A Class of Fifth Order Multipoint Iterative Methods for the Solution of Equations

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The purpose of this paper is to show some fifth order multipoint iterative methods which find new approximations to a zero of a function $f(x)$. These methods require two evaluations of $f(x)$ and two of $f'(x)$ per iteration.

1. Introduction

In order to find new approximations to a zero of a function $f(x)$, several classes of high-order multipoint iterative methods have been studied. In [1, pp. 190–204], Traub showed two classes of fourth order multipoint iterative methods involving two parameters. One of them requires three evaluations of $f(x)$ and one of $f'(x)$ per iteration, the other requires one evaluation of $f(x)$ and three of $f'(x)$ per iteration. In [2], Jarratt showed fourth order multipoint iterative methods involving one parameter and costing one evaluation of $f(x)$ and two of $f'(x)$ per iteration. In [3], we showed fifth order multipoint iterative formulae involving one parameter and costing one evaluation of $f(x)$ and three of $f'(x)$ per iteration. Now, the multipoint iterative method derived from the composition of two Newton methods is given by

$$x_{n+1}=x_n-u(x_n)-u[x_n-u(x_n)], \quad n=0, 1, \cdots, \tag{1.1}$$

where

$$u(x)=\frac{f(x)}{f'(x)}.$$

Since it is well known that Newton's iteration function is of second order for all simple zeros, it follows from Theorem 2–4 [1, pp. 27–28] that (1.1) is of fourth order for all simple zeros.

Then, the iteration function of (1.1) requires two evaluations of $f(x)$ and two of $f'(x)$ per iteration.

In this paper, as stated in [3], we show that for a class of iterative methods of the type

$$x_{n+1}=\phi(x_n), \quad n=0, 1, \cdots, \tag{1.2}$$

where

$$\phi(x)=x-\sum_{k=1}^{4} a_k\phi_k(x)-\psi(x),$$

$$\phi_1(x)=u(x), \quad \phi_2(x)=\frac{f[x-u(x)]}{f'(x)},$$

$$\phi_3(x)=\frac{f(x)}{f'[x+\beta u(x)]}, \quad \phi_4(x)=\frac{f[x-u(x)]}{f'[x+\beta u(x)]},$$

and

$$\psi(x)=\frac{f(x)}{b_1f'(x)+b_2f'[x+\beta u(x)]},$$

fifth order methods can be obtained by suitable choices of the parameters a's, b's and β.

2. Derivation of Formulae

In order to derive fifth order formulae of the type (1.2), we suppose, from now on, that $f(x)$ has a simple zero ζ and continuous fifth derivatives in the range considered. Next, in order to expand $\phi_k(x)$ and $\psi(x)$ into powers of $u(x)$, we introduce the following expansion:

$$\frac{f(x+\alpha u)}{f'(x+\beta u)}=\frac{(1+\alpha)u+\sum_{i=2}^{5}\alpha_i u^i+O(u^6)}{1+\sum_{i=2}^{5}\beta_i u^{i-1}+O(u^5)}=\sum_{i=1}^{5}c_iu^i+O(u^6), \tag{2.1}$$

where

$$u=u(x), \quad A_i=\frac{f^{(i)}(x)}{i!f'(x)}, \quad \alpha_i=\alpha^iA_i, \quad \beta_i=i\beta^{i-1}A_i,$$

$$c_1=1+\alpha, \quad c_i=\alpha_i-\sum_{j=0}^{i-2}c_{j+1}\beta_{i-j}, \quad i=2(1)5.$$

Using (2.1) and expanding $\phi_k(x)$ and $\psi(x)$, we obtain

$$\phi_2(x)=A_2u^2-A_3u^3+A_4u^4-A_5u^5+O(u^6), \tag{2.2}$$

$$\begin{aligned}\phi_3(x)=&\,u-2\beta A_2u^2+4\beta^2A_2^2u^3-3\beta^2A_3u^3-8\beta^3A_2^3u^4+12\beta^3A_2A_3u^4-4\beta^3A_4u^4\\&+(16A_2^4-36A_2^2A_3+16A_2A_4+9A_3^2-5A_5)\beta^4u^5+O(u^6),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\phi_4(x)=&\,A_2u^2-2\beta A_2^2u^3-A_3u^3+4\beta^2A_2^3u^4+(2\beta-3\beta^2)A_2A_3u^4+A_4u^4\\&+[-8\beta^3A_2^4+(-4\beta^2+12\beta^3)A_2^2A_3-(2\beta+4\beta^3)A_2A_4+3\beta^2A_3^2-A_5]u^5+O(u^6),\end{aligned} \tag{2.4}$$

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and

$$\psi(x)=\frac{u}{b_1+b_2}-\frac{2\theta\beta}{b_1+b_2}A_2u^2+\frac{4\theta^2\beta^2}{b_1+b_2}A_2^2u^3-\frac{3\theta\beta^2}{b_1+b_2}A_3u^3-\frac{8\theta^3\beta^3}{b_1+b_2}A_2^3u^4+\frac{12\theta^2\beta^3}{b_1+b_2}A_2A_3u^4$$
$$-\frac{4\theta\beta^3}{b_1+b_2}A_4u^4+\frac{\beta^4}{b_1+b_2}(16\theta^4A_2^4-36\theta^3A_2^2A_3+16\theta^2A_2A_4+9\theta^2A_3^2-5\theta A_5)u^5+O(u^6), \tag{2.5}$$

where

$$\theta=\frac{b_2}{b_1+b_2}.$$

Since the basic sequence [1, pp. 78–84] is given by

$$E_6=x-u-A_2u^2-2A_2^2u^3+A_3u^3-5A_2^3u^4+5A_2A_3u^4-A_4u^4-(14A_2^4-21A_2^2A_3+6A_2A_4+3A_3^2-A_5)u^5,$$

it follows from (1.2), (2.2), (2.3), (2.4) and (2.5) that

$$\phi(x)-E_6=\left(-a_1-a_3-\frac{1}{b_1+b_2}+1\right)u+\left(-a_2+2\beta a_3-a_4+\frac{2\theta\beta}{b_1+b_2}+1\right)A_2u^2+\left(-4\beta^2a_3+2\beta a_4-\frac{4\theta^2\beta^2}{b_1+b_2}+2\right)A_2^2u^3$$
$$+\left(a_2+3\beta^2a_3+a_4+\frac{3\theta\beta^2}{b_1+b_2}-1\right)A_3u^3+\left(8\beta^3a_3-4\beta^2a_4+\frac{8\theta^3\beta^3}{b_1+b_2}+5\right)A_2^3u^4$$
$$+\left[-12\beta^3a_3-(2\beta-3\beta^2)a_4-\frac{12\theta^2\beta^3}{b_1+b_2}-5\right]A_2A_3u^4+\left(-a_2+4\beta^3a_3-a_4+\frac{4\theta\beta^3}{b_1+b_2}+1\right)A_4u^4$$
$$+\left(-16\beta^4a_3+8\beta^3a_4-\frac{16\theta^4\beta^4}{b_1+b_2}+14\right)A_2^4u^5+\left[36\beta^4a_3-(-4\beta^2+12\beta^3)a_4+\frac{36\theta^3\beta^4}{b_1+b_2}-21\right]A_2^2A_3u^5$$
$$+\left[-16\beta^4a_3+(2\beta+4\beta^3)a_4-\frac{16\theta^2\beta^4}{b_1+b_2}+6\right]A_2A_4u^5+\left(-9\beta^4a_3-3\beta^2a_4-\frac{9\theta^2\beta^4}{b_1+b_2}+3\right)A_3^2u^5$$
$$+\left(a_2+5\beta^4a_3+a_4+\frac{5\theta\beta^4}{b_1+b_2}-1\right)A_5u^5+O(u^6).$$

Therefore it follows from Theorem 5-2 [1, pp. 86–87] that for (1.2) to be fifth order, the following system of equations must be satisfied:

$$a_1+a_3+\frac{1}{b_1+b_2}=1,$$
$$a_2-2\beta a_3+a_4-\frac{2\theta\beta}{b_1+b_2}=1,$$
$$4\beta^2a_3-2\beta a_4+\frac{4\theta^2\beta^2}{b_1+b_2}=2,$$
$$a_2+3\beta^2a_3+a_4+\frac{3\theta\beta^2}{b_1+b_2}=1,$$
$$-8\beta^3a_3+4\beta^2a_4-\frac{8\theta^3\beta^3}{b_1+b_2}=5,$$
$$12\beta^3a_3+(2\beta-3\beta^2)a_4+\frac{12\theta^2\beta^3}{b_1+b_2}=-5,$$
$$a_2-4\beta^3a_3+a_4-\frac{4\theta\beta^3}{b_1+b_2}=1. \tag{2.6}$$

The system (2.6) is a set of seven equations with respect to the seven unknowns a_1, a_2, a_3, a_4, b_1, b_2 and β. Suitable calculations lead to the following system equivalent to the system (2.6):

$$a_1+a_3+\frac{1}{b_1+b_2}=1,\quad a_2+a_4=1,$$
$$a_3+\frac{\theta}{b_1+b_2}=0,$$
$$a_3+\frac{\theta^2}{b_1+b_2}=-\frac{3(\beta+1)}{2\beta^2(3\beta+2)},$$
$$a_4=-\frac{6\beta+5}{\beta(3\beta+2)},$$
$$a_3+\frac{\theta^3}{b_1+b_2}=-\frac{24\beta^2+35\beta+10}{8\beta^3(3\beta+2)}. \tag{2.7}$$

The general solution of (2.7) can be expressed in terms of β.

By solving (2.7), the general solution of (2.6) is given as follows:

For

$$\beta(\beta+1)(3\beta+2)(4\beta+5)(11\beta+10)\neq 0,$$
$$a_1=1-\frac{3(\beta+1)}{2\theta\beta^2(3\beta+2)},$$
$$a_2=\frac{(\beta+1)(3\beta+5)}{\beta(3\beta+2)},$$
$$a_3=\frac{3(\beta+1)}{2\beta^2(3\beta+2)(\theta-1)},$$
$$a_4=-\frac{6\beta+5}{\beta(3\beta+2)},$$
$$b_1=\frac{2\theta(\theta-1)^2\beta^2(3\beta+2)}{3(\beta+1)},$$
$$b_2=\frac{2\theta^2(1-\theta)\beta^2(3\beta+2)}{3(\beta+1)}, \tag{2.8}$$

where

$$\theta=\frac{(3\beta+2)(4\beta+5)}{12\beta(\beta+1)}.$$

Since, by (2.8), $(\beta+1)b_1b_2\neq 0$ and the conditions $a_2=a_4=0$ cannot be simultaneouly satisfied, the iteration function of (1.2) requires two evaluations of $f(x)$ and two of $f'(x)$ per iteration.

In the case of (2.8), it follows from Theorem 5–2 [1, pp. 86–87] that the asymptotic error constant of $\phi(x)$, C is given by

$$\begin{aligned}C&=\lim_{x\to\zeta}\frac{\phi(x)-E_6}{u^5(x)}\\&=\left(-16\beta^4a_3+8\beta^3a_4-\frac{16\theta^4\beta^4}{b_1+b_2}+14\right)\tilde{A}_2^4\\&+\left[36\beta^4a_3-(-4\beta^2+12\beta^3)a_4+\frac{36\theta^3\beta^4}{b_1+b_2}-21\right]\tilde{A}_2^2\tilde{A}_3\\&+\left[-16\beta^4a_3+(2\beta+4\beta^3)a_4-\frac{16\theta^2\beta^4}{b_1+b_2}+6\right]\tilde{A}_2\tilde{A}_4\\&+\left(-9\beta^4a_3-3\beta^2a_4-\frac{9\theta^2\beta^4}{b_1+b_2}+3\right)\tilde{A}_3^2\\&+\left(a_2+5\beta^4a_3+a_4+\frac{5\theta\beta^4}{b_1+b_2}-1\right)\tilde{A}_5,\end{aligned}$$

where

$$\tilde{A}_i=\frac{f^{(i)}(\zeta)}{i!f'(\zeta)},\quad i=2(1)5.$$

Using (2.7) and (2.8), we find after a long calculation that

$$\begin{aligned}C&=\frac{1}{6(\beta+1)}(48\beta^3+212\beta^2+299\beta+134)\tilde{A}_2^4\\&-\frac{1}{2}(24\beta^2+65\beta+42)\tilde{A}_2^2\tilde{A}_3\\&+\frac{2}{3\beta+2}(\beta+1)(2\beta+1)\tilde{A}_2\tilde{A}_4\\&+\frac{3}{2}(3\beta+2)(\beta+1)\tilde{A}_3^2.\end{aligned}\qquad(2.9)$$

From (2.9), it can be seen that the asymptotic error constant of $\phi(x)$ does not contain derivatives higher than the fourth. Furthermore, we examine the two sample points in (1.2).

First, since the sample point x_n-$u(x_n)$ is the Newton point of x_n, $x_n-u(x_n)$ is generally nearer to ζ than x_n.

Next, the sample point $x_n+\beta u(x_n)$ must be chosen so that it may be in most cases nearer to the Newton point of x_n than x_n. Therefore, we impose the constraint that

$$-2<\beta<0 \quad\text{and}\quad (\beta+1)(3\beta+2)(4\beta+5)(11\beta+10)\neq 0.$$

Finally, we substitute the parameter β into appropriate values so that the form of (1.2) or the form of the asymptotic error constant may be simplified.

If $\beta=-\frac{5}{6}$, then

$$a_1=\frac{61}{25},\quad a_2=1,\quad a_3=\frac{36}{25},$$

$$a_4=0,\quad b_1=b_2=-\frac{25}{144},$$

$$C=\frac{77}{18}\tilde{A}_2^4-\frac{9}{4}\tilde{A}_2^2\tilde{A}_3+\frac{4}{9}\tilde{A}_2\tilde{A}_4-\frac{1}{8}\tilde{A}_3^2.$$

If $\beta=-\frac{1}{2}$, then

$$a_1=13,\quad a_2=-7,\quad a_3=-4,$$

$$a_4=8,\quad b_1=-\frac{3}{16},\quad b_2=\frac{1}{16},$$

$$C=\frac{21}{2}\tilde{A}_2^4-\frac{31}{4}\tilde{A}_2^2\tilde{A}_3+\frac{3}{8}\tilde{A}_3^2.$$

References

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