Convergence Property of Aitken's Δ^2-Process and the Applicable Acceleration Process

KEN IGUCHI*

We first study the convergence property of Aitken's δ^2-process and then examine why Aitken's δ^2-process is not always successful in the improvement of convergence. Next, acceleration processes are proposed which are established by modifying Aitken's δ^2-process to be applicable for any problem. Then we show which acceleration processes should be used according to the magnitude of the absolutely largest eigenvalue of the iterative matrices. We give five examples to demonstrate the efficiency of the acceleration processes.

1. Introduction

Iterative methods are preferred for solving large sparse systems and large scale eigenvalue problems, However, iterative methods are not frequently used except for these problems because of the slow convergence. In this paper we study Aitken's δ^2-process for vector-valued sequences. As it is well known, application of Aitken's δ^2-process cannot always yield successful results in accelerating the rate of convergence. This is main reason why Aitken's δ^2-process has not been used frequently. Here we propose another acceleration process which is made by modifying Aitken's δ^2-process to be applicable for any problem.

2. Iterative Process and Its Convergence Property

In this section we describe the iterative process in question and its property of convergence.

2.1 Iterative Process

An iterative process considered in this paper is given as follows:

$$y^{(r)}=Cy^{(r-1)}+d \quad \text{for } r=1,2,\cdots \tag{2.1}$$

where C is a square real symmetric matrix of order n and d, a column vector of order n, which are independent of iteration number r.

We assume the vector-valued sequence $\{y^{(r)}\}$ obtained by iterating (2.1) converges to y.

2.2 Convergence Property of the Iterative Process

Suppose that λ_s, $s=1,2,\cdots,n$ are eigenvalues of the matrix C, where the following relation holds:

$$1>|\lambda_1|>|\lambda_2|\geqq\cdots\geqq|\lambda_n| \tag{2.2}$$

and that $x^{(s)}$, $s=1,2,\cdots,n$ are the corresponding eigenvectors to λ_s.

Then, as well known, the general term of the sequence $\{y^{(r)}\}$ is given by

$$y^{(r)}=y-\sum_{s=1}^{n}\lambda_s^r x^{(s)} \quad \text{for } r=0,1,\cdots. \tag{2.3}$$

Hence the error $\varepsilon^{(r)}$ of $y^{(r)}$ is given by

$$\varepsilon^{(r)}=\sum_{s=1}^{n}\lambda_s^r x^{(s)} \quad \text{for } r=0,1,\cdots. \tag{2.4}$$

being defined by

$$\varepsilon^{(r)}=y-y^{(r)} \quad \text{for } r=0,1,\cdots. \tag{2.5}$$

3. Definition of Aitken's Δ^2-Process and Its Convergence Property

We first define Aitken's δ^2-process and then carry out the error analysis in applying Aitken's δ^2-process.

3.1 Definition of Aitken's Δ^2-Process

Now we denote by $y^{(m)}$, $y^{(m+1)}$ and $y^{(m+2)}$($m=0,1,\cdots$) the last three iterates obtained by iterating (2.1) ($m+2$) times, and by $\tilde{y}^{(m)}$ the improved vector. Then, Aitken's δ^2-process is defined as follows (see for example [2], [5]):

$$\tilde{y}^{(m)}=y^{(m+2)}+\omega(y^{(m+2)}-y^{(m)}) \tag{3.1}$$

where

$$\begin{aligned}&\omega=\bar{\lambda}_1^2/(1-\bar{\lambda}_1^2)\\&\bar{\lambda}_1^2=\|y^{(m+2)}-y^{(m+1)}\|^2/\|y^{(m+1)}-y^{(m)}\|^2\end{aligned} \tag{3.2}$$

and the symbol $\|\cdot\|$ denotes the square root of inner product.

Substituting the second of (3.2) into the first, we have the following formula:

$$\omega=-\frac{(y^{(m+2)}-y^{(m+1)},\ y^{(m+2)}-y^{(m+1)})}{(y^{(m+2)}-2y^{(m+1)}+y^{(m)},\ y^{(m+2)}-y^{(m)})}. \tag{3.3}$$

where the symbol (,) denotes the inner product.

3.2 Convergence Property of Aitken's Δ^2-Process

Now putting

$$\alpha_s=\lambda_s/\lambda_1 \text{ for } s=1,2,\cdots,n \tag{3.4}$$

and substituting (3.4) into (2.2), we have the following relation:

*Toyota National Technical College.

$$1>|\alpha_2|\geqq|\alpha_3|\geqq\cdots\geqq|\alpha_n|. \quad (3.5)$$

Substituting (3.4) into (2.4) in case of iterating (2.1) m times, we have the following formula:

$$\varepsilon^{(m)}=\sum_{s=1}^{n}\alpha_s^m\lambda_1^m x^{(s)}. \quad (3.6)$$

Now (3.1) and (3.3) may be written by use of (2.5) respectively as follows:

$$\tilde{\varepsilon}^{(m)}=\varepsilon^{(m+2)}+\omega(\varepsilon^{(m+2)}-\varepsilon^{(m)}) \quad (3.7)$$

$$\omega=-\frac{(\varepsilon^{(m+2)}-\varepsilon^{(m+1)},\ \varepsilon^{(m+2)}-\varepsilon^{(m+1)})}{(\varepsilon^{(m+2)}-2\varepsilon^{(m+1)}+\varepsilon^{(m)},\ \varepsilon^{(m+2)}-\varepsilon^{(m)})} \quad (3.8)$$

where $\quad \tilde{\varepsilon}^{(m)}=y-\tilde{y}^{(m)}. \quad (3.9)$

(3.8) may be written as follows with respective use of (3.6):

$$\omega=\frac{\lambda_1^2}{1-\lambda_1^2}\left\{1-\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}+O\left(\sum_{s=2}^{n}\sum_{t=2}^{n}\theta_s^2\theta_t^2\alpha_s^{2m}\alpha_t^{2m}\right)\right\}. \quad (3.10)$$

(3.7) may be written as follows by use of (3.6) and (3.10) (see [3]):

$$\tilde{\varepsilon}^{(m)}=\lambda_1^{m+2}\left[x^{(1)}\left\{\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}+O\left(\sum_{t=2}^{n}\sum_{s=2}^{n}\theta_t^2\theta_s^2\alpha_t^{2m}\alpha_s^m\right)\right\}\right.$$
$$\left.+\sum_{t=2}^{n}x^{(t)}\left\{-\alpha_t^m\frac{(1-\alpha_t^2)}{(1-\lambda_1^2)}+\alpha_t^m\frac{(1-\alpha_t^2\lambda_1^2)}{(1-\lambda_1^2)}\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}+O\left(\sum_{t=2}^{n}\sum_{s=2}^{n}\theta_t^2\theta_s^2\alpha_t^{2m}\alpha_s^m\right)\right\}\right] \quad (3.11)$$

where

$$\theta_t=\|x^{(t)}\|/\|x^{(1)}\| \quad \text{for } t=2,3,\cdots,n. \quad (3.12)$$

On the other hand in the case of iterating (2.1) $(m+2)$ times with the same initial value as that with which we have obtained (3.11), (3.6) is written as follows:

$$\varepsilon^{(m+2)}=\sum_{s=1}^{n}\alpha_s^{m+2}\lambda_1^{m+2}x^{(s)}$$
$$=\lambda_1^{m+2}x^{(1)}+\sum_{s=2}^{n}\alpha_s^{m+2}\lambda_1^{m+2}x^{(s)}. \quad (3.13)$$

Now let us consider the conditions under which the rate of convergence is improved whenever Aitken's δ^2-process is applied.

We introduce the following functions:

$$V_1(\lambda,\alpha,m)=\alpha^{2m}\frac{(1-\alpha^2)(1-\alpha\lambda)^2}{(1-\lambda^2)(1-\lambda)^2}$$
$$V_2(\lambda,\alpha,m)=|\alpha|^m\frac{(1-\alpha^2)}{(1-\lambda^2)} \quad (3.14)$$
$$V_3(\lambda,\alpha,m)=|\alpha|^m\frac{(1-\alpha^2\lambda^2)}{(1-\lambda^2)}$$

with the ranges of

$$-1<\lambda<1$$
and $\quad (3.15)$
$$-1\leqq\alpha\leqq1.$$

Now it may be considered that Aitken's δ^2-process is effective in improving the rate of convergence when the following inequalities are satisfied: By neglecting the small terms in size of (3.11) and (3.13), and comparing the coefficients of $x^{(1)}$, we have

$$\sum_{s=2}^{n}\theta_s^2V_1(\lambda_1,\alpha_s,m)<1. \quad (3.16)$$

Furthermore, since the second term of (3.11) must not dominate in magnitude the first term of (3.13), we have

$$\theta_t\left\{V_2(\lambda_1,\alpha_t,m)+V_3(\lambda_1,\alpha_t,m)\sum_{s=2}^{n}\theta_s^2V_1(\lambda_1,\alpha_s,m)\right\}<1$$
$$\text{for } t=2,3,\cdots,n. \quad (3.17)$$

However since it rarely happens that all of the eigenvalues of the iterative matrix C are known, Aitken's δ^2-process should be an acceleration process, where (3.16) and (3.17) hold even in the worst case such that $V_t(\lambda_1,\alpha,m)$, $t=1,2,3$ take the maximum values with respect to α.

We now denote the maximum value of $V_t(\lambda_1,\alpha,m)$, $t=1,2,3$ by $Mt(\lambda_1,\zeta,m)$, that is,

$$M_t(\lambda_1,\zeta,m)=\max_{\zeta\leqq\alpha\leqq1}V_t(\lambda_1,\alpha,m) \quad \text{for } t=1,2,3 \quad (3.18)$$

where ζ is a fixed value within $-1\leqq\zeta\leqq0$.

Then the following inequalities ((3.19)(3.20)) instead of (3.16) and (3.17) should be considered for practical applications:

$$\left(\sum_{s=2}^{n}\theta_s^2\right)M_1(\lambda_1,-1,m)<1 \quad (3.19)$$

$$\theta_t\left\{M_2(\lambda_1,-1,m)+M_3(\lambda_1,-1,m)\left(\sum_{s=2}^{n}\theta_s^2\right)M_1(\lambda_1,-1,m)\right\}<1 \quad \text{for } t=2,3,\cdots,n. \quad (3.20)$$

Here we estimate the magnitudes of $M_t(\lambda_1,-1,m)$, $t=1,2,3$, from the graphs of $V_t(\lambda_1,\alpha,m)$, $t=1,2,3$, which are shown in Figs. 1 to 8 in case of $m=1$ and $m=3$, varying the value of α with the range of -1 and 1 for each value of $\lambda_1=\pm0.85$ to ±0.99 with the increment of 0.01.

This is the reason why the quantity of ζ is introduced. As seen from Fig. 2, the value of $M_1(\lambda_1,\zeta,m)$ becomes extremely large in the case of $\zeta=-1$. Another acceleration process will be proposed later, where the value of ζ is so small in the absolute value, say, -0.40 that the magnitude of $M_1(\lambda_1,\zeta,m)$ becomes small.

Now we list the properties of $M_t(\lambda_1,-1,m)$, $t=1,2,3$, which will be discussed later for practical applications:

(i) When the value of λ_1 is positive and close to unity, the magnitude of $M_1(\lambda_1,-1,m)$ is extremely large.

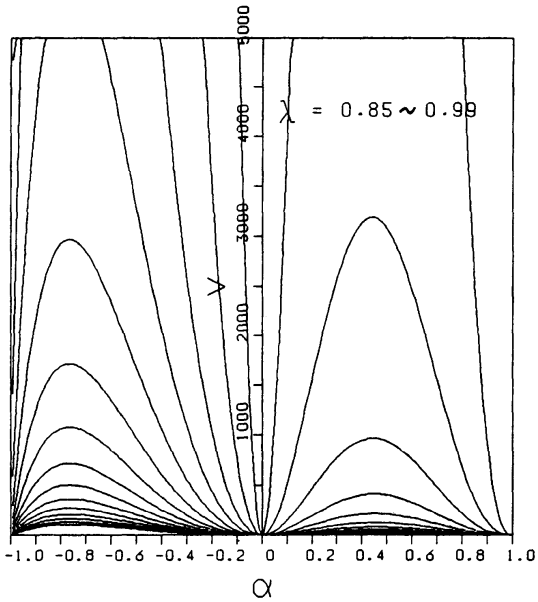

Fig. 1 The graph of the function.
$V_1(\lambda, \alpha, 1)=\alpha^2(1-\alpha^2)(1-\alpha\lambda)^2/(1-\lambda^2)(1-\lambda)^2$
for $\lambda=0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

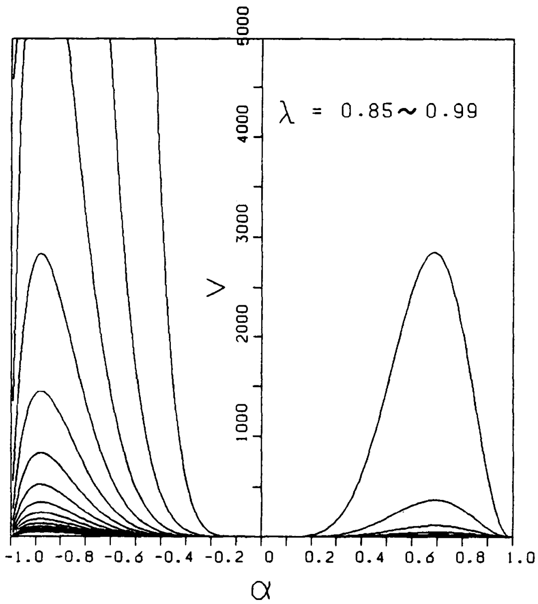

Fig. 2 The graph of the function.
$V_1(\lambda, \alpha, 3)=\alpha^6(1-\alpha^2)(1-\alpha\lambda)^2/(1-\lambda^2)(1-\lambda)^2$
for $\lambda=0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

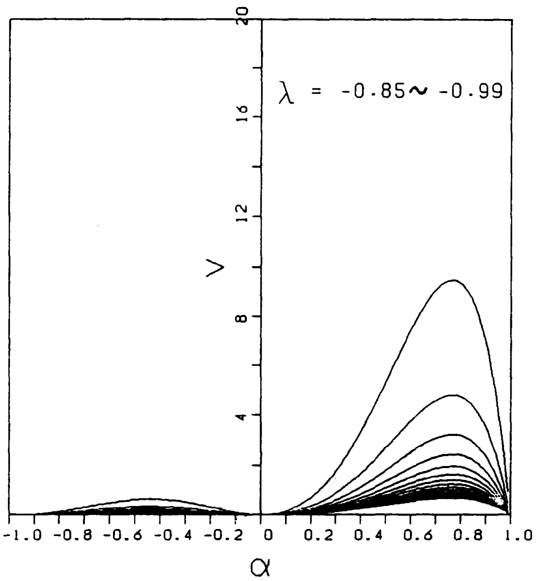

Fig. 3 The graph of the function.
$V_1(\lambda, \alpha, 1)=\alpha^2(1-\alpha^2)(1-\alpha\lambda)^2/(1-\lambda^2)(1-\lambda)^2$
for $\lambda=-0.85$ to -0.99 with the increment -0.01 from the lowest curve to the uppermost curve in order.

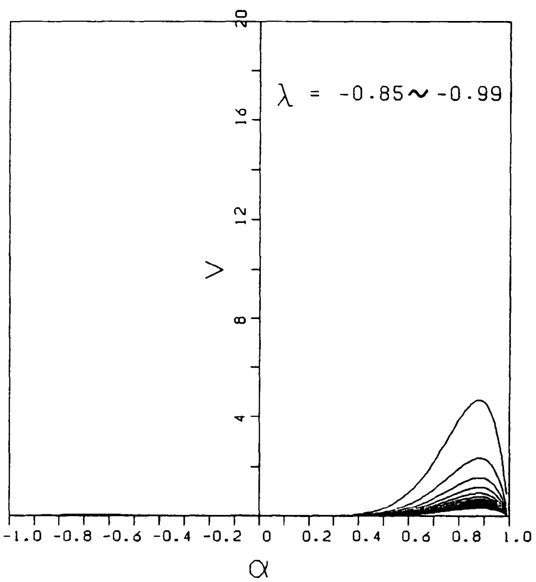

Fig. 4 The graph of the function.
$V_1(\lambda, \alpha, 3)=\alpha^6(1-\alpha^2)(1-\alpha\lambda)^2/(1-\lambda^2)(1-\lambda)^2$
for $\lambda=-0.85$ to -0.99 with the increment -0.01 from the lowest curve to the uppermost curve in order.

(ii) When the value of λ_1 is positive, both magnitudes of $M_2(\lambda_1, -1, m)$ and $M_3(\lambda_1, -1, m)$ are much smaller than that of $M_1(\lambda_1, -1, m)$.

(iii) When the value of λ_1 is positive and close to unity, and one or more of α_s, $s=1, 2, \cdots, n$, is negative and the absolute value is near unity, the magnitude of $M_1(\lambda_1, -1, m)$ is extremely large.

(iv) When the value of λ_1 is negative, the magnitude

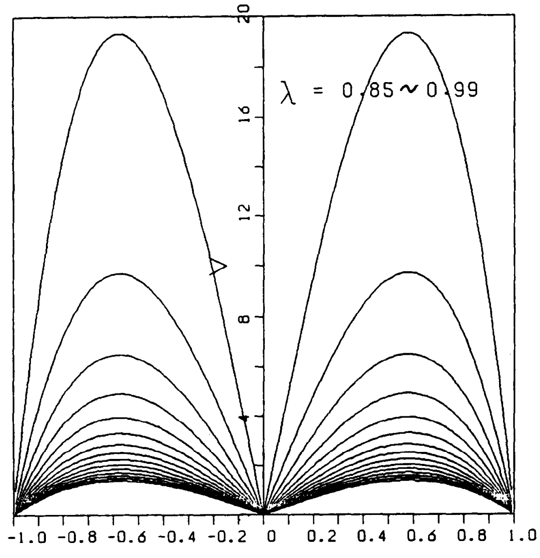

Fig. 5 The graph of the function.
$V_2(\lambda, \alpha, 1) = |\alpha|(1-\alpha^2)/(1-\lambda^2)$
for $\lambda = 0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

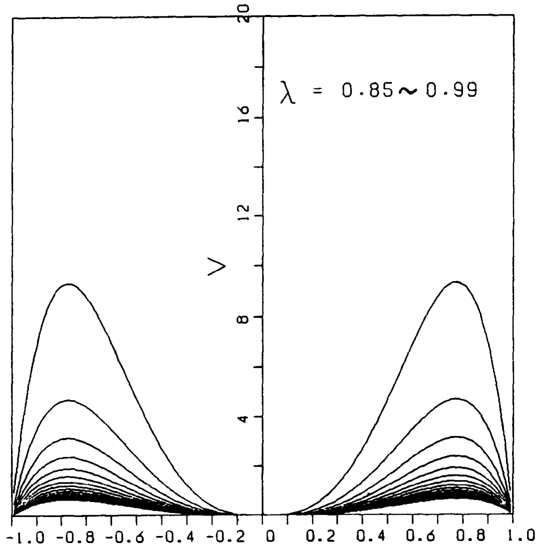

Fig. 6 The graph of the function.
$V_2(\lambda, \alpha, 3) = |\alpha|^3(1-\alpha^2)/(1-\lambda^2)$
for $\lambda = 0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

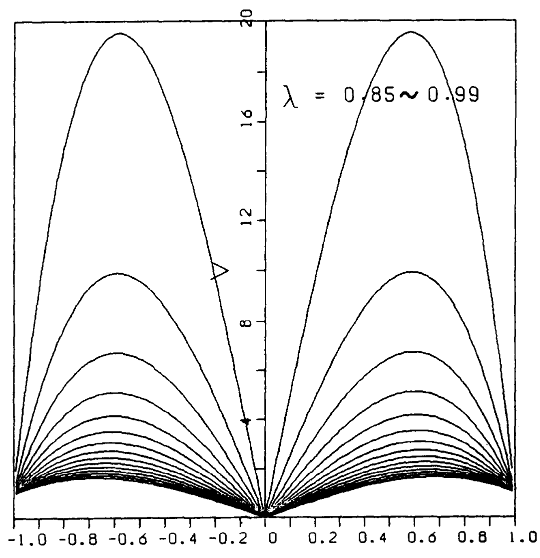

Fig. 7 The graph of the function.
$V_3(\lambda, \alpha, 1) = |\alpha|(1-\alpha^2\lambda^2)/(1-\lambda^2)$
for $\lambda = 0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

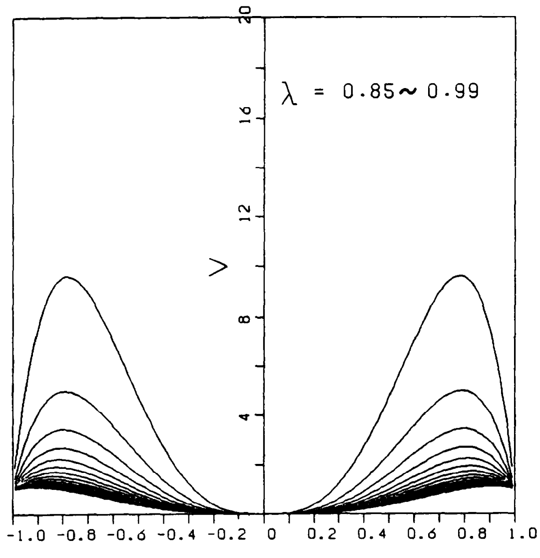

Fig. 8 The graph of the function.
$V_3(\lambda, \alpha, 3) = |\alpha|^3(1-\alpha^2\lambda^2)/(1-\lambda^2)$
for $\lambda = 0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

of $M_1(\lambda_1, -1, m)$ is much smaller than that of $M_1(-\lambda_1, -1, m)$.

(v) When the value of λ_1 is negative, both magnitudes of $M_2(\lambda_1, -1, m)$ and $M_3(\lambda_1, -1, m)$ are smaller than that of $M_1(\lambda_1, -1, m)$.

(vi) The magnitudes of $M_t(\lambda_1, -1, m)$, $t=1, 2, 3$, decrease monotonously as the value of m increases.

Now, (3.19) and (3.20) may be considered as the inequalities obtained at a certain step through the whole calculations, where interating (2.1) $(m+2)$ times with initial iterate $y^{(0)}$ and applying Aitken's δ^2-process to the last three iterates $y^{(m)}$, $y^{(m+1)}$ and $y^{(m+2)}$, we obtain the improved vector $\tilde{y}^{(m)}$. Successively we adopt the $\tilde{y}^{(m)}$ as initial iterate for (2.1) at the next step and repeat the same procedures.

From (3.9) and (3.11), the $\tilde{y}^{(m)}$ is given by

$$\tilde{y}^{(m)}=y+\sum_{t=1}^{n} a_t x^{(t)} \tag{3.21}$$

where

$$a_1 \simeq \lambda_1^{m+2}\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}$$

$$a_t \simeq \lambda_1^{m+2}\left\{-\alpha_t^m\frac{(1-\alpha_t^2)}{(1-\lambda_1^2)}+\alpha_t^m\frac{(1-\alpha_t^2\lambda_1^2)}{(1-\lambda_1^2)}\sum_{s=2}^{n}\theta_t^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}\right\} \quad \text{for } t=2, 3,\cdots, n. \tag{3.22}$$

Here we put, for $s=1, 2,\cdots, n$

$$\begin{aligned} x^{(s)} &:= a_s x^{(s)} \\ \theta_s &:= |a_s/a_1|\theta_s \end{aligned} \tag{3.23}$$

and we proceed with the analysis in the same way as (3.19) and (3.20) were obtained.

In the converging process, the quantities θ_t, $t=2, 3,\cdots, n$ vanish asymptotically as the number of steps increases. Hence by adopting a large value of m, (3.19) and (3.20) can be satisfied for any values of λ_1 and α_s. However, when in a later step the magnitude of θ_t reduces to a small value, the exceedingly large number of iterations of (2.1) must be carried out more than the adequate one that is required for (3.19) and (3.20) to be satisfied. Consequently during the calculations, Aitken's δ^2-process can not be applied. Therefore in all the calculations, the degree of the acceleration of convergence rate reduces considerably. Hence we set the value of m such that $1\leqq m\leqq 3$ through all the calculations. When we adopt such a value of m, it may frequently occur that (3.19) and (3.20) are not satisfied. However, even if the values of θ_t, $t=2, 3,\cdots, n$ at a certain step are not small enough for (3.19) and (3.20) to be satisfied, it may occur that the value of θ_t at the next step becomes small enough for (3.19) and (3.20) to be satisfied when the magnitude of $M_1(\lambda_1, -1, m)$ is comparatively small. When (3.19) and (3.20) are not satisfied at a certain step, it may be considered that the following inequality between the terms of (3.21) hold:

$$|a_t|\,\|x^{(t)}\|<|a_1|\,\|x^{(1)}\| \quad \text{for } t=2, 3,\cdots, n \tag{3.24}$$

and at the next step, as we iterate (2.1) with the initial iterate $\tilde{y}^{(m)}$ and take (2.2) into consideration, we have the value of θ_t small enough for (3.19) and (3.20) to be satisfied.

In all the calculations where Aitken's δ^2-process is applied repeatedly, we have the two following cases where (3.19) and (3.20) are not satisfied: (i) (3.19) and (3.20) are not satisfied successively. This occurs when the magnitude of $M_1(\lambda_1, -1, m)$ is extremely large, and consequently the acceleration of convergence rate is not achieved and the converging sequence even diverges. (ii) (3.19) and (3.20) are satisfied except for a few times. This occurs when the magnitude of $M_1(\lambda_1, -1, m)$ is comparatively small. Therefore in all the calculations, the acceleration of convergence rate is achieved to some extent.

It may be seen from the facts mentioned above that the magnitude of $M_1(\lambda_1, -1, m)$ should not be made extremely large if the acceleration of convergence rate should be gained.

Hence, by taking the properties of $M_1(\lambda_1, -1, m)$ into consideration, the two following conditions (which we refer to as 'condition' later on) should be satisfied:

(i) The absolute value of λ_1 should not be very close to unity.

(ii) Any negative value of α_s, $s=2, 3,\cdots, n$ should not be close to unity in absolute value, for example, larger than 0.60.

However both values of λ_1 and α_s are inherent quantities to the iterative matrix, which determine whether the acceleration of convergence rate is achieved or not. Hence it may be understood that it depends on the problem to be applied whether the acceleration of Aitken's δ^2-process is successful or not.

4. Modified Aitken's Δ^2-Process

In this section we consider the acceleration process which is applied successfully for any problem.

4.1 Analysis of a Modified Aitken's Δ^2-Process

We modify Aitken's δ^2-process to apply successfully to any problem.

We define for later use the polynomial $p_r(\lambda)$ by

$$p_r(\lambda)=\frac{T_r(\lambda c^{-1})}{T_r(c^{-1})}. \tag{4.1}$$

Here, $T_r(x)$ is Chebyshev polynomial generated by

$$T_{r+1}(x)=2xT_r(x)-T_{r-1}(x) \quad \text{for } r=1, 2,\cdots \tag{4.2}$$

with $T_0(x)=1$ and $T_1(x)=x$, where the variable x is defined in the range $|x|\leqq 1$, and c is a constant, such that $0<c<1$.

Of all polynomials of degree r which have the value unity at $\lambda=1$, the $p_r(\lambda)$ has the smallest deviation from zero in the range $|\lambda|\leqq c$, and that is the polynomial which increases or decreases monotonously and rapidly in the range $c\leqq|\lambda|<1$.

The right hand side of (4.1) can be expressed in the form

$$p_r(\lambda)=\sum_{t=0}^{r} b_{t,r}\lambda^t \tag{4.3}$$

where $b_{t,r}$, $t=0, 1,\cdots, r$, are constants.

We have the first few polynomials of (4.3) as follows:

$$\begin{aligned} p_1(\lambda) &= \lambda \\ p_2(\lambda) &= (2\lambda^2 - c^2)/(2-c^2) \\ p_3(\lambda) &= (4\lambda^3 - 3c^2\lambda)/(4-3c^3) \\ p_4(\lambda) &= (8\lambda^4 - 8c^2\lambda^2 + c^4)/(8-8c^2+c^4). \end{aligned} \tag{4.4}$$

Graphs of the $p_2(\lambda)$ with $c=0.80$ and the $p_4(\lambda)$ with $c=0.92$ are pictured in Fig. 9.

Now we define the new vector-valued sequence $\{z_k^{(0)}\}$ as follows:

$$\begin{aligned} z_0^{(t)} &= y^{(t)} \quad \text{for } t=0, 1, \cdots, r \\ z_k^{(0)} &= \sum_{t=0}^{r} b_{t,r} z_{k-1}^{(t)} \quad \text{for } k=1, 2, \cdots \end{aligned} \tag{4.5}$$

where $z_k^{(t)}$ are vectors obtained by iterating (2.1) t times with initial value $z_k^{(0)}$.

The general term of the sequence $\{z_k^{(0)}\}$ is given by (see for example [1] and [4])

$$z_k^{(0)} = y - \sum_{s=1}^{n} \{p_r(\lambda_s)\}^k x^{(s)}. \tag{4.6}$$

It follows that $z_k^{(0)} \to y$ as $k \to \infty$, since in the property of Chebyshev polynomials, $|p_r(\lambda_s)|<1$ for $s=1, 2, \cdots, n$.

Here we assume that q is a fixed integer, such that $2 \leqq q \leqq n$. Suppose that the value of c in (4.1) is specified in the following way with an appropriate choice of q:

$$1 > |\lambda_1| > |\lambda_2| \geqq \cdots \geqq |\lambda_q| \geqq c \geqq |\lambda_{q+1}| \geqq \cdots \geqq |\lambda_n|. \tag{4.7}$$

We can take any value for c as long as the value of q is chosen as mentioned above. Hence the value of q does not necessarily have to be specified.

From now on the degree r of the $p_r(\lambda)$ is assumed to be even (the reason will be mentioned later).

(4.6) is arranged by replacing λ_s in (2.3) by Chebyshev polynomial $p_r(\lambda_s)$. Hence Aitken's δ^2-process will be applied to the sequence $\{z_k^{(0)}\}$, not to the sequence $\{y^{(k)}\}$.

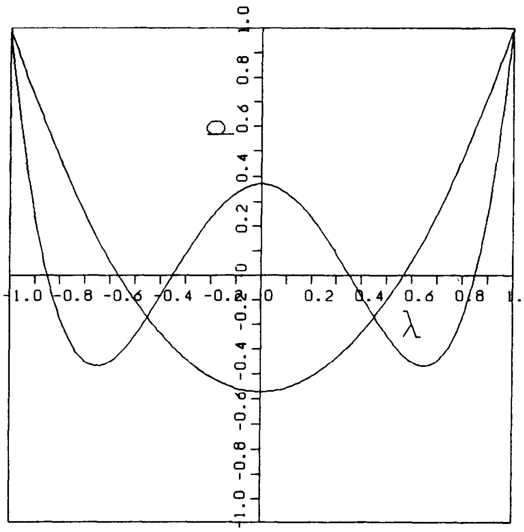

Fig. 9 The graphs of the functions.
$p_2(\lambda)$ for $c=0.8$ and $p_4(\lambda)$ for $c=0.92$

Then taking into consideration the property of the $p_r(\lambda)$ whose degree r is even, we have the following inequality:

$$1 > p_r(\lambda_1) > p_r(\lambda_2) \geqq \cdots \geqq p_r(\lambda_q) \geqq p_r(c) \geqq |p_r(\lambda_s)| \quad \text{for } s=q+1, q+2, \cdots, n. \tag{4.8}$$

The quantities, $p_r(\lambda_s)$ and $p_r(\lambda_s)/p_r(\lambda_1)$, $s=1, 2, \cdots, n$ in (4.6) correspond to those of λ_s and $\lambda_s/\lambda_1(=\alpha_s)$ in (2.3) respectively. However, the magnitudes of $p_r(\lambda_s)$, $s=q+1, q+2, \cdots, n$, are not in the same order as those of s.

Then we have the following inequality:

For $s=1, 2, \cdots, q$,

$$0 < p_r(\lambda_s) < |\lambda_s| \tag{4.9}$$

$$0 < \frac{p_r(\lambda_s)}{p_r(\lambda_1)} < \frac{|\lambda_s|}{|\lambda_1|}. \tag{4.10}$$

(4.9) is trivial so that we omit the proof. (4.10) can be shown as follows: Since the value of the $p_r(\lambda)$ increases or decreases monotonously and rapidly in the range $c \leqq |\lambda| < 1$, where the derivative of $p_r(\lambda)$ with respect to λ (which we denote by $p_r'(\lambda)$) is larger than unity in the absolute value, that is,

$$|p_r'(\lambda)| > 1. \tag{4.11}$$

Since the $p_r'(\lambda)$ is continuous in the range $c \leqq |\lambda| < 1$, we get (4.10), using the mean-value theorem of differential calculus together with (4.9).

Now it is easy to see from (4.8) that the following inequality holds:

For $s=q+1, q+2, \cdots, n$,

$$0 < \frac{|p_r(\lambda_s)|}{p_r(\lambda_1)} \leqq \frac{p_r(c)}{p_r(\lambda_1)}. \tag{4.12}$$

Any negative value of $p_r(\lambda_s)/p_r(\lambda_1)$, $s=q+1, q+2, \cdots, n$, is made small in the absolute value, since the quantity $p_r(c)/p_r(\lambda_1)$ becomes small with a suitable choice of c. Hence the value of ζ of (3.18) is chosen as follows:

$$\zeta = -\frac{p_r(c)}{p_r(\lambda_1)}. \tag{4.13}$$

Then we have for $s=2, 3, \cdots, n$,

$$\zeta \leqq p_r(\lambda_s)/p_r(\lambda_1) \leqq 1 \tag{4.14}$$

in which the value of ζ should be chosen larger than about -0.40 if possible, because as can be seen from Fig. 2, if ζ exceeds the said value, the magnitude of M_1 $(p_r(\lambda_1), \zeta, 3)$ becomes extremely large.

With a smaller value for c in (4.13), the (ii) of 'condition' is satisfied on the one hand, but the (i) of 'condition' is not satisfied on the other hand with the value of $p_r(\lambda_1)$ increasing.

With a larger value for c the results are otherwise.

Hence it may be seen that there exists the appropriate value of c.

We notice in passing that, as seen from Fig. 1, the magnitude of $M_1(p_r(\lambda_1), \zeta, 1)$ does not become small even if the value of ζ is larger than -0.40.

4.2 Determination of the Constants

We determine the degree r of $p_r(\lambda)$ and the value of c in (4.1).

It is desirable for these values to be determined for every problem. However all of the eigenvalues of an iterative matrix are rarely known and it is nearly impossible to have information on these values during the calculations. Hence we determine in advance the values of r and c. We first determine the value of r, since c depends on r.

The value of r should be as large as possible, but cannot be made large, because a large value of r causes the loss of significant value, since the approximate value of $p_r(\lambda_1)$ is required during the calculations for the acceleration process to be applied. As seen from (4.4), the coefficients in Chebyshev polynomials appear with alternating positive and negative signs. In addition, the degree r should be specified for the (ii) of 'condition' to be satisfied.

Hence the most desirable value for the degree of $p_r(\lambda)$ is $r=4$ and $r=2$ at best.

In determining the value of c, we can choose it more freely than that of r.

With a larger value of c, the value of ζ in (4.13) becomes large, and with a smaller value of c, the value of $p_r(\lambda_1)$ becomes large, and these values enlarge the magnitude of $M_1(p_r(\lambda_1), \zeta, m)$.

Hence the value of c should be properly determined in order for the magnitude of $M_1(p_r(\lambda_1), \zeta, m)$ to be small.

We have the following results, calculating the values of $p_r(c)/p_r(\lambda_1)(=|\zeta|)$ with $r=2$ and 4 by varying the value of λ_1 for some values of c:

Here we consider only the case of $m=3$, since the order r of $p_r(\lambda_1)$ is limited now so that the magnitude of $p_r(\lambda_1)$ is not desired to be small when the value of λ_1 is very close to unity.

The appropriate value for c may be roughly estimated from Table 1 and Fig. 2. Here are a few examples:

$$c=0.80 \quad \text{for} \quad r=2 \quad \text{and} \quad c=0.92 \quad \text{for} \quad r=4.$$

4.3 Algorithms of Modified Aitken's Δ^2-Process

In this section we propose three algorithms for the modified Aitken's δ^2-process, based on the analysis described in the preceding sections.

(i) Algorithm of AC3P1 (which is the case of $m=1$ and $r=1$, and which is one of Aitken's δ^2-process).

(1) Compute $y^{(1)}$, $y^{(2)}$ and $y^{(3)}$, using (2.1) with initial value $y^{(0)}$.

(2) Compute $\tilde{y}^{(1)}$, using (3.1) with $y^{(1)}$, $y^{(2)}$ and $y^{(3)}$.

(3) Set $y^{(0)}:=\tilde{y}^{(1)}$.

(4) Repeat the above procedures from (1) to (3) until required accuracy is attained.

(ii) Algorithm of AC5P2 (which is the case of $m=3$ and $r=2$).

(1) Compute $y^{(1)}$ and $y^{(2)}$, using (2.1) with initial value $y^{(0)}$.

(2) Compute $z_1^{(0)}$, using (4.5) and $p_2(\lambda)$ of $c=0.80$ for $y^{(0)}$, $y^{(1)}$ and $y^{(2)}$.

(3) Set $y^{(0)}:=z_1^{(0)}$.

(4) Compute $z_2^{(0)}$, $z_3^{(0)}$, $z_4^{(0)}$ and $z_5^{(0)}$, repeating the procedures from (1) to (3), and set $y^{(t)}:=z_t^{(0)}$ for $t=1, 2, 3, 4, 5$.

(5) Compute $\tilde{y}^{(3)}$, using (3.1) with $y^{(3)}$, $y^{(4)}$ and $y^{(5)}$.

(6) Set $y^{(0)}:=\tilde{y}^{(3)}$.

(7) Repeat the above procedures from (1) to (6) until required accuracy is attained.

(iii) Algorithm of AC5P4 (which is the case of $m=3$ and $r=4$).

The algorithm of AC5P4 is the same as that of AC5P2 except for items (1) and (2), which are replaced respectively by

(1) Compute $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ and $y^{(4)}$, using (2.1) with initial value $y^{(0)}$.

(2) Compute $z_1^{(0)}$, using (4.5) and $p_4(\lambda)$ of $c=0.92$ for $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ and $y^{(4)}$.

4.4 Use of Modified Aitken's Δ^2-Process

We show which acceleration process is to be applied with respect to the magnitude of the absolutely largest eigenvalue λ_1 of an iterative matrix, taking into consideration the error analysis in the preceding sections and the numerical examination of three acceleration processes for some problems.

(i) AC5P2 is recommended in the case of $|\lambda_1|\leqq 0.95$.

Table 1 Magnitude of ζ for some values of c.

c	λ_1	$p_2(\lambda_1)$	$p_2(c)/p_2(\lambda_1)$	c	λ_1	$p_4(\lambda_1)$	$p_4(c)/p_4(\lambda_1)$
0.78	0.999	0.997	0.438	0.90	0.999	0.991	0.304
	0.998	0.994	0.440		0.998	0.983	0.307
	0.997	0.991	0.441		0.997	0.974	0.310
0.80	0.999	0.997	0.472	0.92	0.999	0.991	0.372
	0.998	0.994	0.473		0.998	0.981	0.375
	0.997	0.991	0.475		0.997	0.972	0.379
0.82	0.999	0.997	0.508	0.94	0.999	0.990	0.461
	0.998	0.994	0.510		0.998	0.979	0.464
	0.997	0.991	0.511		0.997	0.969	0.471

(ii) AC5P4 is recommended in the case of $|\lambda_1|>0.95$.

5. Numerical Results and Discussion

We give the results of numerical experiments on five examples.

All of our computations were performed with a FACOM M-200 at the University of Nagoya in double precision for 16 significant decimal figures.

The convergence test

$$|y_i^{(k+1)}-y_i^{(k)}|\leqq\varepsilon,\quad i=1,2,\cdots,n \tag{5.1}$$

can be used to stop the iterations, where $y_i^{(k)}$ is the component of the column vector $y^{(k)}$, and the initial value $y^{(0)}$ was taken $y_1^{(0)}=1$ and $y_i^{(0)}=0$, $i=2,3,\cdots,n$ as the components.

In all examples, the components of the column vector d in (2.1) were taken $d_i=0.01$, $i=1,2,\cdots,n$.

Example 1. The iterative matrix C is (5×5)-symmetric and its elements are as follows:

$c_{11}=\ \ 0.88490\ 413$ $\quad c_{25}=\ \ 0.01236\ 315$
$c_{12}=-0.09917\ 199$ $\quad c_{33}=\ \ 0.87263\ 177$
$c_{13}=-0.01856\ 947$ $\quad c_{34}=-0.06850\ 584$
$c_{14}=\ \ 0.00627\ 197$ $\quad c_{35}=\ \ 0.01737\ 416$
$c_{15}=\ \ 0.00928\ 712$ $\quad c_{44}=\ \ 0.87757\ 832$
$c_{22}=\ \ 0.86642\ 168$ $\quad c_{45}=\ \ 0.09634\ 722$
$c_{23}=-0.08498\ 760$ $\quad c_{55}=\ \ 0.88646\ 410$
$c_{24}=-0.02231\ 328$

C has the values: $\lambda_1=0.998$, $\alpha_2\simeq0.992$, $\alpha_3\simeq0.902$, $\alpha_4\simeq0.802$, $\alpha_5\simeq0.701$.

Table 2 Numerical results for example 1 with $\varepsilon=10^{-5}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	2,833	694	658	316
Computing times (ms)	55	17	17	8

Table 3 Numerical results for example 1 with $\varepsilon=10^{-9}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	7,434	1,234	1,458	636
Computing times (ms)	142	30	40	16

Our experiments show that AC5P4 converges in little more than half the number of iterations required for AC3P1, because the magnitude of $M_1(P_4(0.998),\zeta,3)$ is very small, compared with that of $M_1(0.998,-1,1)$.

Example 2. The iterative matrix C is (5×5)-symmetric and its elements are as follows:

$c_{11}=\ \ 0.71080\ 472$ $\quad c_{25}=0.26800\ 924$
$c_{12}=-0.37433\ 313$ $\quad c_{33}=0.49701\ 126$
$c_{13}=0.27429\ 454$ $\quad c_{34}=-0.38407\ 153$
$c_{14}=0.20856\ 735$ $\quad c_{35}=-0.25496\ 262$
$c_{15}=0.17103\ 900$ $\quad c_{44}=\ \ 0.61246\ 579$
$c_{22}=0.43153\ 421$ $\quad c_{45}=-0.29594\ 966$
$c_{23}=0.48915\ 669$ $\quad c_{55}=\ \ 0.73618\ 402$
$c_{24}=0.36186\ 315$

C has the values: $\lambda_1=0.998$, $\alpha_2\simeq0.992$, $\alpha_3\simeq0.902$, $\alpha_4\simeq0.802$ and $\alpha_5\simeq-0.701$.

Table 4 Numerical results for example 2 with $\varepsilon=10^{-5}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	2,971	—	538	256
Computing times (ms)	57	—	14	7

Table 5 Numerical results for example 2 with $\varepsilon=10^{-9}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	7,572	—	1,018	476
Computing times (ms)	145	—	27	12

Our experiments show that AC3P1 fails in accelerating the rate of convergence. This is because the magnitude of $M_1(0.998,-1,1)$ with $\alpha_5\simeq-0.701$ is extremely large, and furthermore the solution vector $y^{(k)}$ does not converge regularly within the requested tolerance, since the rule of convergence is different with the number of iterations k being either odd or even, when α_5 is negative and its absolute value is near unity.

Example 3. The iterative matrix C is (5×5)-symmetric and its elements are as follows:

$c_{11}=\ \ 0.74811\ 173$ $\quad c_{25}=\ \ 0.21322\ 794$
$c_{12}=-0.31537\ 003$ $\quad c_{33}=\ \ 0.57750\ 137$
$c_{13}=\ \ 0.21949\ 631$ $\quad c_{34}=-0.31645\ 032$
$c_{14}=\ \ 0.16253\ 035$ $\quad c_{35}=-0.20405\ 081$
$c_{15}=\ \ 0.13637\ 788$ $\quad c_{44}=\ \ 0.66927\ 562$
$c_{22}=\ \ 0.52472\ 438$ $\quad c_{45}=-0.25317\ 770$
$c_{23}=\ \ 0.40254\ 903$ $\quad c_{55}=\ \ 0.76838\ 690$
$c_{24}=\ \ 0.28910\ 247$

C has the values: $\lambda_1=0.998$, $\alpha_2\simeq0.992$, $\alpha_3\simeq0.902$, $\alpha_4\simeq0.802$ and $\alpha_5\simeq-0.401$.

Table 6 Numerical results for example 3 with $\varepsilon=10^{-5}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	2,971	955	538	256
Computing times (ms)	57	23	15	6

Table 7 Numerical results for example 3 with $\varepsilon=10^{-9}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	7,572	2,836	1,018	476
Computing times (ms)	146	66	28	12

The eigenvalues of the iterative matrix C for example 3 are all the same values except $\lambda_5(=\alpha_5\lambda_1)$ to those for example 1, but our experiments show that, concerning AC3P1 the number of iterations required for example 3 is more than that for example 1. This is because the magnitude of $M_1(0.998, -1, 1)$ with $\alpha_5 \simeq -0.401$ is large, compared with that of $M_1(0.998, -1, 1)$ with $\alpha_5 \simeq 0.701$.

Example 4. The iterative matrix C is (5×5)-symmetric and its elements are as follows:

$$\begin{array}{llll} c_{11}= & 0.55253\ 013 & c_{25}= & -0.33460\ 717 \\ c_{12}= & 0.26976\ 551 & c_{33}= & 0.26608\ 227 \\ c_{13}= & -0.36349\ 491 & c_{34}= & 0.35698\ 674 \\ c_{14}= & 0.29873\ 510 & c_{35}= & 0.33771\ 504 \\ c_{15}= & 0.22991\ 875 & c_{44}= & 0.42316\ 147 \\ c_{22}= & 0.18183\ 853 & c_{45}= & -0.16999\ 616 \\ c_{23}= & 0.46011\ 816 & c_{55}= & 0.58638\ 760 \\ c_{24}= & -0.48264\ 589 & & \end{array}$$

C has the values: $\lambda_1=-0.990$, $\alpha_2\simeq-0.909$, $\alpha_3\simeq-0.808$, $\alpha_4\simeq-0.707$, $\alpha_5\simeq-0.606$.

Table 8 Numerical results for example 4 with $\varepsilon=10^{-5}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	1,054	—	148	76
Computing times (ms)	20	—	4	2

Table 9 Numerical results for example 4 with $\varepsilon=10^{-9}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	1,971	—	328	256
Computing times (ms)	38	—	9	6

Our experiments show that AC3P1 fails in accelerating the rate of convergence. This is because the irregularity of the convergence due to $\lambda_1=-0.990$ occurring with the number of iterations being either odd or even.

Example 5. The iterative matrix C is (5×5)-symmetric and its elements are as follows:

$$\begin{array}{llll} c_{11}= & 0.36339\ 968 & c_{25}= & 0.01964\ 527 \\ c_{12}= & -0.11152\ 921 & c_{33}= & 0.04565\ 506 \\ c_{13}= & -0.55556\ 445 & c_{34}= & -0.08029\ 714 \\ c_{14}= & -0.00102\ 918 & c_{35}= & 0.55509\ 918 \\ c_{15}= & 0.43531\ 106 & c_{44}= & 0.79053\ 170 \\ c_{22}= & 0.78446\ 830 & c_{45}= & -0.10659\ 578 \\ c_{23}= & 0.09994\ 472 & c_{55}= & 0.36594\ 526 \\ c_{24}= & 0.01212\ 678 & & \end{array}$$

C has the values: $\lambda_1=0.95$, $\alpha_2\simeq0.947$, $\alpha_3\simeq-0.842$, $\alpha_4\simeq0.737$, $\alpha_5\simeq0.632$.

Table 10 Numerical results for example 5 with $\varepsilon=10^{-5}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	142	—	58	56
Computing times (ms)	3	—	2	1

Table 11 Numerical results for example 5 with $\varepsilon=10^{-9}$.

Acceleration process	Not applied	AC3P1	AC5P2	AC5P4
Iteration numbers	321	—	108	116
Computing times (ms)	6	—	3	3

Our experiments show that AC3P1 fails in accelerating the rate of convergence with the same reasons as for example 2, and the number of iterations for AC5P2 is nearly equal to that for AC5P4.

Acknowledgement

The author is grateful to Professor Ichizo Ninomiya and Professor Tatuo Torii for some comments and suggestions to the draft of this paper, and also the author wishes to thank the referee for his comments and suggestions.

References

1. FOX, L. and PARKER, I. B. Chebyshev Polynomials in Numerical Analysis, Oxford University Press, Oxford, 1968.
2. HENRICI, P. Elements of Numerical Analysis, John Wiley & Sons, London, 1964.
3. IGUCHI, K. An Algorithm of an Acceleration Process Covering the Aitken's δ^2-Process., Inf. Process. in Japan, 16 (1976), 89–93.
4. ISSACSON, E. and KELLER, H. B. Analysis of Numerical Methods, John Wiley & Sons, New York, 1966.
5. JENNINGS, A. Matrix Computation for Engineers and Scientists, John Wiley & Sons, London, 1977.

(Received February 7, 1983: revised November 14, 1983)