Algorithms Combining Polyhedra and Separating Polyhedron for CAD

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Abstract

Algorithms to separate a polyhedron into two parts by a plane and to construct a polyhedron by combining two polyhedra are advocated by utilizing some topological properties of objects. In addition, the figures drawn by computer for some simple examples are shown.

1. Introduction

We are often requested to construct three dimensional objects of desirable forms in computer-aided design (CAD) processes. Therefore, it is necessary to develop a method of preparing the data for these objects in conformity with the data structure of them.

B.G.Baumgart proposed a system "GEOMED"[1)] for three dimensional geometric modeling. With this system, various polyhedra may be constructed by combining two dimensional figures and new polyhedron can be made by combining (adding, subtracting or intersecting) them or by cutting a polyhedron with a plane. Especially, a cube, a cylinder or a sphere can be constructed with only one command. Here, each cylinder or sphere is approximated with a polyhedron.

I.C.Braid and C.A.Lang developed a CAD system[2)] for generating three dimensional mechanical components. In this system, some primitive components (for example, cube, cylinder and tetrahedron) can be generated with only a few commands respectively and the other components are constructed by combining them.

M.Hosaka made a geometric modeling system[3)] similar to the two systems mentioned above.

The author has reported a method for generating some symmetrical objects[4)] and for constructing the trajectory which is traced as a polyhedron moves[5)].

In this paper, a method for constructing polyhedron by combining polyhedra or cutting a polyhedron with a plane is discussed. The programs eliminating hidden-lines[6)] and determining interference between various polyhedra[7)], which have been developed by the author, can be applied to this new polyhedron. And some figures drawn by computer are shown.

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2. Definition of Symbols

Consider a polyhedron which is regarded as a set of given points in a three dimensional space so that the elements of this set are the points situated inside and on its boundary faces. We denote this polyhedron by P. Let the number of faces, edge lines and vertices of the polyhedron P be n(S), n(L) and n(V) respectively. We denote the i-th face, the j-th edge line and the k-th vertex of the polyhedron P by S_i (i=1,2,...,n(S)), L_j (j=1,2,...,n(L)) and V_k (k=1,2,...,n(V)) respectively.

Consider the N_i edge lines (Note that the number of these edge lines is equal to that of vertices) of the i-th face S_i. We designate these edge lines and vertices by $L_j(S_i)$ and $V_k(S_i)$ (j,k=1,2,...,N_i) respectively.

Lastly, we define a semi-infinite ray emanating from a point p in arbitrary direction by $R_a(p)$.

3. Data Structure of Polyhedron

The data structure for polyhedron which is utilized in our CAD system is described in this section.

Consider a tetrahedron as shown in Fig.1). It has 4 faces, 6 edge lines and 4 vertices. Each vertex is represented as a set of three real numbers (x,y,z) which represent Cartesian coordinates of a point in three dimensional space.

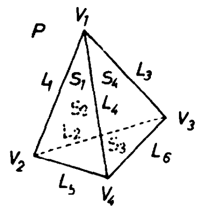


Fig.1 An example of a polyhedron

The vertices are held in a simple list as shown in Table 1-a) and can thus be referred to by the numbers v_i ($1 \leqq v_i \leqq 4$) which indicates their position in the list. We call this list V·table.

The edge lines are likewise held in a simple list as shown in Table 1-b). Each edge line is given by a pair of vertex numbers (v_i, v_j) and is referred to by the line number ℓ_i ($1 \leqq \ell_i \leqq 6$) similar to V·table. We call this list L·table.

We prepare two different lists for faces as shown in Table 1-c) and d). In N·table, the numbers of vertices (or edge lines) of the faces are held. Each face S_i ($1 \leqq i \leqq 4$) is referred to by a face number i. In S.table, the line numbers of the edge lines composing

	x	y	z
1			
2			
3			

a) V·table

1	1	2
2	2	3
3	1	3
4	1	4

b) L·table

1	3
2	3
3	3
4	3

c) N·table

1	1
2	4
3	5
4	1
5	2

d) S·table

Table 1 Data structure of a polyhedron

each face are stored. The edge lines of the i-th face S_i are referred to by the line numbers written in the addresses $M+1, M+2, \ldots, M+K_i$ in S·table. Here, the number K_i and M represent the number of edge lines of the face S_i and $\sum_{j=1}^{i-1} K_j$ respectively.

4. Separating Polyhedron by a Plane

Two polyhedra are obtained by separating from a polyhedron with a plane (hereafter, we call this plane "C-plane"). The method of re-organizing the data structure for these new polyhedra is discussed in this section.

Let R and R^o denote the whole space and the C-plane respectively. Consider that the C-plane separates the whole space R into three subspaces R^p, R^m and R^o (C-plane itself) so that the conditions $f(x,y,z) > 0$, $f(x,y,z) < 0$ and $f(x,y,z) = 0$ hold for all points in R^p, R^m and R^o spaces respectively. Here, the function $f(x,y,z)$ is defined by the expression

$$f(x,y,z) = ax+by+cz+d, \tag{1}$$

the parameters a, b, c and d are some constants and x,y and z represent the coordinates of a point.

Suppose that a polyhedron P is separated into two polyhedra P^p and P^m by the C-plane. We consider whether the i-th edge line L_i of the polyhedron P belongs to P^p or P^m. There are the following three cases:

a) The edge line L_i and the plane R^o have no intersection:

If the condition $f(x_p,y_p,z_p) > 0$ hold for an arbitrarily chosen point $p(x_p,y_p,z_p)$ on the edge line L_i, then the edge line L_i becomes an edge line of the polyhedron P^p. Inversely, if $f(x_p,y_p,z_p) < 0$, L_i belongs to P^m.

b) The edge line L_i and the plane R^o have an intersection:

In this case, the edge line L_i can be separated into two segments L_i^p and L_i^m which are in R^p and R^m spaces respectively. The discussions in case a) are valid for each of the segments L_i^p and L_i^m.

c) The edge line L_i is just on the plane R^o:

Suppose that the line L_i is an edge line of a face S_j which is not on the C-plane. Choose a point p arbitrarily on the edge line L_i and a semi-infinite ray $R_a(p)$ being on the face S_j and not running through any vertex $V_k(S_j)$ $(k=1, 2, \ldots, N_j)$. If the number of intersections between the ray $R_a(p)$ and the lines $L_k(S_j)$ $(k=1,2,\ldots,N_j)$ in R^p space is odd, then the edge line L_i belongs to the polyhedron P^p (Fig.2). If the number of them in R^m space is odd,

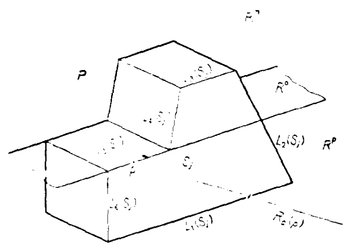


Fig.2 The case where the plane R^o contains a face

then L_i belongs to P^m.

The lines generated by the intersections between the C-plane and the faces of P cleary become the edge lines of both polyhedra P^p and P^m.

Now we can construct each face of the polyhedra P^p and P^m.

5. Algorithm for Combining Polyhedra

In this section, we introduce an algorithm for constructing a union P of polyhedra P_1 and P_2. As the first step, we discuss the problem of determining what part of the j-th edge line $L_{i,j}$ of the polyhedron P_i becomes edge line of P.

We designate the number of intersections between the edge line $L_{i,j}$ and the faces $S_{i',k}$ $(k=1,2,...,n_i\ (S))$ by m. Here, the number i′ is defined by $i' = 3-i$ $(i=1,2)$. Now let's consider the following three cases.

a) The case where m=0:

The edge line $L_{i,j}$ becomes an edge line of the polyhedron P if a point p chosen arbitrarily on $L_{i,j}$ is included in $P_{i'}$. Otherwise, this line is eliminated.

b) The case where $m \neq 0$ and m is finite:

We consider that the edge line $L_{i,j}$ consists of m+1 segments $L_{i,j}^{\ell}$ $(\ell=1,2,...,m+1)$ as shown in Fig.3. The discussion in case a) is valid for each segment.

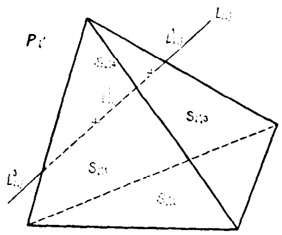


Fig.3 The case where an edge line has intersections with faces

c) The case where m is infinite:

We separate $N(L_{i,j})$ segments $L_{i,j}^{\ell}$ $(\ell=1,2,...,N(L_{i,j}))$ and $N'(L_{i,j})$ segments $L_{i,j}'^{m}$ $(m=1,2,...,N'(L_{i,j}))$ from the edge line $L_{i,j}$ so that each segment $L_{i,j}^{\ell}$ is just on a face of the polyhedron $P_{i'}$ and each segment $L_{i,j}'^{m}$ comes under the case a) or b) as shown in Fig.4.

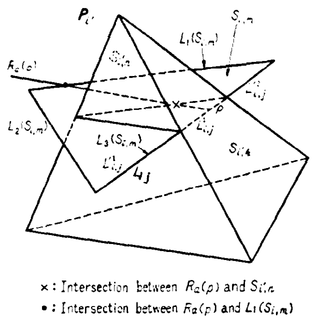


Fig.4 The case where a part of an edge line is on a face

Suppose that a segment $L_{i,j}^{\ell}$ is on a face $S_{i',k}$. We have here the following two cases.

i) The case where the segment does not coincide with any edge line of the face $S_{i',k}$:

Suppose that $L_{i,j}$ is an edge line of a face $S_{i,m}$. Consider an arbitrarily chosen point p on the segment $L_{i,j}^{\ell}$ and a semi-infinite ray

$R_a(p)$ being on the face $S_{i,m}$, neither intersecting with any edge line of P_i, nor running through any vertex of $S_{i,m}$. We designate the number of the intersections between the ray $R_a(p)$ and the lines $L_n(S_{i,m})$ $(n=1,2,\ldots,N_{i,m})$ by m_1 and that between the ray $R_a(p)$ and the faces $S_{i,n}$ $(n=1,2,\ldots,n_{i'}(S))$ by m_2.

If the number m_1+m_2 is odd, then the segment $L^{\ell}_{i,j}$ belongs to the polyhedron P. Otherwise, it is eliminated.

ii) The case where the segment coincides with an edge line of the face $S_{i',k}$:

The segment $L_{i,j}$ is shared by two polyhedra P_i and $P_{i'}$. We designate such faces that contains this segment as an edge line by $S_m(L^{\ell}_{i,j})$ $(m=1,2,\ldots,N(L^{\ell}_{i,j}))$, where $N(L^{\ell}_{i,j})$ is the number of these faces.

Consider a face $S_m(L^{\ell}_{i,j})$ and separate the whole space R into three subspaces : the plane R^o_m containing the face $S_m(L^{\ell}_{i,j})$, subspace being one side of the plane R^o_m and that being on the opposite side of the plane R^o_m .

Consider another face $S_n(L^{\ell}_{i,j})$. Let's take a point p on the segment $L^{\ell}_{i,j}$ and a semi-infinite ray $R_a(p)$ on the plane R^o_m satisfying the conditions : it doesn't run through any vertex of $S_n(L^{\ell}_{i,j})$ and the number of the intersections between this ray and the edge lines of $S_n(L^{\ell}_{i,j})$ is odd. We designate this semi-infinite ray by $R_s(p)$. One of the three subspaces determined by the face $S_m(L^{\ell}_{i,j})$ includes this ray $R_s(p)$. We denote it by $R_m(S_n)$.

We denote the common set of $R_m(S_{m+1})$ and $R_{m+1}(S_m)$, which are determined by the faces $S_m(L^{\ell}_{i,j})$ and $S_{m+1}(L^{\ell}_{i,j})$, by R_m (here, we replace $N(L^{\ell}_{i,j})+1$ by 1). We sort the faces $S_k(L^{\ell}_{i,j})$ $(k=1,2,\ldots,N(L^{\ell}_{i,j}))$ so that these sets R_m $(m=1,2,\ldots,N(L^{\ell}_{i,j}))$ have no intersection with each other.

Now we choose a point p on the segment $L^{\ell}_{i,j}$ arbitrarily. And consider a semi-infinite ray $R_a(p)$, which has no intersection with any edge line $L_{i,k}$ $(i=1,2\ ;\ k=1,2,\ldots,n_i(L))$, in R_m space. We designate the number of the intersections between $R_a(p)$ and the faces $S_{i,k}$ $(i=1,2\ ;\ k=1,2,\ldots,n_i(S))$ by M. If M is odd for all m $(m=1,2,\ldots,N(L^{\ell}_{i,j}))$, then the segment $L^{\ell}_{i,j}$ is eliminated. Otherwise, it becomes to an edge line of the polyhedron P. In the above discussions, we need not consider the case where the subspace $R_m(S_n)$ coincides with the plane R^o_m .

The lines generated by the intersections between the faces $S_{i,j}$ and $S_{i',k}$ clearly belong to P. So we can construct each face of polyhedron P with :

1) the lines generated by the intersections between the faces $S_{i,j}$ $(j=1,2,\ldots,n_i(S))$ and $S_{i',k}$ $(k=1,2,\ldots,n_{i'}(S))$, and

2) the lines which are determined as the edge lines of P in the above discussions.

6. Illustrative Figures Drawn by Computer

Some examples of figures drawn by computer are shown in Fig.5. Fig.5-a) shows original polyhedron, each of Fig. 5-b) and c) shows the polyhedron separated by a plane on which a face of original polyhedron is contained and Fig.5-d) shows the union of two polyhedra : one of them is that shown in Fig.5-b) and the other is that rotated around an axis and moved a little in parallel with it.

The algorithms are implemented by FORTRAN and it is run with the computer HITAC 8800/8700 in the Computer Center of University of Tokyo. The processing time for the example of Fig.5-b) or c) were 2~3 seconds, while that of Fig.5-d) were about 15 seconds including the time for eliminating hidden-lines in all cases.

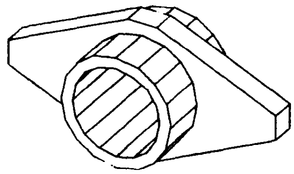
7. Conclusions

In this paper, the algorithms for combining polyhedra and for separating a polyhedron by a plane are advocated as a step to contrive a system with which designer can easily construct the various kinds of objects of his desired forms.

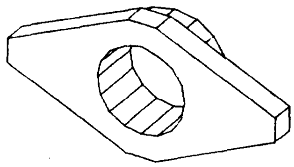
The main advantages of these algorithms are as follows :

1) Our algorithms of combining polyhedra and separating polyhedron are applicable even to concave polyhedra.

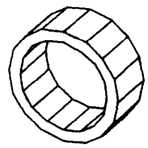
2) The data structures utilized in our system are not so complicated as those now widely used.



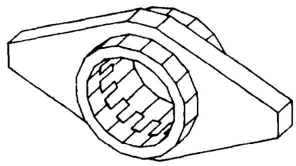
a) Original polyhedron



b) A polyhedron separated from original one



c) A polyhedron separated from original one



d) A union of polyhedra shown in b) and c) (one of them is rotated and displaced)

Fig.5 Separating a polyhedron and combining polyhedra

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