An Algorithm of an Acceleration Process Covering the Aitken's δ^2-process

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Abstract

An algorithm is proposed in which a coefficient ω_n of an acceleration process,

$$x_{i+2}^{(n)} = x_{i+2} + \omega_n(x_{i+2} - x_i)$$

is adjusted automatically so as to yield a rapid convergence. This acceleration process covers the Aitken's δ^2-process in $n \to \infty$. The employment of this algorithm yields a more rapid convergence than that of only the Aitken's δ^2-process.

1. Introduction

Aitken's δ^2-process is well known as an acceleration process for the sequence of linear convergence. This process is successful only after fairly converged. Thus, the careless employment of this process fails. To remedy the defects of this process, an acceleration process was proposed in the reference [4], and a proof for the derivation of this process was shown. Here a numerical analysis of this process is shown and an algorithm for the use of this process with adjusting automatically ω_n, is given.

As a typical example for the algorithm, we consider the power method for the absolutely largest eigenvalue of matrices.

For the sequence of number $\{x_i\}, i = 1,2,\ldots,$ and the appropriate even number n, we define the acceleration process (see[4]),

$$x_{i+2}^{(n)} = x_{i+2} + \omega_n(x_{i+2} - x_i), \tag{1.1}$$

where

$$\omega_n = \sum_{k=1}^{n/2} r_i^{2k}, \tag{1.2}$$

$$r_i = (x_{i+2} - x_{i+1})/(x_{i+1} - x_i). \tag{1.3}$$

The acceleration process eq.(1.1) may be repeatedly used with respect to the i for the sequence $\{x_i\}$ to be accelerated. As will be understood later.

The eq.(1.1) is defined only if the sequence $\{x_i\}$ is satisfied by

$$x_i = x + c_1t_1^i + c_2t_2^i + O(t_3^i), \tag{1.4}$$

where

$$\lim_{i\to\infty} x_i = x, \tag{1.5}$$

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and, c_1 and c_2 are constants,

and

$$1 > |t_1| > |t_2| \geqq |t_3|. \tag{1.6}$$

2. Estimation of the convergence rate

Putting

$$t_2 = \alpha t_1 \tag{2.1}$$

in eq.(1.4), we obtain the formula,

$$x_i = x + (c_1 + c_2\alpha^i)t_1^i + O(t_3^i). \tag{2.2}$$

To simplify the estimation, we neglect the quantity $O(t_3^i)$ in eq.(2.2).

Here putting

$$e_i = x_i - x, \tag{2.3}$$

$$e_{i+2}^{(n)} = x_{i+2}^{(n)} - x, \tag{2.4}$$

and using eq.(2.2) and eq.(1.1), we obtain the formulas:

$$e_i = (c_1 + c_2\alpha^i)t_1^i, \tag{2.5}$$

$$e_{i+2}^{(n)} = e_{i+2} + \sum_{k=1}^{n/2} r_i^{2k}(e_{i+2} - e_i). \tag{2.6}$$

Repeated use of eq.(2.5) yields the formulas:

$$e_{i+1} = ((c_1 + c_2\alpha^{i+1})/(c_1 + c_2\alpha^i))t_1 e_i, \tag{2.7}$$

$$e_{i+2} = ((c_1 + c_2\alpha^{i+2})/(c_1 + c_2\alpha^i))t_1^2 e_i. \tag{2.8}$$

Subtracting eq.(2.5) from eq.(2.7) and eq.(2.8), we obtain the formulas respectively:

$$e_{i+1} - e_i = ((c_1(t_1 - 1) + c_2\alpha^i(\alpha t_1 - 1))/(c_1 + c_2\alpha^i))e_i$$
$$\simeq (t_1 - 1)(1 - \frac{c_2\alpha^i(1 - \alpha)}{c_1(t_1 - 1)} t_1)e_i, \tag{2.9}$$

$$e_{i+2} - e_{i+1} = ((c_1(t_1 - 1) + c_2\alpha^{i+1}(\alpha t_1 - 1))/(c_1 + c_2\alpha^i))t_1 e_i$$
$$\simeq t_1(t_1 - 1)(1 - \frac{c_2\alpha^i(1 - \alpha)((1 + \alpha)t_1 - 1)}{c_1(t_1 - 1)})e_i. \tag{2.10}$$

Substituting eq.(2.9) and eq.(2.10) into eq.(2.6), we obtain the formulas:

$$\frac{e_{i+2}^{(n)}}{e_i} \simeq \frac{c_2}{c_1}\alpha^i(1 - \alpha)^2 t_1^2 (\frac{1 - (n+2)t_1^{n+1} + (n+1)t_1^{n+2}}{(1 - t_1)^2} - \frac{n + 2}{1 - \alpha} t_1^n) + t_1^{n+2}$$

$$\simeq \frac{c_2}{c_1}\alpha^i(1-\alpha)^2 t_1^2\left(\frac{1-(n+2)t_1^{n+1}+(n+1)t_1^{n+2}}{(1-t_1)^2} - \frac{n+2}{1-\alpha}t_1^n\right) + t_1^{n+2}. \quad (2.11)$$

Thus we obtain the following formula for the Aitken's δ^2-process as $n \to \infty$ in eq.(2.11):

$$e_{i+2}^{(\infty)}/e_i \simeq (c_2/c_1)\alpha^i(1-\alpha)^2 t_1^2/(1-t_1)^2. \quad (2.12)$$

Here we obtain the following formula as $i \to \infty$ in eq.(2.11):

$$\lim_{i\to\infty} e_{i+2}^{(n)}/e_i = t_1^{n+2}. \quad (2.13)$$

On the other hand, we obtain the following formula for the Aitken's δ^2-process from eq.(2.12):

$$\lim_{i\to\infty} e_{i+2}^{(\infty)}/e_i = 0. \quad (2.14)$$

The right hand sides of eq.(2.11) and eq.(2.12) may be considered as the convergence rate of the sequence $\{x_{i+2}^{(n)}\}$ to $\{x_i\}$, obtained when eq.(1.1) is employed one time.

Hence, the right hand sides of eq.(2.13) and eq.(2.14) indicate the highest convergence rate.

Roughly speaking, the magnitude of the second term of the right hand side of eq.(2.11) is a increasing function with the value n.

Here, denoting by the $[e_{i+2}^{(n)}/e_i]$ the first term of the right hand side of eq.(2.11), the $[e_{i+2}^{(n)}/e_i]$ decreases with the i since $|\alpha| < 1$, and the ratio $[e_{i+2}^{(n+2)}/e_i]/[e_{i+2}^{(n)}/e_i]$ is independent of the i, and is constant with some fixed n, α, and t_1.

It follows that, when we denote by the A(n) the acceleration process of eq.(1.1) the employment of the acceleration process in the order as $i \to \infty$:

$$A(2) \to A(4) \to \ldots \to A(\infty), \quad (2.15)$$

yields a rapid convergence.

3. Algorithm

We construct an algorithm for the acceleration process, eq.(1.1), based on the facts of the preceding section.

Suppose that the sequence $\{x_i\}$, $i = 0,1,\ldots$, is generated by an iterative function,

$$x_{i+1} = F(x_i), \quad (3.1)$$

which is equivalent to eq.(1.4).

Given three estimates x_1, x_2, and x_3 of a solution x of eq.(3.1).

(1) Set N = 0.

(2) Compute r from

$$r = (x_3 - x_2)/(x_2 - x_1). \quad (3.2)$$

(3) If $N \geqq 6$ (for instance), go to (5); otherwise, go to (4).

(4) Compute x_3' and x_3'' from

$$\omega_{2N} = \sum_{k=1}^{N} r^{2k}, \text{ with } \omega_0 = 0 \text{ for } N = 0, \quad (3.3)$$

$$\omega_{2N+2} = (1.0 + \omega_{2N})r^2, \tag{3.4}$$

$$x_3' = x_3 + \omega_{2N}(x_3 - x_1), \tag{3.5}$$

$$x_3'' = x_3 + \omega_{2N+2}(x_3 - x_1), \tag{3.6}$$

and go to (6).

(5) Compute x_3' from

$$\omega_\infty = r^2/(1 - r^2), \tag{3.7}$$

$$x_3' = x_3 + \omega_\infty(x_3 - x_1). \tag{3.8}$$

(6) Compute x_1, x_2, and x_3 from eq.(3.1) with starting value $x_0 = x_3'$.

(7) Check the convergences. For instance,

if $|x_3 - x_2| < \varepsilon$(tolerance), stop; otherwise, go to (8).

(8) If $N \geqq 6$, go to (2); otherwise, go to (9).

(9) If $|x_3'' - x_3| < |x_3' - x_3|$, set $N = N + 1$, and go to (2); otherwise go to (2) with keeping the value N intact.

The above procedures are repeated until the desired accuracy is attained.

On starting the calculations of eq.(3.1), the above algorithm may be applicable since $\omega_0 = 0$, for $N = 0$ in the step (1).

4. Numerical examples

As a typical example, we consider the power method for the absolutely largest eigenvalue of matrices.

The calculations were performed by the use of the FACOM 230-25 computer of Toyota Technical College with the significant figures of single precision 7.8 .

Here we took $x_0^T = (1,1,\ldots,1)$ as an initial vector for two examples.

Example 1

The matrix

$$A_1 = \begin{pmatrix} 3.8 & 0.8 & -0.4 & 0.0 \\ 0.8 & 2.2 & 0.0 & 0.4 \\ -0.4 & 0.0 & 2.8 & -0.8 \\ 0.0 & 0.4 & -0.8 & 1.6 \end{pmatrix}$$

has an absolutely largest eigenvalue, $\lambda_1 \simeq 4.27916$.

Result: Accelerating by the use of the above algorithm, fifteen iterations were needed to obtain x correct to 6 decimal, while accelerating by the use of the Aitken's δ^2-process, thirty-three iterations were needed.

Example 2

The matrix

$$A_2 = \begin{pmatrix} 10 & 1 & 2 & 3 & 4 \\ 1 & 9 & -1 & 2 & -3 \\ 2 & -1 & 7 & 3 & -5 \\ 3 & 2 & 3 & 12 & -1 \\ 4 & -3 & -5 & -1 & 15 \end{pmatrix}$$

has an absolutely largest eigenvalue, $\lambda_1 \simeq 19.1754$.

Result: Accelerating by the use of the above algorithm, eighteen iterations were needed, while accelerating by the use of the Aitken's δ^2-process, twenty-four iterations were needed. In this case, the employment of the Aitken's δ^2-process gives rise to $|r| > 1$ in eq.(3.2); in which case, the Aitken's δ^2-process was not employed.

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