Short Note

Some Fifth Order Multipoint Iterative Formulae for Solving Equations

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In this paper, we show some fifth order multipoint iterative formulae which find new approximations to a zero of a function $f(x)$. These formulae require one evaluation of $f(x)$ and three of $f'(x)$ per iteration.

1. Introduction

In order to obtain new approximations to a zero of a function $f(x)$, Traub [1, pp. 197–204] showed fourth order multipoint iterative formulae costing one evaluation of $f(x)$ and three of $f'(x)$ per iteration; and Jarratt [2] showed fourth order multipoint iterative formulae costing one evaluation of $f(x)$ and two of $f'(x)$ per iteration. However, we cannot obtain fifth order formulae without increasing the number of derivative evaluations. In this paper, we show that for a class of iterative formulae of the type

$$x_{n+1}=\phi(x_n) \quad n=0, 1, 2, \ldots, \tag{1.1}$$

where

$$\phi(x)=x-a_1u(x)-a_2w_2(x)-a_3w_3(x)-\psi(x),$$

$$u(x)=\frac{f(x)}{f'(x)}, \qquad w_2(x)=\frac{f(x)}{f'[x-u(x)]},$$

$$w_3(x)=\frac{f(x)}{f'[x+\beta u(x)+\gamma w_2(x)]}$$

and

$$\psi(x)=\frac{f(x)}{b_1f'(x)+b_2f'[x-u(x)]},$$

fifth order formulae can be obtained by suitable choices of the parameters a's, b's, β and γ.

2. Derivation of Formulae

In order to obtain fifth order formulae of the type (1.1), we assume that $f(x)$ has a simple zero α and continuous fifth derivatives. Expanding $w_2(x)$, $w_3(x)$ and $\psi(x)$ into powers of $u(x)$, we obtain

$$\begin{aligned} w_2(x)=&\,u+2A_2u^2+4A_2^2u^3-3A_3u^3+8A_2^3u^4-12A_2A_3u^4\\ &+4A_4u^4+(16A_2^4-36A_2^2A_3+16A_2A_4+9A_3^2\\ &-5A_5)u^5+O(u^6), \end{aligned} \tag{2.1}$$

$$\begin{aligned} w_3(x)=&\,u-2\delta A_2u^2+4(-\gamma+\delta^2)A_2^2u^3-3\delta^2A_3u^3\\ &-8(\gamma+\delta^3-2\delta\gamma)A_2^3u^4+6(\gamma+2\delta^3-2\delta\gamma)A_2A_3u^4\\ &-4\delta^3A_4u^4+16(-\gamma+\gamma^2+\delta^4-3\delta^2\gamma+2\delta\gamma)A_2^4u^5\\ &-12(-2\gamma+\gamma^2+3\delta^4-6\delta^2\gamma+4\delta\gamma)A_2^2A_3u^5\\ &+8(-\gamma+2\delta^4-3\delta^2\gamma)A_2A_4u^5+9(\delta^4+2\delta\gamma)A_3^2u^5\\ &-5\delta^4A_5u^5+O(u^6) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \psi(x)=&\frac{1}{b_1+b_2}(u+2\theta A_2u^2+4\theta^2A_2^2u^3-3\theta A_3u^3+8\theta^3A_2^3u^4\\ &-12\theta^2A_2A_3u^4+4\theta A_4u^4+16\theta^4A_2^4u^5\\ &-36\theta^3A_2^2A_3u^5+16\theta^2A_2A_4u^5+9\theta^2A_3^2u^5\\ &-5\theta A_5u^5)+O(u^6), \end{aligned} \tag{2.3}$$

where

$$u=u(x), \quad A_j=\frac{f^{(j)}(x)}{j!f'(x)} \quad j=2(1)5, \quad \delta=\beta+\gamma$$

and

$$\theta=\frac{b_2}{b_1+b_2}.$$

Since the basic sequence [1, pp. 78–88]

$$\begin{aligned} E_6(x)=&\,x-u-A_2u^2-2A_2^2u^3+A_3u^3-5A_2^3u^4\\ &+5A_2A_3u^4-A_4u^4-(14A_2^4-21A_2^2A_3+6A_2A_4\\ &+3A_3^2-A_5)u^5, \end{aligned}$$

it follows from (1.1), (2.1), (2.2) and (2.3) that we obtain

$$\begin{aligned} \phi(x)-E_6(x)=&\left(-a_1-a_2-a_3-\frac{1}{b_1+b_2}+1\right)u\\ &+\left(-2a_2+2\delta a_3-\frac{2\theta}{b_1+b_2}+1\right)A_2u^2\\ &+\left[-4a_2-4(-\gamma+\delta^2)a_3-\frac{4\theta^2}{b_1+b_2}+2\right]A_2^2u^3\\ &+\left(3a_2+3\delta^2a_3+\frac{3\theta}{b_1+b_2}-1\right)A_3u^3\\ &+\left[-8a_2+8(\gamma+\delta^3-2\delta\gamma)a_3-\frac{8\theta^3}{b_1+b_2}+5\right]A_2^3u^4\\ &+\left[12a_2-6(\gamma+2\delta^3-2\delta\gamma)a_3+\frac{12\theta^2}{b_1+b_2}-5\right]A_2A_3u^4\\ &+\left(-4a_2+4\delta^3a_3-\frac{4\theta}{b_1+b_2}+1\right)A_4u^4 \end{aligned}$$

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$$+\left[14-16a_2-16(-\gamma+\gamma^2+\delta^4-3\delta^2\gamma+2\delta\gamma)a_3 - \frac{16\theta^4}{b_1+b_2}\right]A_2^4u^5$$
$$-\left[21-36a_2-12(-2\gamma+\gamma^2+3\delta^4-6\delta^2\gamma+4\delta\gamma)a_3 - \frac{36\theta^3}{b_1+b_2}\right]A_2^2A_3u^5$$
$$+\left[6-16a_2-8(-\gamma+2\delta^4-36\delta^2\gamma)a_3 - \frac{16\theta^2}{b_1+b_2}\right]A_2A_4u^5$$
$$+\left[3-9a_2-9(\delta^4+2\delta\gamma)a_3-\frac{9\theta^2}{b_1+b_2}\right]A_3^2u^5$$
$$-\left[1-5a_2-5\delta^4a_3-\frac{5\theta}{b_1+b_2}\right]A_5u^5+O(u^6).$$

Hence, we can conclude that for (1.1) to be fifth order, the following system of equations must be satisfied:

$$\begin{aligned} &a_1+a_2+a_3+\frac{1}{b_1+b_2}=1\\ &a_2-\delta a_3+\frac{\theta}{b_1+b_2}=\frac{1}{2}\\ &a_2+(-\gamma+\delta^2)a_3+\frac{\theta^2}{b_1+b_2}=\frac{1}{2}\\ &a_2+\delta^2a_3+\frac{\theta}{b_1+b_2}=\frac{1}{3}\\ &a_2-(\gamma+\delta^3-2\delta\gamma)a_3+\frac{\theta^3}{b_1+b_2}=\frac{5}{8}\\ &2a_2-(\gamma+2\delta^3-2\delta\gamma)a_3+\frac{2\theta^2}{b_1+b_2}=\frac{5}{6}\\ &a_2-\delta^3a_3+\frac{\theta}{b_1+b_2}=\frac{1}{4}. \end{aligned} \tag{2.4}$$

The system $(2\cdot 4)$ is a set of seven equations for the seven unknowns a_1, a_2, a_3, b_1, b_2, β and γ. From this system, we obtain the equivalent system

$$\begin{aligned} &\delta=-\frac{1}{2},\quad a_3=\frac{2}{3}\\ &a_1+a_2+a_3+\frac{1}{b_1+b_2}=1\\ &a_2+\frac{\theta}{b_1+b_2}=\frac{1}{6}\\ &a_2+\frac{\theta^2}{b_1+b_2}=\frac{2}{3}\gamma+\frac{1}{3}\\ &a_2+\frac{\theta^3}{b_1+b_2}=\frac{4}{3}\gamma+\frac{13}{24}. \end{aligned} \tag{2.5}$$

The general solution of $(2\cdot 4)$ can be expressed in terms of γ.

For $(16\gamma+5)(4\gamma+1)\neq 0$, the general solution of $(2\cdot 4)$ is given by

$$\left.\begin{aligned} &a_1=\frac{1}{6}\left(1+\frac{4\gamma+1}{\theta}\right),\quad a_2=\frac{1}{\theta-1}\left(\frac{1}{6}\theta-\frac{2}{3}\gamma-\frac{1}{3}\right),\\ &a_3=\frac{2}{3},\quad b_1=-\frac{6\theta(\theta-1)^2}{4\gamma+1},\\ &b_2=\frac{6\theta^2(\theta-1)}{4\gamma+1},\quad \beta=-\gamma-\frac{1}{2}, \end{aligned}\right\} \tag{2.6}$$

where

$$\theta=\frac{16\gamma+5}{4(4\gamma+1)}.$$

Now, the sample point, $x_n-u(x_n)$, is the Newton's point at x_n. Furthermore, since

$$x+\beta u(x)+\gamma w_2(x)=x-\frac{1}{2}u(x)+2\gamma A_2u^2(x)+O[u^3(x)],$$

by suitable choices of γ, the sample point, $x_n+\beta u(x_n)+\gamma w_2(x_n)$, is generally nearer to the zero, α, than x_n. In the case of $(2\cdot 6)$, it follows from Theorem 5–2[1, pp. 86–87] that the asymptotic error constant of $\phi(x)$, c, is given by

$$\begin{aligned} c&=\lim_{x\to\alpha}\frac{\phi(x)-E_6(x)}{u^5(x)}\\ &=-\left[\frac{32}{3}\gamma^2+\frac{8}{3}\gamma-\frac{2}{3}+\frac{1}{6(4\gamma+1)}\right]\tilde{A}_2^4+(8\gamma^2+4\gamma)\tilde{A}_2^2\tilde{A}_3\\ &\quad+\frac{128}{3}\gamma\tilde{A}_2\tilde{A}_4-\frac{3}{8}\tilde{A}_3^2+\frac{1}{24}\tilde{A}_5, \end{aligned} \tag{2.7}$$

where

$$\tilde{A}_j=\frac{f^{(j)}(\alpha)}{j!f'(\alpha)}\quad j=2(1)5.$$

By substituting $\gamma=0$ into $(2\cdot 6)$ and $(2\cdot 7)$, we obtain

$$a_1=\frac{3}{10},\ a_2=-\frac{1}{2},\ a_3=\frac{2}{3},\ b_1=-\frac{15}{32},\ b_2=\frac{75}{32},\ \beta=-\frac{1}{2}$$

and

$$c=\frac{1}{2}\tilde{A}_2^4-\frac{3}{8}\tilde{A}_3^2+\frac{1}{24}\tilde{A}_5.$$

If $\gamma=-\frac{1}{2}$, then

$$a_1=-\frac{1}{18},\ a_2=-\frac{1}{2},\ a_3=\frac{2}{3},\ b_1=\frac{9}{32},\ b_2=\frac{27}{32},\ \beta=0$$

and

$$c=-\frac{1}{2}\tilde{A}_2^4-\frac{64}{3}\tilde{A}_2\tilde{A}_4-\frac{3}{8}\tilde{A}_3^2+\frac{1}{24}\tilde{A}_5.$$

3. Conclusion

The formulae obtained in this paper will be particularly convenient for computing new approximations to a zero of a function $f(x)$, when the evaluation of $f'(x)$ is easily compared with that of $f(x)$.

We shall show fifth order multipoint iterative formulae costing two evaluations of $f(x)$ and two of $f'(x)$ per iteration elsewhere.

References

1. J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall Englewood Cliffs, New Jersey, 1964. MR 29 #6607.
2. P. Jarratt, Some Fourth Order Multipoint Iterative Methods for Solving Equations, *Math. Comp.*, v. 20, 1966, pp. 434–437.

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