Short Note

Acceleration Process for a Positive Definite Iterative Matrix

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Accelerating the rate of the convergence was studied in the context [2] for the iterative process: $y^{(r)}=Cy^{(r-1)}+d$, where the iterative matrix C is a real symmetric. In this paper, an acceleration process is proposed for a real, symmetric and positive definite iterative matrix C. The numerical results for three examples are given to demonstrate the efficiency.

1. Convergence Property of the Iterative Process

We consider in this paper the question of accelerating the convergence rate of the iterative process:

$$y^{(r)}=Cy^{(r-1)}+d \tag{1}$$

where the iterative matrix C is $(n\times n)$-positive definite and d is an n-th column vector. We assume the sequence $\{y^{(r)}\}$, which is generated from (1) has a limiting vector y.

Let us suppose that the matrix C has eigenvalues λ_s $(s=1, 2, \ldots, n)$ with the relation:

$$1>\lambda_1>\lambda_2\geqq\lambda_3\geqq \ldots \geqq\lambda_n>0 \tag{2}$$

and that $x^{(s)}$ $(s=1, 2, \ldots, n)$ are the eigenvectors corresponding to the λ_s.

Then the general term of the sequence $\{y^{(r)}\}$ is given by

$$y^{(r)}=y-\sum_{s=1}^{n}\lambda_s^r x^{(s)} \tag{3}$$

2. Aitken's Δ^2-Process and Applicable Acceleration Process

We consider the calculating process in which the Aitken's δ^2-process, which is defined below by (4), is applied once after every $(m+2)$ $(m=0, 1, \ldots)$ iterations of (1). That is,

$$\tilde{y}^{(m)}=y^{(m+2)}+\omega(y^{(m+2)}-y^{(m)}) \tag{4}$$

where

$$\omega=\tilde{\lambda}_1^2/(1-\tilde{\lambda}_1^2)$$
$$\tilde{\lambda}_1^2=\|y^{(m+2)}-y^{(m+1)}\|^2/\|y^{(m+1)}-y^{(m)}\|^2, \tag{5}$$

denoting by $\|.\|^2$ the inner product.

Then the improved value $\tilde{y}^{(m)}$ can be obtained in the following form (see [2]):

$$\tilde{y}^{(m)}=y-\lambda_1^{m+2}\Bigg[x^{(1)}\Bigg\{\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}$$
$$+O(\sum_{t=2}^{n}\sum_{s=2}^{n}\theta_t^2\theta_s^2\alpha_t^{2m}\alpha_s^m)\Bigg\}$$
$$+\sum_{t=2}^{n}x^{(t)}\Bigg\{-\alpha_t^m\frac{(1-\alpha_t^2)}{(1-\lambda_1^2)}$$
$$+\alpha_t^m\frac{(1-\alpha_t^2\lambda_1^2)}{(1-\lambda_1^2)}\sum_{s=2}^{n}\theta_s^2\alpha_s^{2m}\frac{(1-\alpha_s^2)(1-\alpha_s\lambda_1)^2}{(1-\lambda_1^2)(1-\lambda_1)^2}$$
$$+O(\sum_{t=2}^{n}\sum_{s=2}^{n}\theta_t^2\theta_s^2\alpha_t^{2m}\alpha_s^m)\Bigg\}\Bigg] \tag{6}$$

where $\theta_t=\|x^{(t)}\|/\|x^{(1)}\|$ for $t=2, 3, \ldots, n$.

The magnitude of the second terms in (6) becomes extremely large with the value of λ_1 close to unity as shown in Fig. 2.

Hence we considered, in the context [2], the acceleration scheme for the iterative matrix being real and symmetric, which used Chebyshev polynomials of the r-th degree, $p_r(\lambda_1)$ instead of λ_1 to reduce the magnitude of the second term and the other error terms in (6).

However the acceleration scheme with shifted Chebyshev polynomials for the iterative matrix being real, symmetric and positive definite might be preferable to that with ordinary Chebyshev polynomials, because the former, in addition to its simplicity, can be more frequently applied during the computations than the latter.

Therefore, if the Aitken's δ^2-process is to be used successfully, it should be applied to the sequence $\{z_k^{(0)}\}$ defined by

$$z_0^{(t)}=y^{(t)} \quad (t=0, 1, \ldots, r)$$
$$z_k^{(0)}=\sum_{t=0}^{r}b_{t,r}z_{k-1}^{(t)} \quad (k=1, 2, \ldots) \tag{7}$$

instead of the $\{y^{(k)}\}$, where $z_k^{(t)}$ are vectors obtained by iterating (1) t times with initial value $z_k^{(0)}$ and the $b_{t,r}$ are the coefficients of the shifted Chebyshev polynomials which are defined by,

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$$p_r(\lambda)=\frac{T_r^*(\lambda c^{-1})}{T_r^*(c^{-1})}=\sum_{t=0}^{r} b_{t,r}\lambda^t \tag{8}$$

in the range of $0<\lambda<1$.

Here we assume that the constant c in (8) is a real value in the range of $0<c<\lambda_1$, and the $T_r^*(\lambda)$ are the shifted Chebyshev polynomials which have the relation (see [1]):

$$T_{r+1}^*(\lambda)=2(2\lambda-1)T_r^*(\lambda)-T_{r-1}^*(\lambda) \tag{9}$$

with $T_0^*(\lambda)=1$ and $T_1^*(\lambda)=2\lambda-1$.

The first three $p_r(\lambda)$ are given as follows:

$$\begin{aligned} p_1(\lambda)&=(2\lambda-c)/(2-c)\\ p_2(\lambda)&=(8\lambda^2-8c\lambda+c^2)/(8-8c+c^2)\\ p_3(\lambda)&=(32\lambda^3-48c\lambda^2+18c^2\lambda-c^3)/\\ &\quad(32-48c+18c^2-c^3). \end{aligned} \tag{10}$$

A graph of $p_2(\lambda)$ with $c=0.82$ is pictured in Fig. 1.

Then the general term of the sequence $\{z_k^{(0)}\}$ is given by, after a little manipulation,

$$z_k^{(0)}=y-\sum_{s=1}^{n}\{p_r(\lambda_s)\}^k x^{(s)}. \tag{11}$$

Now, putting

$$\alpha_s=\frac{p_r(\lambda_s)}{p_r(\lambda_1)} \quad (s=1,\ 2,\ \ldots,\ n) \tag{12}$$

$$\zeta=-\frac{p_r(c)}{p_r(\lambda_1)}, \tag{13}$$

we have, for $s=1,\ 2,\ \ldots,\ n$ (see [1]),

$$\zeta\leqq\alpha_s\leqq 1. \tag{14}$$

Here we difine the following function on the range, $-1\leqq\alpha\leqq 1$:

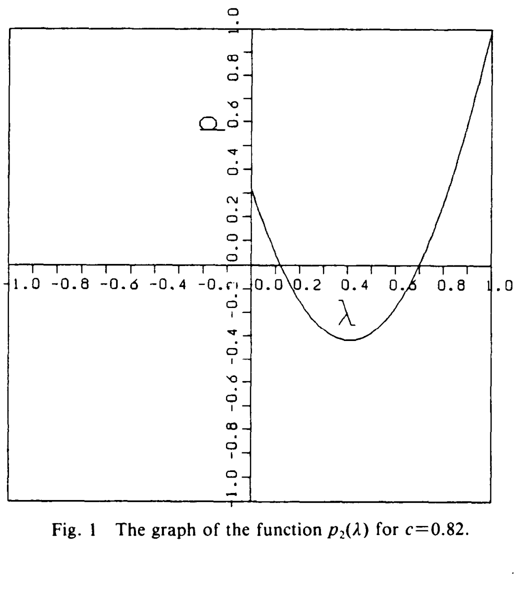


Fig. 1 The graph of the function $p_2(\lambda)$ for $c=0.82$.

$$V(\lambda,\ \alpha,\ m)=\alpha^{2m}\frac{(1-\alpha^2)(1-\alpha\lambda)^2}{(1-\lambda^2)(1-\lambda)^2}. \tag{15}$$

A graph of $V(\lambda,\ \alpha,\ m)$ is pictured in Fig. 2 for each λ between 0.85 to 0.99 with the increment 0.01, changing α from -1 to 1.

Now, we denote by $M(p_r(\lambda_1),\ \zeta,\ m)$ the maximum value of $V(p_r(\lambda_1),\ \alpha,\ m)$ with fixed m and r on the range $\zeta\leqq\alpha\leqq 1$. That is,

$$M(p_r(\lambda_1),\ \zeta,\ m)=\max_{\zeta\leqq\alpha\leqq 1}\{V(p_r(\lambda_1),\ \alpha,\ m)\}. \tag{16}$$

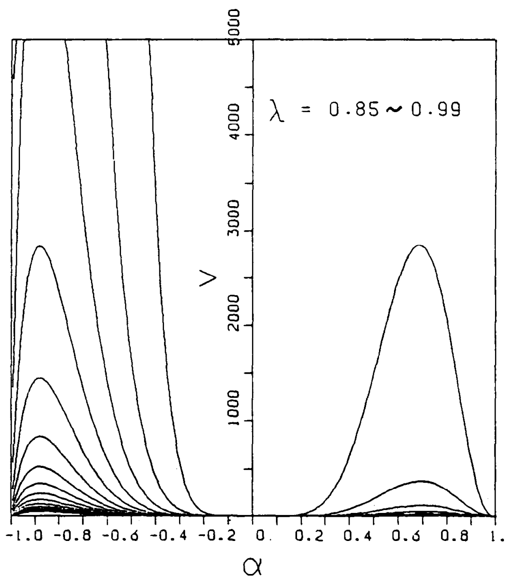
The magnitude of $M(p_r(\lambda_1),\ \zeta,\ m)$ should be made as small as possible if the present acceleration scheme is to be applied successfully.

Here we choose $r=2$ and $m=3$ for the same reasons as shown in the context [2] for the real symmetric iterative matrix.

As shown in Fig. 2, if the negative value of α might be restricted within a larger value than about -0.4, the largest value of $V(\lambda_1,\ \alpha,\ 3)$ in that range of α becomes smaller than or equal to that in the positive range of α for each value of λ_1.

It is known from numerical experiments (see [2]) that the Aitken's δ^2-process can be applied successfully, in the case that the iterative matrix C has eigenvalues, such that, $\alpha_i(=\lambda_i/\lambda_1)\geqq-0.4$, $i=2,\ 3,\ \ldots,\ n$ for each value of λ_1.

Hence when the value of ζ is taken as $\zeta=-0.4$, the following inequality should be satisfied



$$V(\lambda,\ \alpha,\ 3)=\alpha^6\frac{(1-\alpha^2)(1-\alpha\lambda)^2}{(1-\lambda^2)(1-\lambda)^2}$$

for $\lambda=0.85$ to 0.99 with the increment 0.01 from the lowest curve to the uppermost curve in order.

Fig. 2 The graph of the function.

$$\zeta = -\frac{p_2(c)}{p_2(\lambda_1)} = -\frac{c^2}{c^2 - 8c\lambda_1 + \lambda_1^2} \geqq -0.4. \tag{17}$$

We get the following result from (17) together with $0<c<\lambda_1$ after a little manipulation,

$$0 < c \leqq \frac{-8+4\sqrt{7}}{3}\lambda_1 \simeq 0.86\lambda_1. \tag{18}$$

Here the best choice of c to reduce the magnitude of $M(p_2(\lambda_1), \zeta, 3)$ may be the largest one within the allowable range of (18), that is,

$$c = 0.86\lambda_1 \tag{19}$$

which makes $p_2(\lambda_1)$ smallest in the range.

We noticed, from many numerical results, that small variations of the value of c caused much effect on the efficiency of the acceleration process. So, here, we set $c=0.82$ for every value of λ_1 between 0.92 and 0.999.

3. Algorithm of the Acceleration Process

We, from now on, will refer to the acceleration process as AC5SP2. It has the following algorithm:

(1) Calculate $y^{(1)}$ and $y^{(2)}$, using equation (1) with initial value $y^{(0)}$.
(2) Calculate $z_1^{(0)}$, using equation (7) with $y^{(0)}$, $y^{(1)}$ and $y^{(2)}$.
(3) Set $y^{(0)} := z_1^{(0)}$ and calculate $y^{(1)}$ and $y^{(2)}$ from (1).
(4) Calculate $z_t^{(0)}$, $t=2, 3, 4, 5$ in similar way to $z_1^{(0)}$ setting $y^{(0)}$ to the newest $z_t^{(0)}$ like in (3).
(5) Set $y^{(t)} := z_t^{(0)}$, $t=3, 4, 5$.
(6) Calculate $\tilde{y}^{(3)}$, using equation (4) with $y^{(3)}$, $y^{(4)}$ and $y^{(5)}$.
(7) Set $y^{(0)} := \tilde{y}^{(3)}$.
(8) Repeat from (1) to (7) until the required accuracy is attained.

4. Numerical Results

We show numerical results of three examples to demonstrate the efficiency of AC5SP2.

All calculations were performed in double precision of 16 significant figures in decimal on an FACOM M-382 computer at the University of Nagoya.

We here use the convergence test

$$|y_t^{(k+1)} - y_t^{(k)}| \leqq \varepsilon, \ t=1, 2, \ldots, 30 \tag{20}$$

to stop the iterations, where ε is an error tolerance.

Initial values for all examples were taken as

$$y^{(0)} = (1, 0, \ldots, 0)^T,$$

denoting by T the transpositions of the raw vector to the column.

The elements of the column vector d in (1) were taken as $d_t=0.01$, $t=1, 2, \ldots, 30$ for all examples, and the iterative matrices C in (1) are set up below for three examples.

We first explain the matrices required to set up the iterative matrices C.

A is a 30×30 real symmetric matrix and its elements a_{ij} are given as follows:

$$a_{ij} = \begin{cases} 6 & \text{for} \quad i=j \\ 3 & \text{for} \quad |i-j|=1 \\ 1 & \text{for} \quad |i-j|=2 \\ 1 & \text{for} \quad |i-j|=3 \\ 0 & \text{for} \quad |i-j|\geqq 4. \end{cases}$$

X is a 30×30 orthogonal matrix which is composed of the n eigenvectors corresponding to the eigenvalues of A, computed by Jennings method (see [3]) in decreasing order of size.

B is a 30×30 diagonal matrix and its diagonal elements b_{ii} (>0) are varied for each example. The values used are given in the examples.

Now, matrix C is given by

$$C = XBX^T$$

which is a 30×30 real, symmetric and positive definite matrix.

Example 1. The matrix B is a 30×30 diagonal matrix and its elements are given as follows:

$b_{11}=0.999$ $b_{22}=0.998$ $b_{33}=0.997$ $b_{44}=0.996$
$b_{55}=0.995$ $b_{66}=0.994$ $b_{77}=0.993$ $b_{88}=0.992$
$b_{99}=0.991$ $b_{10\,10}=0.99$ $b_{11\,11}=0.95$ $b_{12\,12}=0.90$
$b_{13\,13}=0.85$ $b_{14\,14}=0.80$ $b_{15\,15}=0.75$ $b_{16\,16}=0.70$
$b_{17\,17}=0.65$ $b_{18\,18}=0.60$ $b_{19\,19}=0.55$ $b_{20\,20}=0.50$
$b_{21\,21}=0.45$ $b_{22\,22}=0.40$ $b_{23\,23}=0.35$ $b_{24\,24}=0.30$
$b_{25\,25}=0.25$ $b_{26\,26}=0.20$ $b_{27\,27}=0.15$ $b_{28\,28}=0.10$
$b_{29\,29}=0.05$ $b_{30\,30}=0.03$.

Table 1 Numerical results for example 1 with $\varepsilon=10^{-5}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	3798	528	610	718	656
Computing Times (ms)	1373	201	225	271	243

(Please refer to the context [2] for AC3P1, AC5P2 and AC5P4)

Table 2 Numerical results for example 1 with $\varepsilon=10^{-9}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	>10000	1168	1447	1398	1616
Computing Times (ms)	> 3614	444	537	530	599

Example 2. The matrix B is a 30×30 diagonal matrix and its elements are given as follows:

$b_{11}=0.96$ $b_{22}=0.95$ $b_{33}=0.94$ $b_{44}=0.93$
$b_{55}=0.92$ $b_{66}=0.91$ $b_{77}=0.90$ $b_{88}=0.89$
$b_{99}=0.88$ $b_{10\,10}=0.87$ $b_{11\,11}=0.86$ $b_{12\,12}=0.85$

The other diagonal elements are the same as in the example 1.

Table 3 Numerical results for example 2 with $\varepsilon=10^{-5}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	112	48	49	58	56
Computing Times (ms)	40	18	18	21	22

Table 4 Numerical results for example 2 with $\varepsilon=10^{-9}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	291	78	109	108	96
Computing Times (ms)	106	29	40	40	35

Example 3. The matrix B is a 30×30 diagonal matrix and its elements are given as follows:

$b_{11}=0.92$ $b_{22}=0.91$ $b_{33}=0.90$ $b_{44}=0.89$
$b_{55}=0.88$ $b_{66}=0.87$ $b_{77}=0.86$ $b_{88}=0.85$
$b_{99}=0.84$ $b_{10\,10}=0.83$ $b_{11\,11}=0.82$ $b_{12\,12}=0.81$
$b_{13\,13}=0.80$

The other diagonal elements are the same as in the example 1.

Table 5 Numerical results for example 3 with $\varepsilon=10^{-5}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	68	28	34	38	56
Computing Times (ms)	24	11	13	14	21

Table 6 Numerical results for example 3 with $\varepsilon=10^{-9}$.

Acceleration Process	not applied	AC5SP2	AC3P1	AC5P2	AC5P4
Iteration Numbers	165	58	70	78	76
Computing Times (ms)	60	22	26	30	28

5. Conclusion

It is observed from the results of the three examples that AC5SP2 is superior to the others in the iteration numbers and the computing times required until condition (20) is satisfied, and that AC3P1 is comparable in those results to AC5SP2. But the application of AC3P1 yields poor results in accuracy, compared with those of AC5SP2 and AC5P4.

The application of AC3P1 for the iterative matrix with λ_1 very close to unity stops the computations as if condition (20) is satisfied even though the required accuracy is not attained.

So, in the present case, we recommend AC5SP2 with $c=0.86\lambda_1$ in all the positive range of λ_1, especially in $\lambda_1>0.92$, and we recommend AC3P1 in the range of $\lambda_1\leqq 0.92$.

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