Combination of Two Discrete Fourier Transforms of Data Sampled Separately at Different Rates

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This paper is motivated by a demand for increasing the frequency-resolution in power spectrum analysis using ordinary data-acquisition-equipment, even if the speeds of sampling and quantization is so moderate as to yield confusion between the low and high frequencies.

For this purpose, in our proposal, a given record is sampled separately at slightly different rates, e.g., $T/2N$ and $T/2(N-n)$ seconds respectively for every period T seconds, where $1 \leq n < N$. From these time-series of samples, two Fourier coefficients are computed separately by the usual Discrete Fourier Transform (DFT). To improve this effect, the two discrete Fourier coefficients are combined. By solving simple algebraic equations of the corresponding order-coefficients, i.e., the frequency-components in the original record can be distinguished from one another. The maximum frequency that can be defined in this version is $[2(N-n)-1]/T$ Hz, not more than two times as high as that anticipated by a single version of the usual DFT.

This algorithm has instant effects not only on improving the frequency-resolution, but on damping the so-called Gibbs phenomenon near discontinuous points of a reconstructed rectangular wave.

1. Introduction

There occurs sometimes a difficulty in investigating power spectrums of a random record. This results from poor conversion of an analog to digital converter used for quantizing data values. Obviously, the frequency-resolution depends upon the rate of sampling within the fundamental period of a given record. Precisely speaking, for a periodic record with the fundamental period T, the highest frequency (Nyquist frequency) that can be defined by sampling at a rate of $T/2N$ seconds is N/T Hz. Higher frequency-components than N/T Hz in the original data are folded back into the frequency-range from 0 to N/T Hz in an accordion-fashion and confused with data in this frequency-range [1]. In detail, frequencies $(2mN/T) \pm f$ for $m=1, 2, 3, \cdots$, are aliased with any frequency f in the range $0 \leq f \leq N/T$ Hz.

A straightforward method of improving the frequency-resolution is to increase the rate of sampling so that the maximum frequency can be defined. However, the time-interval of sampling cannot be so reduced as to be less than the required time for quantization of each data. Real improvement can be obtained by utilizing improvised equipment which is slower in the speed of sampling and quantizing.

Two kinds of approaches [2] ~ [4] have been developed. In the approaches [2], [3] of the first kind, a given record is sampled at the even and odd numbered points alternatively with each equal space and quantized to shape two time-series of N samples. Thereby, the given record is substantially sampled at $2N$ equally spaced points within its fundamental period. Thus, the rates of sampling in each time-series can be slow down to be one-half of the required rate for the cut-off frequency. In other words, a split into two time-series of samples can improve the frequency-resolution by two times as much as that anticipated by sampling at an ordinary rate. Nevertheless, we shall fail when the one of two time-series of samples cannot be delayed from the other by one-half of the time-interval of sampling exactly. In the second approach [4], a given record is divided into two records so that one of them leads to the other by $\pi/2$ radian. They are then sampled simultaneously at the same-numbered points. This system can surely define the maximum frequency N/T Hz at a rate of N/T samples per second. But it is difficult to perform the phase-shift of $\pi/2$ radian without influence on the amplitudes of all inherent frequencies in the original record.

This paper develops a technique to increase the frequency-resolution without speeding up the rate of sampling and without employing special equipment, such as a timing-pulse generator for the alternative data-acquistion and a phase-shifter without influencing the amplitude-gain. This algorithm is displayed with its application.

2. DFT in a Single Version

Let $x(t)$ be a stationary random record of the form:

$$x(t) = \sum_{r=-\infty}^{\infty} a_r \varepsilon^{j2\pi rt/T}, \quad j=\sqrt{-1}, \tag{1}$$

with a fundamental period T. The Fourier coefficients a_r for $r=0, \pm 1, \pm 2, \cdots$, are analytically given by

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$$a_r=\frac{1}{T}\int_0^T x(t)\varepsilon^{-j2\pi rt/T}dt. \tag{2}$$

If this $x(t)$ is sampled at $2N$ equally spaced points within its fundamental period, a time-series of these samples $\{x_k\}$, $k=0, 1, 2,\cdots, 2N-1$, is defined in a discrete version:

$$x_k=x\left(\frac{kT}{2N}\right)=\sum_{r=-N}^{N} A_r\varepsilon^{j\pi rk/N}. \tag{3}$$

Then, the Fourier coefficients A_r for $r=0, \pm1, \pm2,\cdots, \pm N$, are given by

$$A_r=\frac{1}{2N}\sum_{k=0}^{2N-1} x_k\varepsilon^{-j\pi rk/N}. \tag{4}$$

We have now determined Fourier coefficients a_r and A_r. An investigation can clarify the relationship between a_r and A_r:

$$A_0=a_0+\sum_{m=1}^{\infty}(a_{2mN}+a_{-2mN}),$$
$$A_r=a_r+\sum_{m=1}^{\infty}(a_{r+2mN}+a_{r-2mN}). \tag{5}$$

Equation (5) can be illustrated in an accordion-fashion of Fig. 1. This illustration is made by placing the original Fourier coefficients a_r in ascending order and by folding this coefficient-string at every N terms. This displays a detail of confusion between the low and high frequencies in the discrete Fourier transform. For example, in a discrete version dealing with $2N$ samples per the fundamental period, the original Fourier coefficients a_r for $r=0, \pm1, \pm2,\cdots, \pm N$, aliase with the high-order coefficients $a_{r\pm 2mN}$, where $m=1, 2, 3,\cdots$. Consequently, a true power spectrum-density function of Fig. 2-(a) is aliased by the folded power spectrum-density function of Fig. 2-(b).

The aliasing phenomenon is unavoidable in the discrete Fourier transform of a series of samples acquired at far apart points. Fortunately, we can distinguish between these confused frequency-components by well handling the aliasing effects. This paper is just concerned with such an approach.

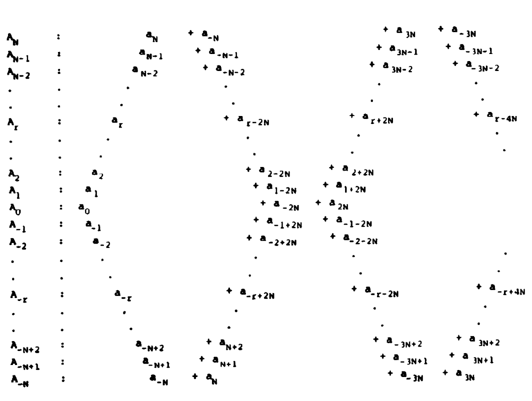

Fig. 1 Aliased Fourier coefficient-series due to folding.

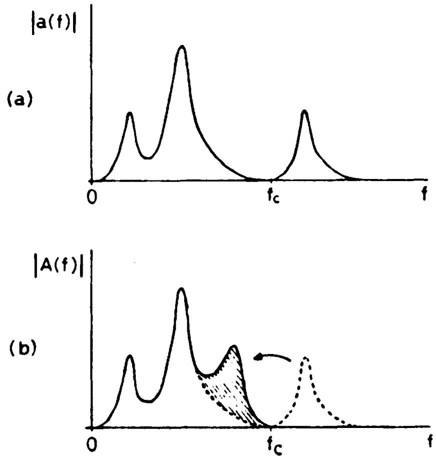

Fig. 2 Aliased power spectrum due to folding:
(a) True spectrum.
(b) Aliased spectrum.
Where f_c: Cut-off frequency.

3. DFT in a Dual Version

The confusion between the low and high frequency-components can be solved without increasing the speed of sampling and without employing any special equipment. For this purpose, the original record $x(t)$ is separatetly sampled at slightly different speeds respectively. Thereby, $x(t)$ is split into two time-series of samples. Let sample the original record at a rate of $T/2N$ seconds as shown in Fig. 3-(a). At the same time, let sample it again at a rate of $T/2(N-n)$, where $1\leq n<N$, seconds as shown in Fig. 3-(b). Note that the original record is thereby sampled at $4(N-n)$ unequally spaced points within the fundamental period T, as illustrated in Fig. 3-(c)..

From data of these $2(N-n)$ samples, another Fourier coefficients A'_r for $r=0, \pm1, \pm2,\cdots, \pm(N-n)$, is computed by

$$A'_r=\frac{1}{2(N-n)}\sum_{l=0}^{2(N-n)-1} x_l\varepsilon^{-j\pi rl/(N-n)}. \tag{6}$$

The relationship between a_r and A'_r can be written as

$$A'_0=a_0+\sum_{m=1}^{\infty}[a_{2m(N-n)}+a_{-2m(N-n)}],$$
$$A'_r=a_r+\sum_{m=1}^{\infty}[a_{r+2m(N-n)}+a_{r-2m(N-n)}]. \tag{7}$$

Either A_r or A'_r is an approximation of the original Fourier coefficients a_r, though different in their values from the others. The difference is caused by the fact that any A_r for $0\leq r\leq N$ is confused with the original coefficients $a_{r\pm 2mN}$ of high orders, while A'_r of the rth order is done with $a_{r\pm 2m(N-n)}$, where $m=1, 2, 3,\cdots$. These compounded aliasing phenomena can be investigated by an illustration of Fig. 4. The combination of two folded coefficient-strings serves us information on a_r

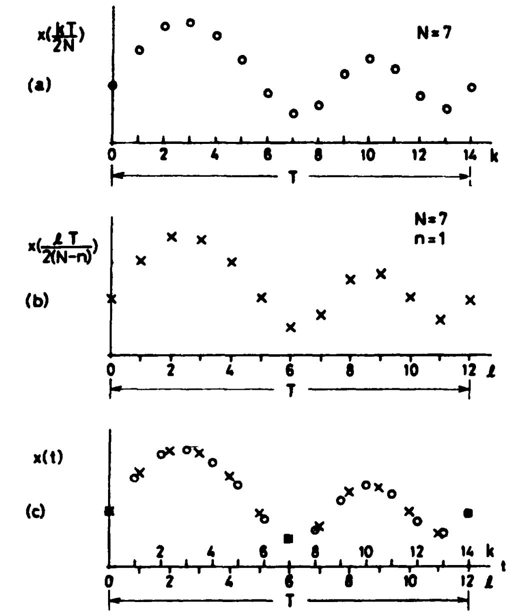

Fig. 3 Combination of two time-series of samples which are sampled at slightly different rates:
(a) Time-series sampled at a rate of $2N/T$ samples, where $N=7$.
(b) Time-series sampled at a rate of $2(N-n)/T$ samples, where $N=7$, $n=1$.
(c) Combination of (a) and (b).

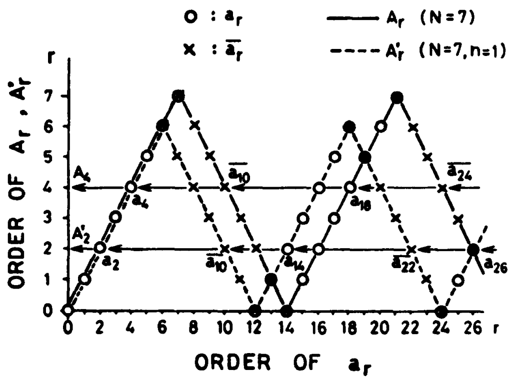

Fig. 4 Aliased two Fourier coefficient-series due to folding at slightly different pitches.

turned in onto A_r and A'_r. An effect of this combination should be understood as follows: A time-series of $4(N-n)$ samples has been provided, as though each data were sampled at unequally spaced points within the fundamental period of the given record. Acquisition of many discrete data leads to an potentiality of high frequency-resolution in power-spectrum analysis using DFT.

That potentiality becomes reality if we could solve simultaneous relations between Eqs. (5) and (7) with unknowns $a_0, a_1, a_2, \cdots, a_{2(N-n)-1}$.

Suppose that the magnitude of the coefficients a_r of higher orders than the $[2(N-n)-1]$th is so small as to be negligible. Then, the difference between a_r and A_r for $0 \leq r \leq 2n$ in the magnitude can be handled as to be too little to distinguish. Thereby, we can solve the confusion of A_r and A'_r with a_r, and define new coefficients A^*_r ($\simeq a_r$) for $r=0, \pm 1, \pm 2, \cdots, \pm[2(N-n)-1]$ and $s=1, 2, 3, \cdots, N-2n-1$, as follows:

$$
\begin{aligned}
&A^*_i = A_i, && \text{for } r=i,\ i=0, 1, \cdots, 2n,\\
&A^*_{2(N-n)-s} = \overline{A'_s - A^*_s}, && \text{for } r=2(N-n)-s,\\
&A^*_{s+2n} = A_{s+2n} - \overline{A^*_{2(N-n)-s}}, && \text{for } r=s+2n,\\
&A^*_N = \overline{A'_{N-n} - A^*_{N-n}}, &&
\end{aligned}
\tag{8}
$$

where $\overline{A'_s - A^*_s}$: complex conjugation of $A'_s - A^*_s$,
$\overline{A^*_{2(N-n)-s}}$: complex conjugation of $A^*_{2(N-n)-s}$,
$\overline{A'_{N-n} - A^*_{N-n}}$: complex conjugation of $A'_{N-n} - A^*_{N-n}$.

4. Discussion

In this algorithm, a time-series of $4(N-n)$ samples, as though sampled at unequally spaced points, is made of a total of $2N$ and $2(N-n)$ samples at equally spaced points, though different in the rate of spacing each to each. It is because data of $2n$ samples are overlapped to those in this connection. It should be stressed that the original data are sampled at so far apart points as to be capable to define the maximum frequency of at most N/T Hz and $(N-n)/T$ Hz respectively. Nevertheless, we can evaluate Fourier coefficients of up to the $2(N-n)$th order except for the imaginary part of its highest order. The maximum frequency that can be defined in our algorithm becomes $[2(N-n)-1]/T$ Hz.

The relations of Eq. (8) have been based on the condition that the amplitude of higher frequencies than $[2(N-n)-1]/T$ Hz are all so small as to be negligible. If not so, these frequencies will necessarily be aliased with lower frequencies than $[2(N-n)-1]/T$ Hz, resulting in error on a task for the theoretical Fourier coefficients a_r with $r=0, \pm 1, \pm 2, \cdots, \pm[2(N-n)-1]$. Its effect on our Fourier coefficients A^*_r for $r=0, 1, 2, \cdots, 11$, in $N=7$, $n=1$, is illustrated in Fig. 5. If our task persisted in obtaining the precise values of a_r, we should use a low-pass filter prior to sampling of the original record and suppress those aliasing phenomena. But we shall otherwise discover a benefit of those phenomena.

To demonstrate this, let us reconstruct a 10-term approximation of a periodic rectangular wave in different methods. The wave of Fig. 6-(a) is reconstructed by the theoretical Fourier coefficients a_r of odd-numbered order r up to the 19th. Their values can of course be given in a continuous version of Fourier Transform. Figure 6-(b) shows a reconstructed wave based on the ordinary DFT method without prefiltering. The curve of Fig. 6-(c) is based on the relations of Eq. (8) without condition of its premise. Note that the so-called Gibbs oscillation is thoroughly damped and no overshoot is seen near discontinuous points of the wave in Fig. 6-(c). That is the reconstruction using A^*_r obtained by a technique of

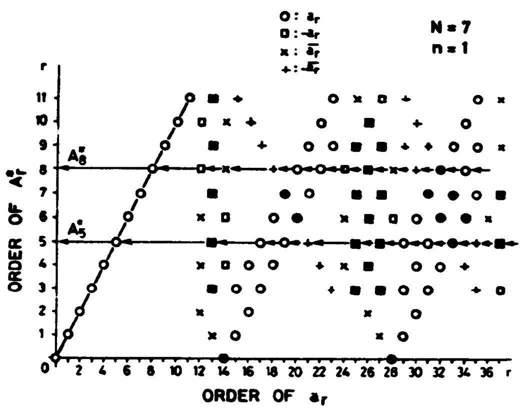

Fig. 5 Effect of remaining high order-components in our dual DFT version.

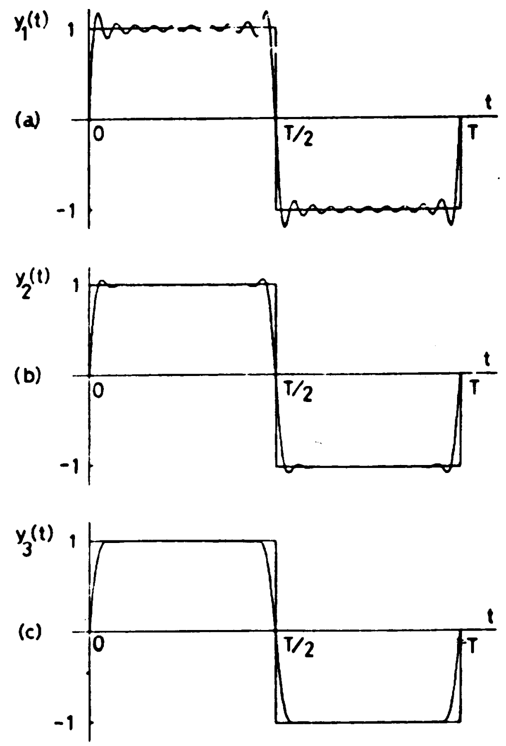

Fig. 6 Rectangular waves reconstructed by different versions using terms of up to the 19th order:
(a) Continuous version using a_r, $r=\pm1, \pm3, \pm5, \cdots, \pm19$.
(b) Single discrete version using A_r, $r=\pm1, \pm3, \pm5, \cdots, \pm19$, $N=20$.
(c) Dual discrete version using A_r^*, $r=\pm1, \pm3, \pm5, \cdots, \pm19$, $N=11$, $n=1$.

the dual DFT version.

To investigate this closely, let us compare our reconstruction with the wave by the σ-factor method of Lanczos [5] and also with that by the well-known Fejer's correction [6]. Figure 7 is given for this purpose. We can herewith observe that our wave has more leveled top than both of those reconstructed by the Lanczos and Fejer's methods and has more broad shoulders than that corrected by the Fejer's weight. A rise-time of our wave is faster than the Fejer's wave, though slower than that in the Lanczos' reconstruction. This implies that our algorithm can provide Fourier coefficients converging more rapidly in their high order terms than those corrected by the Lanczos' factors. The Fejer's correction provide too decreased values of low and high order Fourier coefficients to well reconstruct a rectangular wave.

In addition to this figure, numerical values of different Fourier coefficients a_r, A_r, A_r^*, the Lanczos' correction $\sigma_r a_r$ and the Fejer's correction $\omega_r a_r$ are shown in Table 1, where σ_r, ω_r designate the Lanczos and Fejer's weights, respectively.

5. Conclusions

The following attributes are considered as characteristics of this algorithm:

(1) Facility for utilizing improvised equipment which is too slow in speed of sampling and quantizing to define the highest frequency-component in the original data.

(2) Separate sampling with each local clock-pulse.

(3) Permission to perform each sampling so moderately as to yield confusion between the low and high frequencies in the original data.

(4) Ability to distinguish between these confounding frequencies by simple algebraic operations.

(5) Frequency-resolution improved about two times as much as that anticipated by a single version of the discrete Fourier transform.

Fig. 7 Damping and rise-time in different 10-term approximations to a rectangular wave. These curves are based on data given in Table 1.

Table 1 Different Fourier coefficients of a periodic rectangular wave.

order r	FT $\mathrm{Im}(a_r)$	DFT (single) $N=20$ $\mathrm{Im}(A_r)$	DFT (dual) $N=11$ $n=1$ $\mathrm{Im}(A_r^*)$	Lanczos' method $\sigma_r \times \mathrm{Im}(a_r)$	Fejer's method $\omega_r \times \mathrm{Im}(a_r)$
1	−0.6366	−0.6353	−0.6323	−0.6340	−0.6048
3	−0.2122	−0.2083	−0.2000	−0.2044	−0.1804
5	−0.1273	−0.1207	−0.1086	−0.1146	−0.0955
7	−0.0909	−0.0816	−0.0671	−0.0737	−0.0591
9	−0.0707	−0.0585	−0.0428	−0.0494	−0.0389
11	−0.0579	−0.0427	−0.0270	−0.0331	−0.0260
13	−0.0490	−0.0306	−0.0161	−0.0214	−0.0171
15	−0.0424	−0.0217	−0.0086	−0.0127	−0.0106
17	−0.0374	−0.0120	−0.0037	−0.0064	−0.0056
19	−0.0335	−0.0039	−0.0009	−0.0018	−0.0017

Where $\mathrm{Re}(a_r)$, $\mathrm{Re}(A_r)$ and $\mathrm{Re}(A_r^*)=0.$, $r=1, 3, 5, \cdots, 19$,
Re: Notation of the real part,
Im: Notation of the imaginary part,
Lanczos' factors: $\sigma_r = \dfrac{\sin (r\pi/20)}{r\pi/20}$, $r=1, 3, \cdots, 19$,
Fejer's weights: $\omega_r = \dfrac{20-r}{20}$, $r=1, 3, \cdots, 19$.

(6) Accuracy almost equal to that achieved by the alternative sampling at each exactly equal space.

(7) Solution-times relatively long (i.e., the number of samples cannot be chosen as some power of two).

(8) Unnecessity for providing any special equipment, e.g., phase-shifters, clock-pulse generators for simultaneous or alternative sampling.

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