A Matching Transformation of a Bipartite Graph

JUN NUMATA*

Abstract

A matching transformation is defined between two complete matchings on a bipartite graph, where the bipartite graph corresponds to a certain matrix and each complete matching corresponds to each term of its determinant. Here, some structual properties of the set of complete matchings are investigated, and a method for generating the set of complete matchings by the matching transformation is introduced. The properties of a circuit in a bipartite graph is also discussed with respect to the alternation of matching edges and nonmatching edges.

1. *Introduction*

The present paper introduces the concept of matching transformations among the set of complete matchings on a bipartite graph and discusses properties of matching transformations. The concept of matching transformations is useful for generating all complete matchings on a bipartite graph and for the calculation of determinants. Furthermore, we will investigate the properties of circuits on a bipartite graph by means of alternating paths whose successive edges alternatly belong to and do not belong to a given complete matching. If a ring sum of two complete matchings M_i and M_j on a bipartite graph forms an elementary circuit, then the matchings will be said to be matching-transformable to each other. Since any two complete matchings can be related to each other by a matching transformation, the structual properties of the set of complete matchings can be clarified in terms of matching transformations. An undirected bipartite graph is one of graphical expressions of a matrix, and, at the same time, a matrix can be expressed in terms of a directed graph (often called a signal flow graph or a flow graph). The relation between a directed graph and a bipartite graph is made clear through an "augmented graph".

2. *Basic Concepts*

Throughout this paper, the notation and terminology are based mainly on references [1] and [3]. We will confine our attention to undirected graphs

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* SONY Corporation Research Center

$G\{E, V\}$ where E is the set of edges and V is the set of vertieces, i.e. the graphs for which the orientations of edges are of no significance. Generally, a set of edges E is expressed as a subset of the cartesian product $V \times V$ of the vertex set V with itself. A graph $G\{E, V\}$ is a bipartite graph, if the vertex set V is the union of two disjoint subsets V_L and $V_R (V_L \cap V_R = \phi,\ V_L \cup V_R = V)$ and the relation

$$E \subseteq V_L \times V_R \tag{1}$$

is satisfied. V_L and V_R are called the left vertex set and the right vertex set of the bipartite graph. A binary operation $\odot$ on two subgraphs $G_1\{E_1, V_1\}$ and $G_2\{E_2, V_2\}$ of a graph $G\{E, V\}$ is defined by

$$G_1\{E_1, V_1\} \odot G_2\{E_2, V_2\} = G_3\{E_1 \odot E_2, V_1\} \tag{2}$$

when $V_1 = V_2$, where operator $\odot$ in $E_1 \odot E_2$ is a set-teeoretical operator such as $\cup$ (union), $\cap$ (intersection),—(difference) and $\odot$ (ring sum). *A* matching M on a bipartite graph G is a subgraph $M\{E_M, V_M\}$ of $G\{E, V\}$ such that every vertex of V_M is an end point of at least one edge of E_M, and that no two edges of E_M have a vertex of V_M incommon. An edge is said to be a matching edge if it belongs to E_M, and otherwise, it is said to be a nonmatching edge. If $|E_M|$(=the number of matching edges of $M\{E_M, V_M\}$) is maximum among all the matchings on G, then M is called a maximum matching on G, and if $V_M = V$ then the matching is called a complete matching on G. If every edge (v^l, v^r) of G satisfies $v^l \in V_L'$ or $v^r \in V_R'$ then the pair $[V_L', V_R']$ is said to be a cover of G, where $V_L' \subseteq V_L$ and $V_R' \subseteq V_R$. If $|V_L'| + |V_R'|$ is minimum among all the possible cover of G, $[V_L', V_R']$ is called a minimum cover of G. If $[V_L, \phi]$ and $[\phi, V_R]$ are the minimum covers $[V_L', V_R']$ of a bipartite graph G and no other minimum covers exist, then G is called an irreducible bipartite graph. Let $G[E, V_L \cup V_R]$ be a bipartite graph and let V_L and V_R be partitioned as $V_L = V_{L1} \cup V_{L2} \cup \cdots \cup V_{Ln}$ and $V_R = V_{R1} \cup \cdots \cup V_{R2} \cup \cdots \cup V_{Rn}$ into the union of disjoint subsets. The induced graph $G^\triangle\{E^\triangle, V^\triangle{}_L \cup V^\triangle{}_R\}$ with respect to the partition is the graph whose vertices $l_i{}^\triangle \in V_L{}^\triangle$ and $r_j{}^\triangle \in V_R{}^\triangle$ correspond to V_{Li}, V_{Rj}, respectively $(i, j = 1, \cdots, n)$, where an edge $(l_i{}^\triangle, r_j{}^\triangle)$ exists if and only if $(V_{Li} \times V_{Rj}) \cap E$ is not empty. Let E_1 and E_2 be two disjoint subsets of E of a graph $G\{E, V\}$, a path P (a circuit C) is said to be an alternating path (circuit) with respect to $[E_1, E_2]$, if, in its edge sequence, the edges belonging to E_1 and those belonging to E_2 appear alternately. Especially if $\rho(P(v_i, v_j))$ (the length of P) is odd and both v_i and v_j are the end points of the edges of E_1 (or E_2) P is called an alternating path with respect to $[E_1{}^*, E_2]$ (or $[E_1, E_2{}^*]$). An algorithm [6] is known to obtain a complete matching on a bipartite graph by means of alternating paths. It is obvious by the definition of a complete matching that each term of the determinant of a matrix corresponds to a complete matching on the corresponding bipartite graph. The following two theorems are fundamental in our discussion.

Theorem 2–1. If G is a bipartite graph with vertex sets V_L and V_R such that $|V_L|=|V_R|=n$ and if G contains a circuit C of length $\rho(C)=2n$, then G is an irreducible bipartite graph [5].

Theorem 2–2. Let $G\{E, V_L \cup V_R\}$ be a bipartite graph. Suppose there is a partition $V_L=V_{L1}\cup V_{L2}\cup\cdots\cup V_{Lk}$, $V_R=V_{R1}\cup V_{R2}\cup\cdots\cup V_{Rk}$ of V_L and V_R into disjoint subsets such that each of the subgraphs which is composed of the edges $E\cap(V_{Li}\times V_{Ri})$ $(i=1, 2, \cdots, k)$ is irreducible. And $G^{\triangle}$ be the induced bipartite graph by the partition. Then, G is irreducible if and only if $G^{\triangle}$ is irreducible [5].

The property that there exist at least two disjoint complete matchings in an irreducible bipartite graph, is frequently used in the following discussion. In the following sections, a bipartite graph means an irreducible bipartite graph.

3. *Matching Transformation*

Let M_1 and M_2 be two complete matchings of a bipartite graph G. By definition, the degree of every vertex in the subgraph M_1 is 1 and so is M_2. Therefore, the degree of every vertex in the subgraph which is obtained by the union of two subgraphs $M_1\{E_1, V_1\}$ and $M_2\{E_2, V_2\}$ is 1 or 2, so that the subgraph which is formed by $M_1 \oplus M_2$ consists of disjoint alternating circuits with respect to $[E_1, E_2]$.

Definition 3–1. Two complete matchings M_1 and M_2 on a bipartite graph G are said to be adjacent to each other, if $M_1 \oplus M_2$ forms a circuit C. The transformation of M_1 and M_2 (or, conversely, M_2 to M_1) is said to be the matching transformation with respect to C.

In other words, if $M_i\{E_i, V_i\}$ is a complete matching on a bipartite graph $G\{E, V\}$ and if a circuit $C\{E_c, V_c\}$ is an alternating circuit with respect to $[E_i, E-E_i]$, then the transformation of M_i to $M_j[E_j, V]=M_j[E_i \oplus E_c, V]$ is the matching transformation with respect to C.

Definition 3–2. A circuit of a bipartite graph $G\{E, V\}$ is said to be a matching circuit of G, if it is an alternating circuit with respect to $[E_1', E_2']$ and there exist complete matchings $M_1\{E_1, V\}$ and $M_2\{E_2, V\}$ where $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$.

Theorem 3–1. If two matching circuits C_1 and C_2 of a bipartite graph $G\{E, V\}$ form alternating circuits with respect to $[E_M, E-E_M]$ where E_M is the set of all matching edges of a complete matching M and $C_1 \cap C_2$ forms a path P, then P is an alternating path with respect to $[E_M^*, E-E_M]$. In general, if $C_1 \cap C_2$ consists of disjoint paths $P_1, P_2, \cdots, P_k$, then every $P_i (i=1, \cdots, k)$ forms an alternating circuit with respect to $[E_M^*, E-E_M]$.

Let $M_i\{E_i, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$. If the matching transformation with respect to a circuit C changes M_i to $M_j\{E_j, V\}$ and if there exist alternating paths with respect to $[E_i, E-E_i]$, $[E_i^*, E-E_i]$ and

$[E_i, (E-E_i)^*]$ in C then these paths are alternating paths with respect to $[E_j, E-E_j]$, $[E_j, (E-E_j)^*]$ and $[E_j^*, E-E_j]$, respectively. These properties follow from the relation $M_i \oplus M_j = C$, which in turn entail the following theorems.

Theorem 3-2. Let $M\{E_M, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$ and if matching circuits C_1 and C_2 form alternating circuits with respect to $[E_M, E-E_M]$ then $C_1 \oplus C_2$ forms a matching circuit or a set of matching circuits.

Theorem 3-3. Let $M_i\{E_i, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$ and there exist matching circuits C_1, C_2 and $C_k'(k=1, \cdots, p; C_1 \oplus C_2 = C_1' \cup C_2' \cup \cdots \cup C_p'$ where C_k''s are disjoint) where C_1 and C_2 are matching circuits with respect to $[E_i, E-E_i]$. If $M_j\{E_j, V\}$ is obtained from M_i by the matching transformation with respect to C_1, then C_1 and C_k' $(k=1, \cdots, p)$ form alternating circuits with respect to $[E_j, E-E_j]$.

Theorem 3-4. Let $M_i\{E_i, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$, and C_1, C_2, and C_k' $(k=1, \cdots, p;$ $C_1 \oplus C_2 = C_1' \cup C_2' \cup \cdots \cup C_p'$ where C_k''s are disjoint) be matching circuits and C_1 and C_2 are alternating circuits with respect to $[E_i, E-E_i]$. The complete matching which is obtained by the sequence of the matching transformations with respect to C_1 and C_k' $(k=1, \cdots, p)$ is the same as that which is obtained by the matching transformation with respect to C_2.

4. *Generation of Complete Matchings*

Let us present a method of generating all complete matchings on a bipartite graph. The method is useful also for calculating the determinant of a matrix and for solving a signal-flow graph. By virtue of the preceeding section, complete matchings are related to one another by matching transformations. Now, we will show that the set of all complete matchings can be obtained by successive matching transformations with respect to a prescribed set of circuits.

Definition 4-1. A set of circuits is said to be basic circuits for a complete matching $M\{E_M, V\}$ on a bipartite graph $G\{E, V\}$ if the set contains all the alternating circuits with respect to $[E_M, E-E_M]$ and them only.

Theorem 4-1. If M is a complete matching on a bipartite graph G then any complete matching except M itself can be obtained by a sequence of matching transformations with respect to circuits belonging to the set of basic circuits for M.

By the theorem 4-1, the set of all complete matchings of a bipartite graph can be generated by the successive matching transformations with respect to all the sets of disjoint basic circuits. The basic circuits may be obtained, e.g., by the methods presented in references [7] and [8]. A concrete procedure of generating all complete matchings using the pushdown memory is as following.

Let the set of basic circuits be $C_1, C_2, \cdots, C_{k-1}, C_k$. We will denote the i-th set of disjoint basic circuits appearing in the procedure by $N_i=\{C_{i1}, C_{i2}, \cdots, C_{in}\}$ where $i_1<i_2<\cdots<i_n$. First, determine an arbitrary complete matching M_0 on the bipartite graph under consideration and then enumerate the set of basic circuits for M_0. Then, store M_0 on the top of the pushdown memory and set $i=1$, $n=1$, $C_{i1}=C_1$ and $N_1=\{C_1\}$.

(I) Put $\alpha=C_{in}$ and transform the matching stored on the top of the pushdown memory with respect to α to obtain new matching M_i. If $N_i=\{C_k\}$, the process terminates. Otherwise, store M_i to the pushdown memory and go to (IV) or (II) according as $\alpha=C_k$ or $\alpha\neq C_k$.

(II) Replace α by the basic circuit next in the sequence and put tentatively $N_{i+1}=N_i\cup\{\alpha\}$. If the circuits in N_{i+1} are disjoint return to (I), otherwise go to (III).

(III) If $\alpha\neq C_k$ then go to (II), and if $\alpha=C_k$ then replace N_i by $N_i-\{C_{in}\}$ and α by new C_{in}, and then pop up the entry on the top of the pushdown memory, and go to (II).

(IV) Replace N_i by $N_i-\{C_{in-1}, C_{in}\}$, set $\alpha=C_{in-1}$, pop up the pushdown memory and go to (II).

It is obvious that all the complete matchings are generated in (I) of the above procedure. The above procedure may be applied to the calculation of determinants, where each complete matching corresponds to a term of the determinant. The sign of a term can successively be decided by starting from M_0 with the positive sign and changing the sign in (I) when $(1/2)\cdot\rho(\alpha)$ is even.

5. *Properties of a Circuit in a Bipartite Graph*

We will investigate properties of an alternating circuit with respect to $[E_M, E-E_M]$ in a bipartite graph $G\{E, V\}$ having a complete matching $M\{E_M, V\}$.

Theorem 5-1. Let $M\{E_M, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$. Then any matching circuit is expressible as a ring sum of alternating circuits with respect to $[E_M, E-E_M]$.

Theorem 5-1 can be extended to an arbitrary circuit of a bipartite graph.

Lemma 5-1. Let M_1 and M_2 be arbitrary complete matchings on a bipartite graph $G\{E, V_L\cup V_R\}$. If $M_1\cup M_2$ is expressible as a sum $C_1\cup C_2\cup\cdots\cup C_n$ then the induced graph is obtained with respect to the partition $V_L=V_{L1}\cup V_{L2}\cup\cdots\cup V_{Ln}$, $V_R=V_{R1}\cup\cdots V_{R2}\cup\cdots\cup V_{Rn}$ where $V_{Li}\cup V_{Ri}=C_i$ $(i=1, \cdots, n)$. Taking two complete matchings in $G^{\vartriangle}$ and continuing similar procedure, we will finally have an induced graph with one and only one edge.

Theorem 5-2. If a bipartite graph $G\{E, V_L\cup V_R\}$ $(|V_L|=|V_R|)$ is irreducible then between arbitrary two verties $l_i, l_j\in V_L$ (or $r_i, r_j\in V_R$) there exist at least two alternating paths with respect to $[E_M, E-E_M]$, where E_M is a set of matching edges.

Theorem 5-3. Let $M\{E_M, V\}$ be a complete matching on a bipartite graph $G\{E, V_L \cup V_R\}$. For alternating paths P_1 and P_2 between $l_i \in V_L, r_j \in V_R$ with respect to $[E_M^*, (E-E_M)]$ and $[E_M, (E-E_M)^*]$, the ring sum $P_1 \oplus P_2$ is expressible as the ring sum of a set of alternating circuits with respect to $[E_M, E-E_M]$.

Theorem 5-4. Let $M\{E_M, V\}$ be a complete matching on a bipartite graph $G\{E, V\}$. Then any circuit is expressible as the ring sum of a set of alternating circuits with respect to $[E_M, E-E_M]$.

Theorem 5-4 is equivalent to a theorem in [9] for a directed graph that any circuit in a strongly connected directed graph is expressible as a ring sum of directed circuits.

6. *Conclusion*

In this paper, we introduced the concept of matching transformation on a bipartite graph and, on the basis of it, investigated the properties of an irreducible bipartite graph. These properties are useful for calculating determinants. The result are applicable to a reducible bipartite graph by neglecting inadmissible edges [4], [5].

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