An Approximation of a Curve with Circular Arcs

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1. *Introduction*

This paper treats the data reduction and reconstruction of 2-dimensional graphics composed of curved lines.

Several proposed methods [1] representing a curve as a concatenation of straight line segments are not successful in the preservation of smoothness of curves. There exists another algorithm which connects given points smoothly with circular arcs [2]. This algorithm is very simple on the procedure for restoring curves, but sometimes causes undesirable oscillations.

In this paper we present an optimum sampling method which prevents the oscillations in the process of reconstructing an original curve with circular arcs. We assume a treated curve is given as an explicit function of x such as $y=f(x)$.

2. *Cause of the Oscillation*

The algorithm in the paper [2] takes the assumption that each segment of circular arcs should have the same tangents at both ends as the adjacent ones'. Therefore, the restored shape of a curve is determined at once by an initial tangent and sampled points. The deviation in each segment from an original curve accumlates from segment to segment and result in inducing such oscillations as seen in Fig. 1. This oscillatory phenomenon is remarkable in case that the original curve decreases its curvature monotonically as shown in Fig. 2 (a).

Let the initial tangent of an original curve be M_1 and the end points of the first circular arc be P_1 and P_2; then the problem is to select the next end point P_3 of the second circular arc. Assuming that M_2 and m_2 are the tangent of the original curve and an approximate curve at point P_1, the latter is determined directly from circular arc $\widehat{P_1P_2}$. For the simplicity we call the tangents at the end points of the remarked circular arc the incoming and the outgoing tangent, according to the sequence to be connected. At this point, we select a position of P_3 for chord $\overline{P_2P_3}$ to turn outside of m_2, the sign of the curvature of the approximate curve will be opposite to the original curve and the oscillations start.

In case that the original curve increases its curvature monotonically as shown in Fig. 2 (b), we can select the position of P_3 very easily so that the chord

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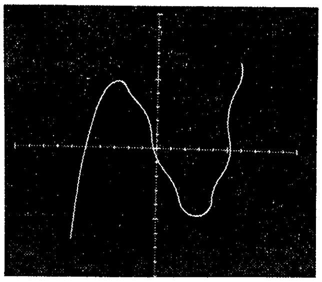

Fig. 1 Oscillation induced by calcatenation of circular arcs.

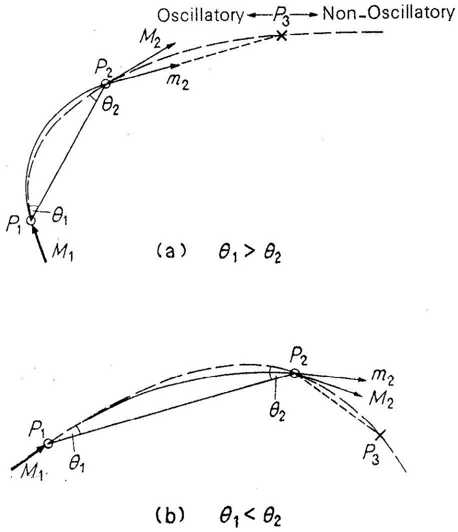

Fig. 2 Sampling method for an approximate curve with circular arcs.

turns inside of m_2 because m_2 turns outside of M_2. But it should be noticed that the similar condition to the previous case occurs in next segment P_3P_4 whenever P_3 may be selected where m_3 turns inside too much, and the oscillations begin.

From these considerations, we find it very difficult to select appropriate points successively from the original curve without causing the oscillation.

3. *Sampling Algorithm which Prevents the Oscillation*

Now, we consider our procedure, that is, encoding the original curve as chain of primary sampling points which approximate the curve and of secondary sampling points in the manner that the latters interlace the formers.

We prove that there exists a point Q (secondary sampling point) which can prevent the oscillation between points P_1 and P_2 of the original curve, where Fig. 3 shows the optimum position of Q.

At first, let us formulate a relation between Q and m_2. We assume that the original curve has continuous derivatives of at least first and second order in the segment P_1P_2 and is convex. These conditions will not impose special requirements on primary sampling points. Hence θ_1 is greater than θ_2; moreover

$$0<\theta_2<\theta_1<\pi/2 \qquad (1)$$

The other case $\theta_2>\theta_1$ will be mentioned later.

From Fig. 3 tangents M_1, M_2 and n_1 $(=\overline{P_1Q})$ and n_2 $(=\overline{QP_2})$ are shown as follows

$$\left.\begin{array}{ll} M_1=\tan\theta_1 & n_1=\tan\beta \\ M_2=-\tan\theta_2 & n_2=-\tan\gamma \end{array}\right\} \qquad (2)$$

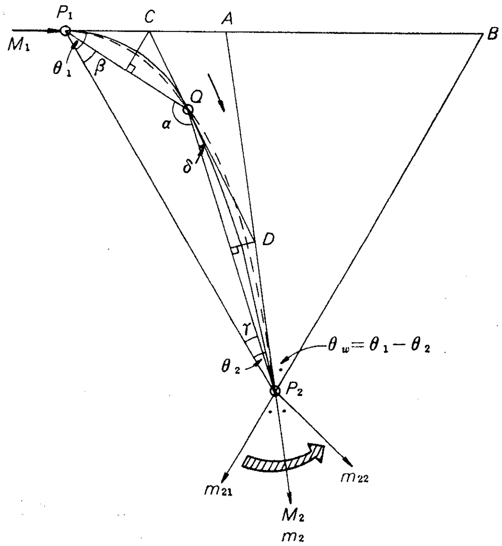

Fig. 3 Secondary sampling points which prevent the oscillation ($\theta_1>\theta_2$).

Using these notations of M_1, n_1 and n_2, the expression of m_2 is

$$m_2=\frac{-2(n_2-n_1)(n_1n_2+1)+M_1\{(n_1-n_2)^2-(n_1n_2+1)^2\}}{(n_1-n_2)^2-(n_1n_2+1)^2+2M_1(n_2-n_1)(n_1n_2+1)} \tag{3}$$

Here, if we let $\alpha=\angle P_1QP_2$, α represents a continuous function obviously in an open interval where $P_1<Q<P_2$, and we obtain

$$\tan\alpha=-\tan(\beta+\gamma)=\frac{n_2-n_1}{1+n_1n_2} \tag{4}$$

from $\alpha=\pi-(\beta+\gamma)$.

We can show that $\tan\alpha$ (<0) is a continuous function in this interval (P_1, P_2) using following inequalities

$$\pi/2<\pi-\theta_1<\alpha<\pi-\theta_2<\pi \tag{5}$$

These inequalities are derived from the relation between Q and α. From (5), m_2 can be expressed in terms of a variable $\tan\alpha$;

$$m_2=\frac{-2\tan\alpha+M_1(\tan^2\alpha-1)}{\tan^2\alpha-1+2M_1\tan\alpha} \tag{6}$$

Here, if we introduce a new variable t $(=\tan\alpha-1/\tan\alpha)$, we obtain

$$m_2(t)=\frac{-2+M_1t}{t+2M_1} \tag{7}$$

It is easy to show that $m_2(t)$ is a continuous and monotonically increasing function in the following interval

$$t_l=-M_1+1/M_1<t<M_2-1/M_2=t_u \tag{8}$$

these are derived from (5), where t_l and t_u correspond to the limit of t when Q approaches P_1 and P_2, respectively. Now if we let

$$f(t)=m_2(t)-M_2 \tag{9}$$

we can derive following equations

$$\left.\begin{aligned} f(t_l) &= \lim_{t\to t_l+0} f(t) = -(\tan\theta_1 - \tan\theta_2) \\ f(t_u) &= \lim_{t\to t_u-0} f(t) = \frac{1+M_2^2}{\tan\theta_2 + \dfrac{1}{\tan(\theta_1-\theta_2)}} \end{aligned}\right\} \tag{10}$$

where $\tan\theta_2$ and $\tan(\theta_1-\theta_2)$ are positive. Hence it has been proved that a root of $f(t)=0$ exists in the interval (t_l, t_u), that is, the root of $f\{t(\alpha)\}=0$ exists in the range of (5).

Next we consider this process geometrically by using Fig. 3. Let $\theta_x = \angle DP_2P_1$ in order to investigate a geometrical relation between m_2 and Q, then

$$\theta_x = \angle DP_2Q + \angle QP_2P_1 = 2\pi - \theta_1 - 2\alpha \tag{11}$$

From (5) and $\theta_w = \theta_1 - \theta_2$, we have

$$-\theta_w < \theta_x - \theta_2 < \theta_w \tag{12}$$

Namely, as Q moves along the original curve from P_1 to P_2, m_2 turns from m_{21} to m_{22} as shown in Fig. 3. Hence m_2 varies throughout of the range between θ_w and $-\theta_w$, and the center of the range is M_2. In other words, we can say that the deviation up to the extent of $\pm\theta_w$ of the incoming tangent can be canceled in the segment.

As an example of this process, Fig. 4 shows a graph of angles $(\theta_x-\theta_2)$ between tangents of the approximate curve and of the original curve $y=-x^2$.

Now it should be noticed that Q will not be a point of inflexion in the approximate curve and hence will not yield an oscillation in the segment when Q induces the equation $m_2 = M_2$.

On the other hand, when the curvature of the original curve increases monotonically (i.e. $\theta_1 < \theta_2$), m_2 varies over the range between θ_w and $-\theta_w$ similarly to the above case, but the orientation of m_2 changes from the outside to the inside

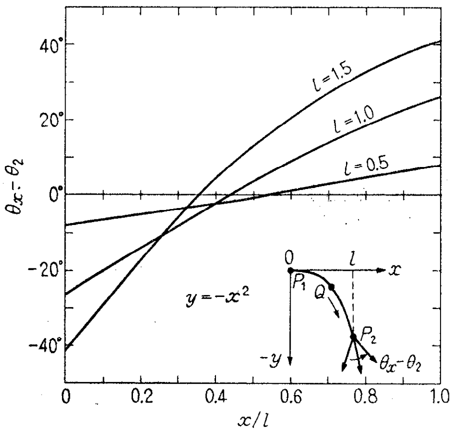

Fig. 4 Angles $(\theta_x-\theta_2)$ between tangents of an original curve and of an approximate curve $(y=-x^2)$.

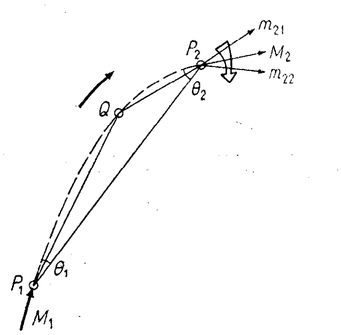

Fig. 5 Secondary sampling points which prevent the oscillation $(\theta_1<\theta_2)$.

of the original curve with Q moving from P_1 to P_2 as shown in Fig. 5. Consequently, there exists Q for which the equation $m_2=M_2$ is satisfied. Moreover, this Q never yields the oscillation in the segment.

From these considerations, it has been proved that there exists Q which prevents the oscillatory phenomenon.

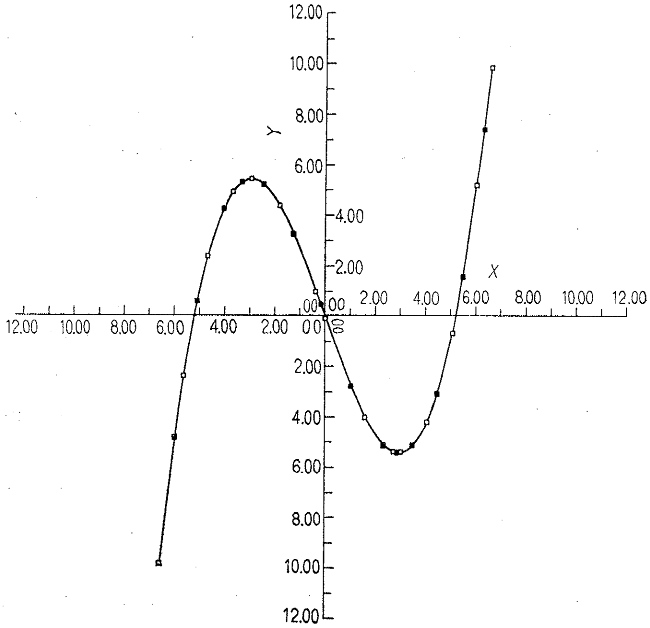

Fig. 6 Example of a curve restored by the proposed method

$y=0.1x(x^2-27)$

Total sampled points: 29

(□: primary 15, ■: secondary 14)

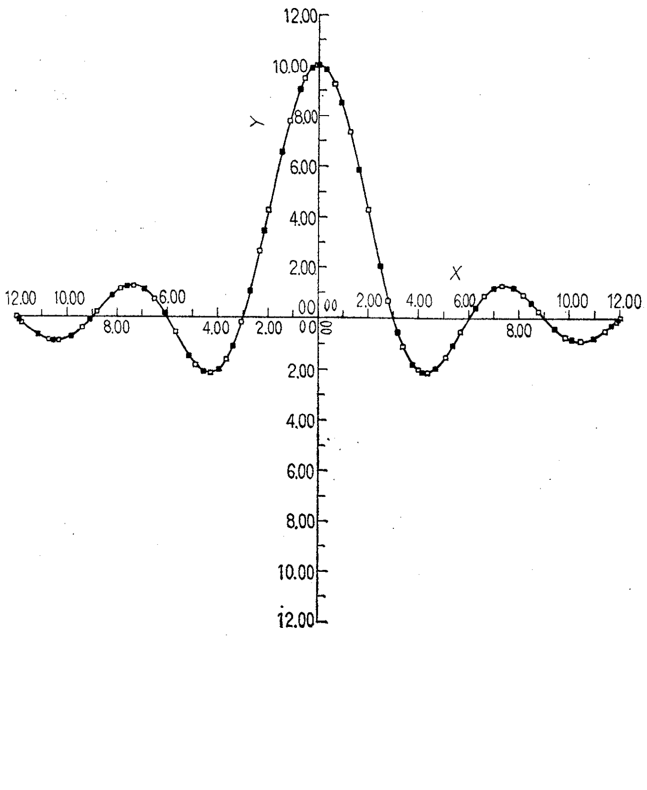

Fig. 7 Example of a curve restored by the proposed method

$$y=10\sin\left(\frac{\pi x}{3}\right)\Big/\left(\frac{\pi x}{3}\right)$$

Total sampled points: 75

(□: primary 38, ■: secondary 37)

The procedure to obtain Q is reduced to the method which searches a root of a equation $f(x)=0$; we take any value, say, $x'=(x_1+x_2)/2$, in the interval $[x_1, x_2]$ where $f(x_1)\cdot f(x_2)<0$ (we put $f(x_1)<0$, $f(x_2)>0$). By substituting x' in $f(x)$, we obtain on replacing x_1 by x' a narrower interval $[x', x_2]$ if $f(x')<0$, whereas if $f(x')>0$, we obtain a narrower interval $[x_1, x']$, and so on.

4. *Examples of Approximated Curves*

Fig. 6, 7 show examples of curves restored by proposed method. Fig. 6 is a function $y=0.1x(x^2-27)$, and Fig. 7 is a function $y=10\sin(\pi x/3)/(\pi x/3)$, where we adopt a least square approximation for primary sampling points.

5. *Conclusion*

We have proposed a method which approximates curves given by algebraic functions. Furthermore, the proposed method can be available for a problem to connect scattered points smoothly by fitting appropriate explicit functions to those points.

When we process graphics by computers this method is one of the efficient means not only for the representation of smooth curves, but also for the data reduction of figures. Then this method can contribute to the transmissions, the storages and the processing, of figures.

References

[1] For example, B. D. Fried: Solving Mathematical Problems, pp. 131~178, McGRAW-HILL, 1967; H. Freeman: On the Encoding of Arbitrary Geometric Configrations, *I. R. E. Trans. on EC*, Vol. EC-10, No. 2, pp. 260~268, June 1961.

[2] Hosaka, M. and M. Endo: On Generation, Storage and Processing of Curve-connected Patterns, *J. Information Processing Society of Japan*, Vol. 6, No. 3, pp. 129~139, May 1965.