one first determines the distribution of the roots of the characteristic equation so that the response of the system satisfies the required conditions, and then determines the physical constants corresponding to it. This process in the design of a linear system is greatly simplified through the hybrid system shown in Fig. 5.

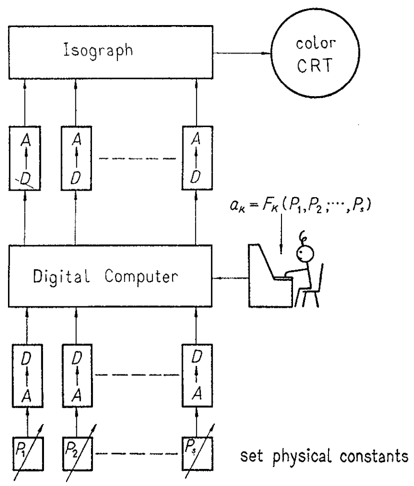

Fig. 5. Proposed hybrid system for designing linear system.

The user of the design system in Fig. 5 sets the physical constants of the linear system on the s dials which are converted into digital information before transmitted to the minicomputer. The user also inputs the definition of the functions F_k's through the typewriter in conversational mode. By design, the software on the minicomputer requests the user to input the functions F_k's one by one, and the user is allowed to redefine any of the functions by depressing the interrupt button. The output of the minicomputer is the coefficients of the characteristic equation and the Isograph immediately displays the distribution of the roots, immediately indicating the effect of the variation in the physical constants. Thus the distribution of the roots of the characteristic equation for a particular setting of the physical constants will be found within one second or so [14], and the design process of determining the root locus corresponding to the variation in the physical constants will be greatly expedited.

7. *Conclusion*

A scheme for displaying the roots of algebraic equations on the z-plane have been confirmed by digital simulation, and automation of several types of transformation of the variable has been proposed and confirmed by digital simulation. The hybrid system proposed in the present paper will expedite the design process for linear systems.

8. *Acknowledgement*

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