A Synthesis Theory of Curves and Surfaces for CAD

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Abstract

Based on a model of manual curve construction process, curve and surface synthesis methods, which are useful in computer aided geometrical design, are developed. Besides computational methods, manual graphical procedures for shape synthesis are presented. And concise and useful expressions for Bézier's curve and surface can also be introduced from our basic equation.

1. Introduction

This paper deals with methods of synthesizing curve and surface which can be used in computer aided geometric design. In design process, shape of a curve or a surface has to be controlled locally as well as globally to meet designers' requirements which are not all mathematically expressed. Therefore, interactive and graphical construction procedures have to be built in the method.

Bézier(1) used a polygon to control designer's intended global shape of a curve segment, whereas Riesenfeld(2) adopted a series of points for local control of connected curve segments. Both of their methods give simple means for constructing and controlling the shape of curve. However, their derivation processes are not straightforward and geometric characteristics of constructed curves are not all explicitly shown. Accordingly, we developed new methods(3) which are physically understandable and more flexible in use without losing the favourable features of their methods. Our methods are based on manual procedures of drawing, in which one draws a new curve over the old ones by controlling the movement of his pencile. We simply formulated the simulated procedures. With our methods, a curve with intended geometric features can be constructed numerically as well as graphically. And concise expressions for Bézier curve and surface are also derived from our theory. Useful relations and applications can easily be obtained with these expressions.

2. Basic equation, its solution and characteristics

A new curve segment is generated from moving average of the adjacent old curve

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segments. A curve segment $\boldsymbol{R}_{i,n}(t)$ is denoted in a parametric vector form, where i and n mean the i-th segment and the n-th generation and the parameter t varies between 0 and 1. $\boldsymbol{R}_{i,n}(t)$ is defined as follows.

$$\boldsymbol{R}_{i,n}(t)=\int_t^1 w(\tau)\boldsymbol{R}_{i,n-1}(\tau)d\tau+\int_0^t w(\tau)\boldsymbol{R}_{i+1,n-1}(\tau)d\tau \quad ,t\in[0,1]. \tag{2.1}$$

where w(t) is a weight and the following condition is assumed.

$$\int_0^1 w(\tau)d\tau=1 \ . \tag{2.2}$$

At the initial stage of curve generation, a segment $\boldsymbol{R}_{i,0}(t)$ is defined to be an isolated point $\boldsymbol{P}_i$.

$$\boldsymbol{R}_{i,0}(t)=\boldsymbol{P}_i f_{0,0}(t), \quad f_{0,0}(t)\equiv 1 \ . \tag{2.3}$$

Generally, $\boldsymbol{R}_{i,n}(t)$ is written as follows.

$$\boldsymbol{R}_{i,n}(t)=\sum_{j=0}^{n} \boldsymbol{P}_{i+j} f_{j,n}(t) \ . \tag{2.4}$$

Next conditions can be derived from eq.(2.1).

$$\left.\begin{aligned} f_{j,n}(t)&=\int_t^1 w(\tau)f_{j,n-1}(\tau)d\tau+\int_0^t w(\tau)f_{j-1,n-1}(\tau)d\tau \\ f_{0,n}(t)&=\int_t^1 w(\tau)f_{0,n-1}(\tau)d\tau \\ f_{n,n}(t)&=\int_0^t w(\tau)f_{n-1,n-1}(\tau)d\tau \end{aligned}\right\} \tag{2.5}$$

and

$$\sum_{j=0}^{n} f_{j,n}(t)=\int_0^1 w(\tau)\sum_{j=0}^{n-1} f_{j,n-1}(\tau)d\tau=1 \ . \tag{2.6}$$

Eq.(2.6) can be easily proved by mathematical induction and this result shows that eq.(2.4) is unaffected by coordinate transformation. It can be proved that the influence of a vertex on the shape is limited within n+1 successive segments and has one maximum. This means that local shape of the curve can be modified smoothly.

The weight w(t) in eq.(2.5) is used to alter the influence functions. Eq.(2.5) and eq.(2.6) hold even if w(t)'s in every generation stages are different. If all the w(t)'s are 1, the influence function is obtained in a closed form.

$$\begin{aligned} f_{i,n}(t)&=\frac{1}{n!}\sum_{j=0}^{n-i}(-1)^j\cdot {}_{n+1}C_j\cdot(n-i-j+t)^n \\ &=\frac{1}{n!}\sum_{j=0}^{i}(-1)^j\cdot {}_{n+1}C_j\cdot(i-j+1-t)^n \ . \end{aligned} \tag{2.7}$$

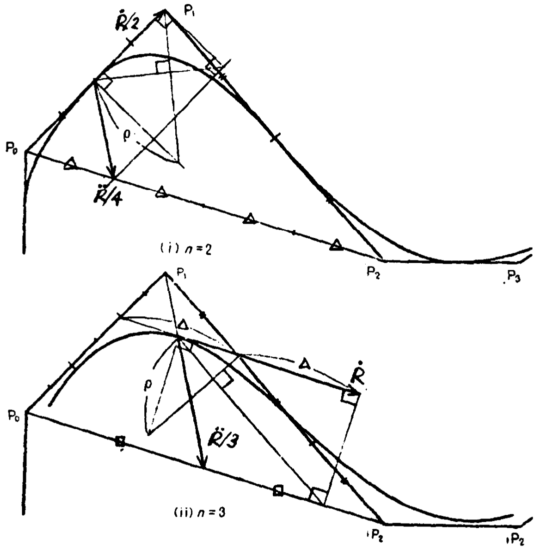

Fig. 1 Quadratic curve-(i) and cubic curve-(ii) defined by given data points $\{P_i\}$.

3. Practical method of shape synthesis

In the following discussion w(t)=1 is assumed for simplicity. As the next relations hold from eq.(2.1),

$$\dot{R}_{i,n}(t)=R_{i+1,n-1}(t)-R_{i,n-1}(t),\quad \ddot{R}_{i,n}(t)=R_{i+2,n-2}(t)-2R_{i+1,n-2}(t)+R_{i,n-2}(t) \tag{3.1}$$

Geometrical features of the curve such as position, tangent, radius of curvature at each connecting point are easily obtained graphically.

When n=3, the above relations are shown in Fig.1. When a sequence of points $\{\boldsymbol{R}_i\}$ on a curve is given and their successive distances are not extremely different, the corresponding vertices $\{\boldsymbol{P}_i\}$ can be approximately given by the following formula.

$$\boldsymbol{P}_i=2\sqrt{3}\sum_{j=1}^{4}\boldsymbol{Q}_j\alpha^j,\qquad \alpha=-2+\sqrt{3},$$

where $\boldsymbol{Q}_j$ is the middle point of $\boldsymbol{R}_{i-j}$ and $\boldsymbol{R}_{i+j}$. This approximation can be extended to surface construction such as given below.

$$\boldsymbol{P}_{i,j}=3\sum_{k=-4}^{4}\sum_{l=-4}^{4}\boldsymbol{S}_{i+k,j+l}\cdot\alpha^{|k|+|l|}, \tag{3.2}$$

where $\boldsymbol{P}_{i,j}$'s are vertices and $\boldsymbol{S}_{i,j}$'s are the corresponding points on the surface.

4. Relation to Bezier's curve

In this section we consider one curve segment. We modify eq.(3.1) slightly as shown below.

$$\dot{\boldsymbol{R}}_{0,n}(t)=n\{\boldsymbol{R}_{1,n-1}(t)-\boldsymbol{R}_{0,n-1}(t)\}. \tag{4.1}$$

We removed the continuity conditions of two curves $\boldsymbol{R}_{1,n-1}(t)$ and $\boldsymbol{R}_{0,n-1}(t)$. Now, let sequence of points $\{\boldsymbol{P}_i\}$ be given. Shift operator E for subscript of $\boldsymbol{P}_i$'s is introduced to simplify the notation. The algebraic rule of integer exponent can be applied on E. Then the solution of eq.(4.1) is

$$\boldsymbol{R}_{0,n}(t)=(1-t+tE)^n\boldsymbol{P}_0. \tag{4.2}$$

$$\boldsymbol{R}_{0,n}(t)=(1-t+tE)\boldsymbol{R}_{0,n-1}(t)=(1-t)\cdot\boldsymbol{R}_{0,n-1}(t)+t\boldsymbol{R}_{1,n-1}(t). \tag{4.3}$$

Bézier defined a curve as a vectorial sum of polygon sides $\{\boldsymbol{a}_i\}$ multiplied by the influence functions shown in eq.(4.6), which he gave without much explanation. His expression can be derived naturally from eq.(4.2) as follows. Since $\boldsymbol{P}_1-\boldsymbol{P}_0=(E-1)\boldsymbol{P}_0=\boldsymbol{a}_1$ by setting

$$x\equiv t(1-E),\quad \phi(x)\equiv\{1-(1-x)^n\}/x, \tag{4.4}$$

We obtain

$$\boldsymbol{R}_{0,n}(t)=\boldsymbol{P}_0+t\phi(x)\cdot\boldsymbol{a}_1. \tag{4.5}$$

As the function $\phi(x)$ is a polynomial of degree (n-1), it is expressed by Taylor expansion around x=t, as follows.

$$\boldsymbol{R}_{0,n}(t)=\boldsymbol{P}_0+\sum_{i=1}^{n}\frac{-(-t)^i}{(i-1)!}\cdot\phi^{(i-1)}(t)\cdot\boldsymbol{a}_i. \tag{4.6}$$

This coincides with Bézier's expression.

By the binomial expansion of eq.(4.2) with respect to 1-t and tE, the following expression is obtained.

$$\boldsymbol{R}_{0,n}(t)=\sum_{i=0}^{n} {}_nC_i \cdot t^i(1-t)^{n-i} \cdot \boldsymbol{P}_i \ . \qquad (4.7)$$

This is the expression by Forrest[4]. Eq.(4.2) or eq.(4.3) indicates directly that a point on the resultant curve is determined by the linear interpolation between the corresponding points of the previous two curves.

Other useful formulae can be also derived easily from eq.(4.2). By expanding with respect to t,

$$\boldsymbol{R}_{0,n}(t)=\sum_{i=0}^{n} {}_nC_i t^i(E-1)^i\boldsymbol{P}_0 = \sum_{i=0}^{n} {}_nC_i t^i(1-E^{-1})^i\boldsymbol{P}_i \ . \qquad (4.8)$$

where $(E-1)$ and $(1-E^{-1})$ are the forward and the backward difference operator, Δ and ∇, respectively. By differentiating eq.(4.2),

$$\boldsymbol{R}_{0,n}^{(i)}(t)=\frac{n!}{(n-i)!} \cdot (1-t+tE)^{n-i} \cdot (E-1)^i \cdot \boldsymbol{P}_0 \ . \qquad (4.9)$$

This shows that the curve $\boldsymbol{R}_{0,n}^{(i)}(t)$ has Bezier's expression. The values of $\boldsymbol{R}_{0,n}^{(i)}(t)$ at t=0 and 1 are expressed in terms of $\{\boldsymbol{P}_i\}$, vice versa.

$$\left.\begin{aligned}
&\boldsymbol{R}_{0,n}^{(i)}(0)=\frac{n!}{(n-i)!}(E-1)^i \cdot \boldsymbol{P}_0 \\
&\boldsymbol{R}_{0,n}^{(i)}(1)=\frac{n!}{(n-1)!}(1-E^{-1})^i\boldsymbol{P}_n \\
&\boldsymbol{P}_i=\{1+(E-1)\}^i\boldsymbol{P}_0=\frac{1}{n!}\sum_{j=0}^{i}(n-j)! \cdot {}_iC_j \cdot \boldsymbol{R}_{0,n}^{(j)}(0) \\
&\boldsymbol{P}_{n-i}=\{1-(1-E^{-1})\}^i\boldsymbol{P}_n=\frac{1}{n!}\sum_{j=0}^{i}(-1)^j \cdot (n-j)! \cdot {}_iC_j \cdot \boldsymbol{R}_{0,n}^{(j)}(1) \ .
\end{aligned}\right\} \qquad (4.10)$$

The relation between these $\{\boldsymbol{P}_i\}$ and $\{\boldsymbol{P}_i\}$ in eq.(2.4) (now written as $\{\boldsymbol{P}_i'\}$), which give the same curve, is written as follows.

$$P_i=\frac{1}{n!}\sum_{j=0}^{n} P_j' \sum_{k=0}^{j}(-1)^{j-k} {}_{n+1}C_{j-k}(k+1)^{n-i}k^i . \qquad (4.11)$$

A surface expression for rectangular meshes (similar to eq.(4.2)) is given as follows.

$$\boldsymbol{S}(u,v)=(1-u+uE_1)^m(1-v+vE_2)^n\boldsymbol{P}_{0,0} \ , \qquad (4.12)$$

where E_1, E_2 are shift operators for subscripts i, j.

A surface for triangular meshes is expressed as follows.

$$\boldsymbol{S}(u,v,w)=(u+vE_1+wE_2)^n\boldsymbol{P}_{0,0} \ , \qquad (4.13)$$

where u+v+w=1 and $u,v,w\in[0,1]$.

5. Connecting conditions and curvature

In this section, we treat the curvature and show a method of calculation of radius of curvature. Radius of curvature, which has the dimension of length, is the important quantity to characterize the shape of a curve. Curvature vector $\boldsymbol{n}/\rho$ of a parametric curve $\boldsymbol{R}(t)$ is given as follows.

$$\frac{\boldsymbol{n}}{\rho}=\frac{1}{\dot{\boldsymbol{R}}^2}\{\ddot{\boldsymbol{R}}-(\ddot{\boldsymbol{R}}\cdot\boldsymbol{t})\boldsymbol{t}\}=\frac{(\ddot{\boldsymbol{R}}\cdot\boldsymbol{n})\boldsymbol{n}}{\dot{\boldsymbol{R}}^2}, \qquad (5.1)$$

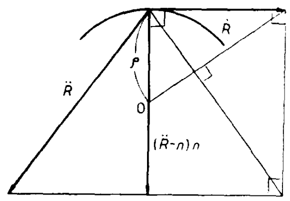

Fig. 2 Graphical relation between $\dot{R}$, $\ddot{R}$ and ρ.

where ρ, $\boldsymbol{t}$, $\boldsymbol{n}$ are the radius of curvature, the unit tangent vector, the unit normal vector, respectively. As the curvature is the normal component of $\ddot{\boldsymbol{R}}$ divided by $\dot{\boldsymbol{R}}^2$, that is, $|\dot{\boldsymbol{R}}|$ is a mean proportional, there is a simple geometric relation as shown in Fig.2. If the characteristic polygon $\{\boldsymbol{P}_i\}$ is known, $\dot{\boldsymbol{R}}$ and $\ddot{\boldsymbol{R}}$ at the ends of curve segment are obtained as shown in eq.(3.1) or eq.(4.10). Therefore, we can easily connect curves with continuity of curvature, though $\ddot{\boldsymbol{R}}$ has discontinuity at the connecting points. Even a number of curves of degree 2 can be connected with continuity of curvature. A curve of degree 3 can be determined completely with curvatures and tangents being given at the both ends. This is because there are two unknowns in Bézier's polygon: magnitudes of two sides, and there are two relations between curvature and unknown variables. Its solving method is not complicated.

We presented a new theory and practical methods of constructing curve and surface which are used in CAD. More thorough discussion about connection of surfaces will be given in another paper.

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