A Numerical Method for Extraction of Multiple Roots of Algebraic Equation

MISAKO ISHIGURO*

1. *Introduction*

A new method is given for the numerical solution of algebraic equation with multiple roots. First we describe that the multiple roots of the equation are determined by existing methods with very small accuracy. So, we provide the practical algorithm to extract the multiple roots accurately by the repeated use of EUCLID algorithm that is called HERMITE's method [5], and give the computer program. Computer results of the new method is also given and compared with that of existing one.

2. *Disadvantage of BAIRSTOW's Method when Multiple Roots Exist*

The quadratic divisor of polynomial $f(x)$ is obtained as follows.

For appropriate initial values p and q, $f(x)$ is divided by $x^2+p\cdot x+q$,

$$f(x)=(x^2+p\cdot x+q)\cdot Q(x)+r\cdot x+s. \tag{1}$$

$Q(x)$ is also divided by $x^2+p\cdot x+q$,

$$Q(x)=(x^2+p\cdot x+q)\cdot p(x)+u\cdot x+v, \tag{2}$$

and evaluate $\Delta p=\frac{1}{\Delta}(r\cdot v-u\cdot s)$, $\Delta q=\frac{1}{\Delta}[s\cdot(v-p\cdot u)+q\cdot u\cdot r]$,

$$\Delta=v\cdot[v-p\cdot u]+q\cdot u^2. \tag{3}$$

$p+\Delta p$ and $q+\Delta q$ are more accurate approximations to p and q. Thus, we get an accurate divisor $x^2+p\cdot x+q$. Then $f(x)$ is divided by $x^2+p\cdot x+q$ and process is repeated for new $f(x)$.

Let us suppose that the degree of the polynomial $f(x)$ is even and $f(x)=0$ has multiple roots. We shall consider the case when $p(x)$ has a quadratic divisor $x^2+p\cdot x+q$ with multiplicity greater than two. From (1) and (2),

$$f(x)=(x^2+p\cdot x+q)[(x^2+p\cdot x+q)\cdot p(x)+u\cdot x+v]+r\cdot x+s. \tag{4}$$

$$f'(x)=(x^2+p\cdot x+q)[(x^2+p\cdot x+q)\cdot p'(x)+2(2x+p)\cdot p(x)+2u+v] \\ +(2v-p\cdot u)\cdot x+p\cdot v-2u\cdot q+r. \tag{5}$$

Since $(x^2+p\cdot x+q)^2$ is a divisor of $f(x)$, both $f(x)$ and $f'(x)$ are divided by $x^2+p\cdot x+q$ without remainder. Then

$$r\to 0,\ s\to 0,\ 2v-p\cdot u\to 0,\ p\cdot v-2u\cdot q+r\to 0. \tag{6}$$

So, we get $p\risingdotseq\frac{2v}{u}$, $q\risingdotseq\frac{v^2}{u^2}$, (7)

This paper first appeared in Japanese in Joho-Shori (Journal of the Information Processing Society of Japan), Vol. 13, No. 1 (1972), pp. 2~7.

* Japan Atomic Energy Research Institute

therefore Δ in formulae (3) vanishes in the limit:

$$\Delta = v \cdot (v - p \cdot u) q \cdot u^2 \to 0,$$

accordingly, the denominator of expression (3) is too small to calculate Δp and Δq correctly.

This fact is obvious from Table 1. The calculation is carried out for thirteen sample problems shown in column 1. Both BAIRIS, included in FACOM SSL [2] and ROOTP [4] are computer subprograms for solution of equation by BAIRSTOW's method. We must give up almost all problems by BAIRIS. Because, for extremely small Δ, next approximations of p and q are not determented from (3) but assigned to the appropriate values for the convenience of calculation. While, ROOTP is improved as in the following.

(1) The coefficients are normalized to avoid the floating point overflow. In some case, reciprocals of roots are computed.

(2) The simultaneous NEWTON and BAIRSTOW iteration is used.

(3) In calculating Δp and Δq, various unusual conditions, such as the very small Δ, are considerd.

But the roots found always contain the error in proportion to the multiplicity and calculation takes much time for the loss of convergence.

We know more intuitively that, in NEWTON's method, same situation occurs since roots are found by the recurrence formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{8}$$

Consequently, it is obvious both NEWTON and BAIRSTOW methods are useful only for the case when equation has no multiple roots. Real multiple roots are successfully extracted by applying STÜRM's theorem. But it is remarkable that, in existing methods, multiple roots are not always obtained accurately.

In [3] the following method is described. If a root is bi-multiple, more accurate values of roots are sought from the equation

$$f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{1}{2} f''(x_n)(x_{n+1} - x_n)^2 = 0. \tag{9}$$

Analogously, for a tri-multiple root, a third order equation is found and so on. In practice it is rarely happens that the multiplicity of a root is known beforehand.

3. *Separation of Polynomial into the Divisors with Same Multiplicity*

Equation $p(x) = 0$ is given. Let us assume that the polynomial $p(x)$ has the form

$$p(x) = p_1(x) \cdot p_2(x)^2 \cdots p_l(x)^l \cdots p_m(x)^m, \quad m \geqq 1. \tag{10}$$

l-multiple divisors are gathered and written by $p_l(x)^l$. And we see every equation $p_l(x) = 0$ $(l = 1, \cdots, m)$ has no multiple root and $p_1(x), p_2(x), \cdots, p_m(x)$ are

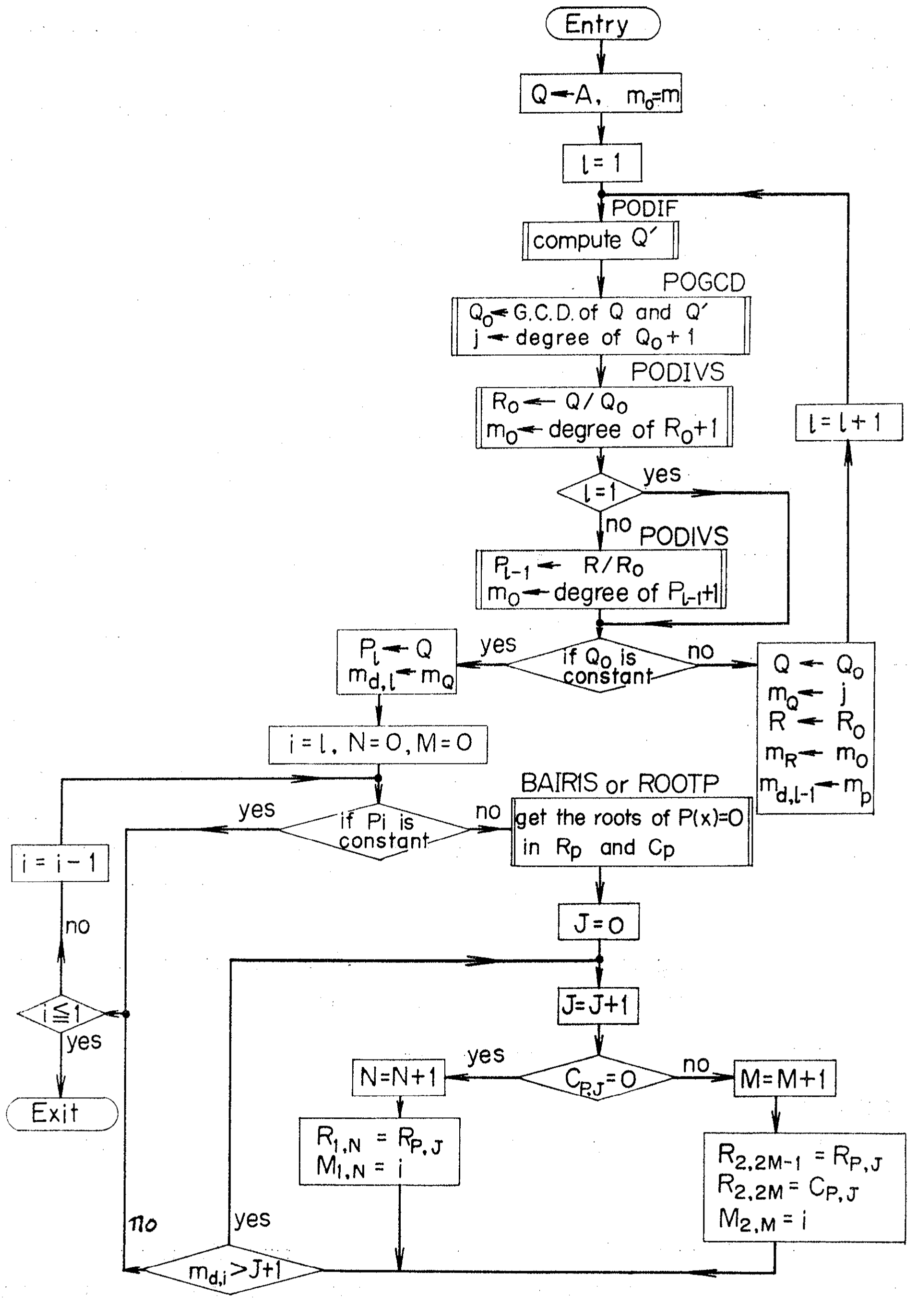

Fig 1 Flow chart of MROOT

Table 1

Problem \ Subroutine	BAIRIS		MROOT-BAIRIS		ROOTP		MROOT-ROOTP	
$x^4+4x^2+4=(x^2+2)^2$	ILL=2		$\pm 1.4142\,i$ (2)	8	$0.00209\pm 1.4168\,i$ $0.00209\pm 1.4116\,i$	13	$\pm 1.4142\,i$ (2)	11
x^6-12x^2-16 $=(x-2)(x+2)(x^2+2)^2$	ILL=2		± 2.0000 $\pm 1.4142\,i$ (2)	11	± 2.0000 $0.00125\pm 1.4160\,i$ (2)	22	± 2.00000 $\pm 1.4142\,i$ (2)	14
$x^4-2x^2+1=(x-1)^2(x+1)^2$	± 1.0000 (2)	234	± 1.0000 (2)	8	± 1.0000 (2)	6	± 1.0000 (2)	9
$x^6+4x^4+5x^2+2$ $=(x^2+1)^2(x^2+2)$	Overflow		$\pm 1.0000\,i$ (2) $\pm 1.4142\,i$	14	$0.00591\pm 1.0059\,i$ $-0.00577\pm 0.99411\,i$ $0.00138\pm 1.4142\,i$	24	$\pm 1.0000\,i$ (2) $\pm 1.4142\,i$	15
$x^8-x^7+7x^6-6x^5+18x^4-12x^3$ $+2x^3+2x^2-8x+8$ $=(x^2+2)^3(x^2-x+1)$	Overflow		$\pm 1.4142\,i$ (3) $0.50000\pm 0.86602\,i$	22	$-0.01260\pm 1.4159\,i$ $0.00758\pm 1.4244\,i$ $0.00502\pm 1.4023\,i$ $0.50000\pm 0.86602\,i$	44	$\pm 1.4142\,i$ (3) $0.50000\pm 0.86602\,i$	23
$x^4+2x^3+3x^2+2x+1$ $=(x^2+x+1)^2$	ILL=2		$-0.50000\pm 0.86603\,i$ (2)	8	$-0.50115\pm 0.86753\,i$ $-0.49885\pm 0.86452\,i$	15	$-0.50000\pm 0.86603\,i$ (2)	8
$x^3+3x^2+3x+1=(x+1)^3$	ILL=1		-1.0000 (3)	7	$-1.0087\pm 0.01502\,i$ -0.98266	13	-1.0000 (3)	7
$x^5-x^4+x^2-x^2-2x+2$ $=(x-1)^2(x+1)(x^2+2)$	Overflow		1.0000 (2) -1.0000 $\pm 1.4142\,i$	12	1.0000 (2) -1.0000 $\pm 1.4142\,i$	11	1.0000 (2) -1.0000 $\pm 1.4142\,i$	14
$x^7+x^6+x^5-x^4-x^3-x^2$ $=x^2(x-1)(x^2+x+1)^2$	ILL=2		0.0000 (2) 1.0000 $-0.50000\pm 0.86603\,i$ (2)	18	0.0000 (2) 1.0000 $-0.49974\pm 0.86665\,i$ $-0.50026\pm 0.86540\,i$	44	0.0000 (2) 1.0000 $-0.50000\pm 0.86603\,i$ (2)	16
$x^8-x^7+6x^6-5x^5+13x^4-8x^3$ $+12x^2-4x+4$ $=(x^2+2)^2(x^2-x+1)(x^2+1)$	ILL=2		$\pm 1.4142\,i$ (2) $0.50000\pm 0.86602\,i$ $\pm 1.0000\,i$	28	$0.00014\pm 1.4136\,i$ $-0.00014\pm 1.4149\,i$ $0.50000\pm 0.86603\,i$ $\pm 1.0000\,i$	47	$\pm 1.4142\,i$ (2) $0.50000\pm 0.86602\,i$ $\pm 1.0000\,i$	32
$x^4-2x^3+3x^2-2x+1$ $=(x^2-x+1)^2$	ILL=2		$0.50000\pm 0.86603\,i$ (2)	8	$0.50115\pm 0.86752\,i$ $0.49885\pm 0.86452\,i$	12	$0.50000\pm 0.86603\,i$ (2)	9
$x^8+2x^6+3x^4+2x^2+1$ $=(x^2+x+1)^2(x^2-x+1)^2$	Overflow		$\pm 0.50000\pm 0.86603\,i$ (2)	20	$-0.50044\pm 0.86670\,i$ $-0.49956\pm 0.86536\,i$ $0.50004\pm 0.86498\,i$ $0.49996\pm 0.86708\,i$	393	$\pm 0.50000\pm 0.86603\,i$ (2)	20
$12x^5+12x^4+24x^3+24x^2$ $=12x^2(x+1)(x^2+2)$	Overflow		0.0000 (2) -1.0000 $\pm 1.4142\,i$	12	0.0000 (2) -1.0000 $\pm 1.4142\,i$	5	0.0000 (2) -1.0000 $\pm 1.4142\,i$	14

( ) multiplicity of the root, condition for convergence is 10^{-5}

relatively prime. Let $Q_1(x)$ be the greatest common divisor of $p(x)$ and $p'(x)$, derivative of $p(x)$, then from (10)

$$Q_1(x)=p_2(x)\cdot p_3(x)^2\cdots p_l(x)^{l-1}\cdots p_m(x)^{m-1}.$$

And let $Q_i(x)$ be that of $Q_{i-1}(x)$ and $Q'_{i-1}(x)$, analogously,

$$Q_i(x)=p_{i+1}(x)\cdot p_{i+2}{}^2(x)\cdots p_l(x)^{l-i}\cdots p_m(x)^{m-i},$$

$$Q_{m-1}(x)=p_m(x) \text{ and } Q_m(x) \text{ is a constant.}$$

From this, we give an algorithm to get the formula (10).

$$\begin{aligned}
&Q_0(x)=p(x),\\
&Q_l(x)=\text{G.C.D.}[Q_{l-1}(x),\ Q'_{l-1}(x)]\ (l=1, 2, \cdots, m-1),\\
&Q_m(x) \text{ is a constant } (=1),\\
&R_l(x)=Q_{l-1}(x)/Q_l(x)\ (l=1, 2, \cdots, m-1),\ R_m(x)=Q_{m-1}(x),\\
&p_{l-1}(x)=R_{l-1}(x)/R_l(x)\ (l=2, 3, \cdots, m),\ p_m(x)=R_m(x).
\end{aligned} \tag{11}$$

After all, we solve the equations with single root,

$$p_l(x)=0,\ (l=1, 2, \cdots, m) \tag{12}$$

in place of equation $p(x)=0$. Computer subprogram MROOT (Fig. 1) is made to evaluate the expressions (11). Effectiveness is shown by the computing results (Table 1) in the cases when MROOT is used together with BAIRIS and ROOTP.

4. *Concluding Remark*

There are many methods for solving algebraic equations. But the methods do not exist which are equally effective for all eqations. Therefore, method proposed here gives one of the useful mean for equation with multiple roots. The calculation is carried out on FACOM 230-60 computer.

References

[1] Computer Hand Book: p. 5-45~5-47, Information Processing Society of JAPAN, Omusya.
[2] The Numerical Methods of FACOM SSL, p. 32-39.
[3] Zaguskin, V. L.: Handbook of Numerical Methods for the Solution of Algebraic and Transcendental Equations, Pregamon Press.
[4] Ellenberger, K. W.: Algorithm 30, Numerical Solution of the Polynomial Equation, ANL-7054, p. 18-19.
[5] Takagi, T: Introduction to Analysis, p. 244-245, Iwanami.