NP-Completeness of the Hamiltonian Cycle Problem for Bipartite Graphs

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We consider the problem of determining whether a bipartite graph G has a hamiltonian cycle. We show that this problem is NP-complete for two classes of bipartite graphs: 2-connected cubic bipartite planar graphs; and 3-connected cubic bipartite graphs. Hence the hamiltonian cycle problem for these classes of graphs, or any larger class containing all such graphs, is probably computationally intractable.

1. Introduction

The hamiltonian cycle problem, formulated by Irish mathematician William Rowan Hamilton, asks whether there is a cycle in a graph passing through each vertex exactly once. Such a cycle is a hamiltonian cycle of a graph. Many attempts have been made to characterize the graphs which contain hamiltonian cycles. While providing characterizations in various special cases, none of these results has led to an efficient algorithm for identifying such graphs in general. In fact Garey, Johnson and Tarjan [3] and Krishnamoorthy [4] have shown that this prpblem is "NP-complete" even if restricted to a class of planar graphs, i.e. planar cubic 3-connected graphs, or to a class of bipartite graphs. Now attention has shifted to special cases with more restricted structure. In this paper we restrict our attention to bipartite graphs, and show that the problem remains NP-complete for two more restricted classes of such graphs: 2-connected cubic bipartite planar graphs; and 3-connected cubic bipartite graphs. Our proof is based on the transformation-technique employed by Garey, Johnson and Tarjan [3]. Note that even the existence of nonhamiltonian graphs is not yet known for the class of 3-connected cubic bipartite planar graphs; while Barnette conjectures that there is no such nonhamiltonian graph (See [2, p. 248]).

2. The Case of 2-Connected Cubic Bipartite Planar Graphs

In this section we show that the hamiltonian cycle problem is NP-complete for the class of 2-connected cubic bipartite planar graphs. For our purposes, the only nontrivial requirement is that we show how a known NP-complete problem can be "transformed" in polynomial time into this restricted hamiltonian cycle problem [1]. This "known" NP-complete problem will be "the satisfiability problem".

Let F be any well formed formula containing atomic variables and the connectives $\wedge$ (and), $\vee$ (or) and $-$ (not). F is "satisfiable" if there exists some assignment of the values "true" and "false" to the variables which makes F true under the standard interpretation of the connectives. We shall show how to construct, in polynomial time, a 2-connected cubic bipartite planar graph G so that F is satisfiable if and only if G contains a hamiltonian cycle. It suffices to consider only formulas F in conjunctive normal form with three literals per clause. That is, we may assume that F has the form $(p_{11} \vee p_{12} \vee p_{13}) \wedge (p_{21} \vee p_{22} \vee p_{23}) \wedge \cdots \wedge (p_{m1} \vee p_{m2} \vee p_{m3})$, where each $(p_{i1} \vee p_{i2} \vee p_{i3})$ is called a "clause" and each p_{ij}, called a "literal", is either an atomic variable or the negation of an atomic variable. We assume that F contains n atomic variables, denoted $x_1, x_2, \cdots, x_n$.

Although we employ fully the construction-technique of [3], we must modify some of their component-graphs, including "required-edge graph", "exclusive-or", "2-input or" and "3-input or" so that a constructed graph results in a 2-connected cubic bipartite planar graph G.

We use the graph G_1 depicted in Fig. 1(a) as a

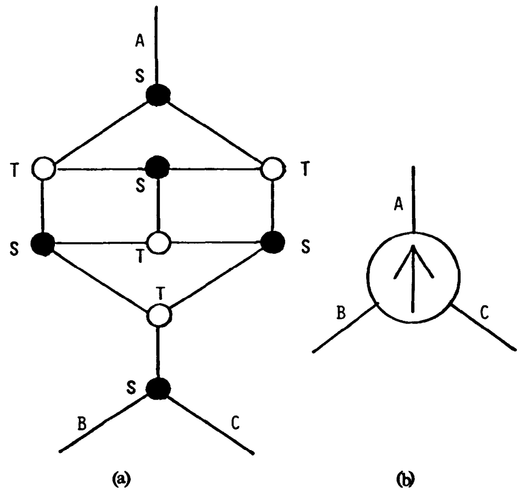

Fig. 1 Required-edge graph G_1. (a) Graph, and (b) Abbreviation.

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"**required-edge graph**" of our case. Note that G_1 can be a vertex-induced subgraph of a 2-connected cubic bipartite planar graph G. Any hamiltonian cycle in such a graph G must use the edge marked A. Thus G_1 acts like a single degree-3 vertex which has one "specified" edge that is required to be used in any hamiltonian cycle of G. We use the symbol of Fig. 1(b) as an abbreviation

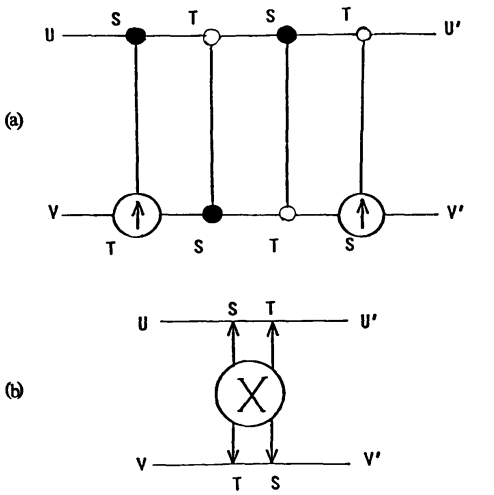

Fig. 2 Exclusive-or graph. (a) Graph, and (b) Abbreviation.

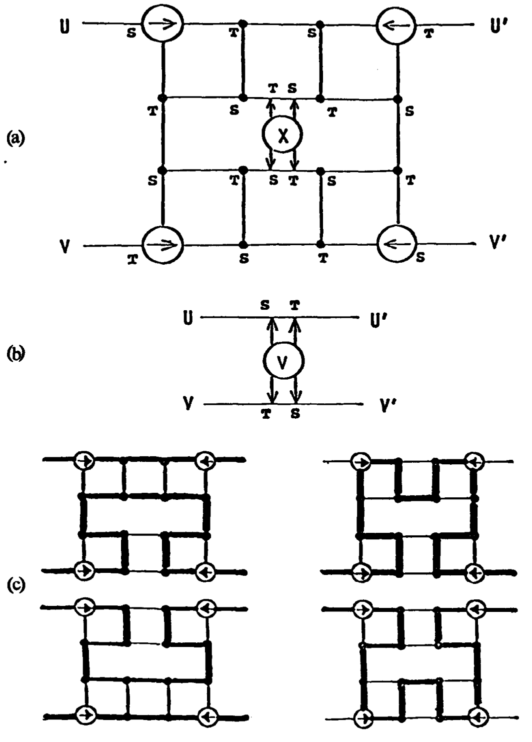

Fig. 3 2-input or graph. (a) Graph, (b) Abbreviation, and (c) Possible local states.

of G_1. Each vertex of G_1 is marked "S" or "T" in a way that no edge joins two vertices of the same mark so that one can easily recognize G_1 to be bipartite. From now on such marking will be used for bipartite graphs.

For an "**exclusive-or graph**" we use the graph depicted in Fig. 2(a) which is the same as in [3] except for G_1 being used as "required-edge graphs". In a graph which contains this subgraph as a vertex-induced subgraph, this subgraph acts like two separate edges, one joining v to v' and the other joining u to u', with the constraint that exactly one of these edges must occur in any hamiltonian cycle. This will be represented by the abbreviation shown in Fig. 2(b). Since the structure of our exclusive-or is the same as in [3], it has the property that two "exclusive-or lines" joining different pairs of edges may cross each other without destroying the planarity of the graph G which will corresponds to the formula F.

In addition to the exclusive-or, we will also use the "**2-input or graph**" of Fig. 3(a) which is different from that of [3]. Any hamiltonian cycle in a graph G which contains this graph as a vertex-induced subgraph must

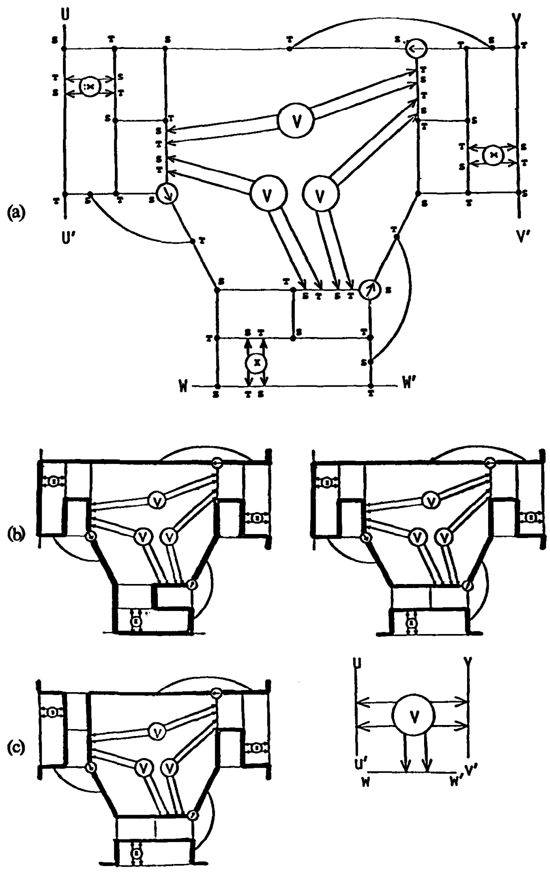

Fig. 4 3-input or graph. (a) Graph, (b) Possible local states, and (c) Abbreviation.

appear locally in one of the states in Fig. 3(c). Thus this subgraph acts like two separate edges, one joining v to v' and the other joining u to u', with the constraint that at least one of these edges must occur in any hamiltonian cycle of G. This will be represented by the abbreviation shown in Fig. 3(b).

Finally we use the graphs in Figs. 1, 2 and 3 to construct the "**3-input or**" shown in Fig. 4 which is slightly different from that of [3]. This subgraph acts like three separate edges, one joining v to v', one joining u to u' and one joining w to w', with the constraint that at least one of these three edges must occur in any hamiltonian cycle of G. We can verify this fact by showing that any hamiltonian cycle in a graph G which contains this graph as a vertex-induced subgraph must occur locally as one of the states shown in Fig. 4(b) or the symmetric one.

With these components we can construct a graph G which corresponds to a formula F. Since the construction of G is the same as in [3] except for **our** components being used, we show only an example of G corresponding to a formula $F=(x\vee y\vee z)\wedge(x\vee\bar{y}\vee w)\wedge(\bar{x}\vee\bar{z}\vee\bar{w})$ in Fig. 5.

Although the constructed graph G is not planar, "crossings of exclusive-or lines" can be made planar so that a planar graph G results in (See [3, Fig. 4]). One can verify that the graph G is 2-connected cubic and planar. The bipartite property of G can be recognized from the vertex marking with "S" and "T" of G and its components. By the same reason as in [3], the construction of G requires polynomial time, and G has a hamiltonian cycle if an only if F is satisfiable. This completes the proof.

3. The Case of 3-connected Cubic Bipartite Graphs

In this section we show that the three-satisfiability problem is also transformed into the hamiltonian cycle problem restricted to 3-connected cubic bipartite graphs. We shall construct a 3-connected cubic bipartite graph in polynomial size, so that the graph is hamiltonian if and only if F is satisfiable. The construction is the same as that of preceding section except for the graph G_2 in Fig. 6(a) being used instead of G_1 as a required-edge graph. Horton has used G_2 in order to construct a 3-connected cubic bipartite graph which is not hamiltonian (See [2, p. 240]). Any hamiltonian cycle Z in a graph which contains G_2 as a vertex-induced subgraph must appear locally in Fig. 6(b) or (c), that is, Z must contain the edge marked A. This fact can be verified via a straightforward but tedious argument (See [2, Exercise 4. 2.14]). Hence it follows that G_2 can act a required-edge graph.

Acknowledgement

We wish to thank Dr. T. Asano for valuable sugges-

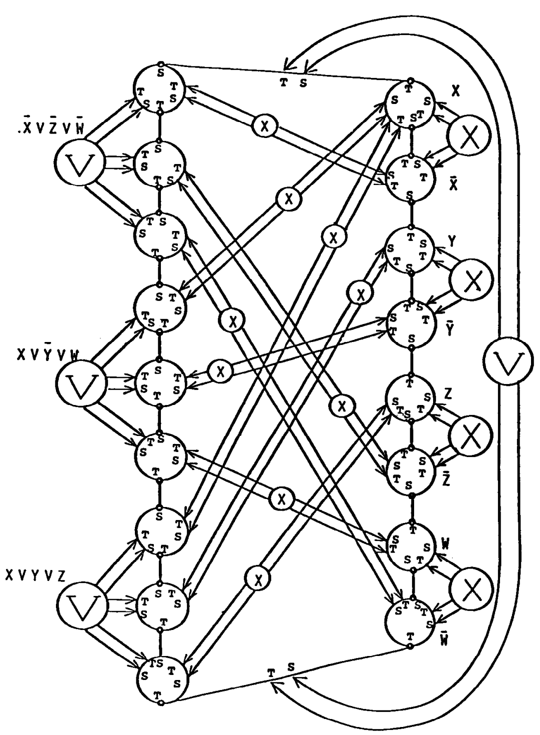

Fig. 5 Complete construction for $F=(x\vee y\vee z)\wedge(x\vee\bar{y}\vee w)\wedge(\bar{x}\vee\bar{z}\vee\bar{w})$.

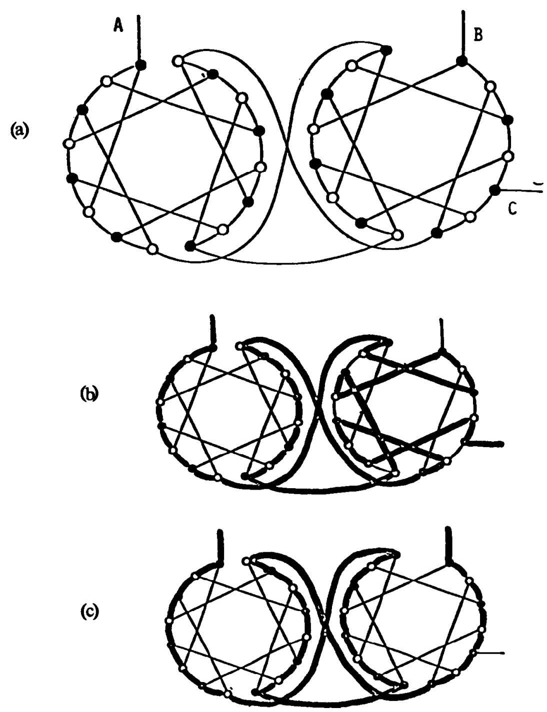

Fig. 6 Required-edge graph G_2. (a) Graph, and (b), (c) Possible local states.

tions on the subjects. This work was supported in part by the Grant in Aid and Scientific Research of the Ministry of Education, Science and Culture of Japan under Grants: Coopertive Research (A)435013(1979) and YSE(A)475235(1979).

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(Received March 31, 1980; revised June 7, 1980)