The Reachability Problem for Quasi-Ground Term Rewriting Systems

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A term rewriting system is said to be a quasi-ground system if, for every rewrite rule in the system, the left-hand-side is linear and the right-hand-side is a ground term. This paper shows that the reachability problem for quasi-ground systems (i.e., the problem of deciding, for a quasi-ground system and two terms, whether one of the terms can reduce to the other) is decidable, and there exists an efficient algorithm solving this problem under the assumption that the number of the variable occurrences appearing in the left-hand-side for each rewrite rule is bounded by a fixed constant.

1. Introduction

The reachability problem for term rewriting systems (for short, TRS) is the problem of deciding, for a given TRS E and two terms M and N, whether M can reduce to N by applying the rules of E. It is well-known that this problem is undecidable for general TRS's: it reduces to the halting problem for Turing machines. Togashi-Noguchi [8] have shown, however, that reachability for ground TRS's is decidable in a polynomial time. In this paper, we extend the above result by showing that reachability is solvable for quasi-ground TRS's. Here, a TRS is said to be a quasi-ground system if, for every rewrite rule in the system, the left-hand-side is linear and the right-hand-side is a ground term.

The reachability problem for TRS's is closely related to the Church-Rosser property (i.e., confluence) problem for TRS's. For ground TRS's, the decidability of the former problem was used to show the decidability of the latter problem [4]. The result of this paper can also be used to show the decidability of the Church-Rosser property for quasi-ground TRS's (see [6]). Moreover, in [5], the decidability result of this paper can be used to obtain a sufficient (and decidable) condition insuring call-by-need computations in TRS's (defined in [2]).

To obtain the decidability of reachability for quasi-ground TRS's, we prove that, if a term P can reduce to a term Q, then there exists a reduction sequence γ from P to Q such that the size of each rule instance used in γ is bounded by some constant depending on only the sizes of P and rewrite rules. That is, there exists a finite bound of the number of rule instances which we should consider to check whether term P can reduce to term Q. Thus, the reachability problem for quasi-ground TRS's is reducible to that for ground TRS's. Using an algorithm of deciding the latter problem, we show that reachability for quasi-ground TRS's can be checked in a polynomial time under the assumption that the number of the variable occurrences appearing in the left-hand-side for each rewrite rule is bounded by a fixed constant.

2. Preliminaries

We use definitions and notations similar to those in [1, 4]. Let X and F be the sets of variables and operation symbols, respectively. Let T be the set of terms constructed from X and F. We use ε to denote the empty string and ϕ to denote the empty set. For a set Y, we use $\|Y\|$ to denote the cardinality of Y.

For a term M in T, we use $O(M)$ to denote the set of occurrences (positions) of M, and M/u to denote the subterm of M at occurrence u, and $M[u\leftarrow N]$ to denote the term obtained from M by replacing the subterm M/u by N. Let $O_X(M)$ be the set of variable occurrences, i.e., $O_X(M)=\{u\in O(M)\,|\,M/u\in X\}$. Let $V(M)$ be the set of variables occurring in M. We use $h(M)$ to denote the height of M and $|M|$ to denote the size of M.

Example 1. Let M be a term $f(g, x)$ where f, $g\in F$ and $x\in X$. Then, $O(M)=\{\varepsilon, 1, 2\}$, $M/1=g$, $M/\varepsilon=M$, $M[2\leftarrow g]=f(g, g)$, $O_X(M)=\{2\}$, $V(M)=\{x\}$, $h(M)=1$ and $|M|=3$.

For a term M, let sub (M) be the set of subterms of M, i.e., $\text{sub}(M)=\{M/u\,|\,u\in O(M)\}$. This definition is naturally extended to that for subsets of T: $\text{sub}(\Gamma)=\bigcup_{M\in\Gamma}\text{sub}(M)$ for $\Gamma\subseteq T$. For example, if $M=f(g, x)$, then sub $(M)=\{M, g, x\}$.

The set of occurrences $O(M)$, where $M\in T$, is partially ordered by the prefix ordering: $u\leq v$ iff $\exists w\ uw=v$. In this case we denote w by v/u. If $u\not\leq v$ and $v\not\leq u$, then u and v are said to be disjoint, denoted $u|v$. If $u\leq v$ and $u\neq v$, then $u<v$.

A term M is said to be linear if no variable occurs more than once in M, and a ground term if there is no variable occurring in M, i.e., $V(M)=\phi$. A rule $\alpha\rightarrow\beta$ is a directed equation over terms where $V(\beta)\subseteq V(\alpha)$ and

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$\alpha \notin X$. Rule $\alpha \rightarrow \beta$ is said to be a ground rule if α and β are ground terms, and a quasi-ground rule if α is linear and β is a ground term.

A term rewriting system (TRS) is a finite set of rules $E = \{\alpha_i \rightarrow \beta_i \mid 1 \leq i \leq n\}$ for some $n > 0$ where $\alpha_i, \beta_i \in T$. A substitution is a mapping $\sigma: X \rightarrow T$ and σ is extended to a mapping from terms to terms: $\sigma(fM_1 \ldots M_m) = f\sigma(M_1) \ldots \sigma(M_m)$ for $f \in F$ where m is the arity of f. A term M reduces to N at occurrence u iff $M/u = \sigma(\alpha)$ and $N = M[u \leftarrow \sigma(\beta)]$ for some substitution σ and rule $\alpha \rightarrow \beta \in E$. In this case, M/u is called the redex and u is called the redex occurrence of this reduction. We denote this reduction by $M \xrightarrow[E]{u} N$. In this notation, u and E may be omitted (i.e., $M \rightarrow N$) and $\rightarrow$ is regarded as a relation over T. Let $\rightarrow^+$ and $\rightarrow^*$ be the transitive closure and the reflexive-transitive closure of $\rightarrow$, respectively. A term M is reachable from N iff $N \rightarrow^* M$. Let $\rightarrow^0$ be the identity relation and let $\rightarrow^k = \rightarrow \cdot \rightarrow^{k-1}$ for $k > 0$. Let $\rightarrow^{(k)} = \bigcup_{i=0}^{k} \rightarrow^i$ for $k \geq 0$.

We define the size of system E by $\Sigma_{\alpha_i \rightarrow \beta_i \in E} (|\alpha_i| + |\beta_i|)$, denoted size($E$).

Henceforth, we are dealing with a fixed quasi-ground system $E = \{\alpha_i \rightarrow \beta_i \mid 1 \leq i \leq n\}$ such that every $\alpha_i \rightarrow \beta_i$ is a quasi-ground rule, $1 \leq i \leq n$. Let $L_E = \{\alpha_i \mid 1 \leq i \leq n\}$, $R_E = \{\beta_i \mid 1 \leq i \leq n\}$ and $h_E = \text{Max } \{h(\alpha_i), h(\beta_i) \mid 1 \leq i \leq n\}$. (Note that R_E is the set of ground terms β_i.)

Let Red (E) be the set of redexes, i.e., Red (E) $= \{\sigma(\alpha) \mid \alpha \in L_E$ and $\sigma: X \rightarrow T\}$. For a subset Γ of T, let Red (E, Γ) be the subset of Red (E) whose elements are obtained from mappings $\sigma: X \rightarrow \Gamma$, i.e., Red ($E$, Γ) $= \{\sigma(\alpha) \mid \alpha \in L_E$ and $\sigma: X \rightarrow \Gamma\}$. In the next section, we will show a technical lemma which says that, if a term P can reduce to an instance $\sigma(\alpha)$ of some left-hand-side α in L_E (where σ is a substitution), then P can also reduce to another instance $\sigma'(\alpha)$ of the same α in L_E which belongs to Red (E, sub ($\{P\} \cup R_E$)). As we will often use this set Red (E, sub ($\{P\} \cup R_E$)), we use $\Delta_E(P)$ as the abbreviation, hereafter.

Notation. We use $\Delta_E(P)$ to denote Red (E, sub ($\{P\} \cup R_E$))

Further, we use $E(P)$ to denote the set of rules which have form $\sigma(\alpha) \rightarrow \beta$ where $\sigma(\alpha) \in \Delta_E(P)$ and $\alpha \rightarrow \beta \in E$, i.e., $E(P) = \{\sigma(\alpha) \rightarrow \beta \mid \sigma(\alpha) \in \Delta_E(P)$ and $\alpha \rightarrow \beta \in E\}$. We will regard $E(P)$ as the set of ground rules by regarding variables in X as constants. (That is, any instance $\sigma'(\sigma(\alpha))$ of $\sigma(\alpha)$, where $\sigma(\alpha) \rightarrow \beta \in E(P)$, is not allowed unless σ' is the identity mapping.) Using the above technical lemma, we will show that, if a term P can reduce to a term Q, i.e., $P \xrightarrow[E]{} {}^* Q$, then P can also reduce to Q by applying (only) the ground rules $E(P)$. Thus, the reachability problem for quasi-ground TRS's is reducible to that for ground TRS's.

The following lemma is a technical one used in the next section. It says that $\Delta_E(P) \subseteq \Delta_E(M)$ holds if P is a subterm of some term in $\{M\} \cup R_E$.

Lemma 1. Let P be a term in sub ($\{M\} \cup R_E$) where $M \in T$. Then, $\Delta_E(P) \subseteq \Delta_E(M)$ holds.

The proof is obvious by the definition of $\Delta_E(P)$.

3. Decidability of reachability

In this section, we will show that the reachability problem for quasi-ground TRS's is reducible to that for ground TRS's. To show this, we prove a technical lemma which says that, if a term P can reduce to an instance $\sigma(\alpha)$ of some left-hand-side α in L_E (where σ is a substitution), then P can also reduce to another instance $\sigma'(\alpha)$ of the same α in L_E which belongs to $\Delta_E(P)$.

Lemma 2. Let E be a quasi-ground system. Let $P \rightarrow^k Q$ for $k \geq 0$ where $Q = \sigma(\alpha)$ for some substitution σ: $X \rightarrow T$ and $\alpha \in L_E$. Then, there exists a term $Q' \in \Delta_E(P)$ such that $P \rightarrow^{(k)} Q'$ and $Q' = \sigma'(\alpha)$ for some substitution σ'.

Proof. Let γ: $P = P_1 \xrightarrow{v_1} P_2 \xrightarrow{v_2} \ldots P_k \xrightarrow{v_k} P_{k+1} = Q$. Let $Q = \sigma(\alpha)$ for substitution σ and $\alpha \in L_E$. Let $O_X(\alpha) = \{u_1, \ldots, u_n\}$. For each u_i, $1 \leq i \leq n$, we define

$$c(u_i) = \begin{cases} \max \{j \mid v_j < u_i, \ 1 \leq j \leq k\} \\ \qquad \text{if } \exists\, v_j \text{ such that } v_j < u_i \\ 0 \qquad \text{otherwise} \end{cases}$$

For the m-th reduction $P_m \xrightarrow{v_m} P_{m+1}$ in γ following the $c(u_i)$-th reduction (where $c(u_i) < m \leq k$), if $v_m \geq u_i$, then this reduction is not essential in order to obtain a redex of form $\sigma'(\alpha)$, since TRS E is left-linear. So, in this case we omit this reduction from γ. Let γ': $P \rightarrow^{(k)} Q'$ be the new reduction sequence obtained from γ by removing all redexes $P_m \xrightarrow{v_m} P_{m+1}$ satisfying that $m > c(u_i)$ and $v_m \geq u_i$ for some $u_i \in O_X(\alpha)$. Obviously, $Q' = \sigma'(\alpha)$ holds for some σ' and we can prove that $Q' \in \Delta_E(P)$ holds, since for each $u_i \in O_X(\alpha)$, if $c(u_i) = 0$, then $Q'/u_i = P/u_i$, otherwise $Q'/u_i \in$sub (R_E) holds by the definition of γ'. Thus, this lemma holds. □

We are now ready to show the main lemma which says that, if a term P can reduce to a term Q, i.e., $P \xrightarrow[E]{} {}^* Q$, then P can also reduce to Q by applying (only) the ground rules $E(P)$ ($= \{\sigma(\alpha) \rightarrow \beta \mid \sigma(\alpha) \in \Delta_E(P)$ and $\alpha \rightarrow \beta \in E\}$).

Lemma 3. Let E be a quasi-ground system. Let P be a term in sub ($\{M\} \cup R_E$) where $M \in T$. Then, $P \xrightarrow[E]{} {}^* Q \Leftrightarrow P \xrightarrow[E(M)]{} {}^* Q$. Here, it is assumed that $E(M) = \{\sigma(\alpha) \rightarrow \beta \mid \sigma(\alpha) \in \Delta_E(M)$ and $\alpha \rightarrow \beta \in E\}$ is the set of ground rules.

Proof. The proof of ($\Leftarrow$): Obvious.

The proof of ($\Rightarrow$). Let $P \xrightarrow[E]{} {}^k Q$ for some $k \geq 0$. Let $|P| = n$. We will prove this only-if-part by induction on (k, n). Here, we define a noetherian ordering $>$ on (k, n) as follows:

$(k, n) > (k', n')$ iff $(k > k') \vee (k = k' \wedge n > n')$

Basis. The case of $k = 0$ is obvious. Let $k = 1$ and $P \xrightarrow[E]{u} Q$. Obviously, $P/u \in$sub ($\{M\} \cup R_E$) holds by the assumption of this lemma. Thus, $P/u = \sigma(\alpha)$ holds for some $\alpha \in L_E$ and mapping σ: $X \rightarrow$sub ($\{M\} \cup R_E$). This means that $P/u \in \Delta_E(M)$ and $P/u \rightarrow Q/u \in E(M)$. Hence, $P \xrightarrow[E(M)]{} Q$ holds, as claimed.

Induction Step: The case of $k > 1$.

Let $P = P_1 \xrightarrow[E]{u_1} P_2 \xrightarrow[E]{u_2} \ldots P_k \xrightarrow[E]{u_k} P_{k+1} = Q$. We first consider the case of $|P|=1$. In this case, note that $P \to P_2$ is a rule in E, so that $P_2 \in R_E$. Thus, $P_2 \in \mathrm{sub}(\{M\} \cup R_E)$. Hence, the induction hypothesis insures that $P_2 \xrightarrow[E(M)]{} {}^*Q$. Also, $P \xrightarrow[E(M)]{} P_2$ holds by the proof of Basis. It follows that $P \xrightarrow[E(M)]{} {}^*Q$ holds.

In the case where $|P|>1$, there are two cases depending on whether $\varepsilon = u_i$ for some i, $1 \le i \le k$.

Case 1: $\varepsilon \ne u_i$ for any i, $1 \le i \le k$. Let $P = f\bar{P}_1 \ldots \bar{P}_m$ for some operation $f \in F$ and terms $\bar{P}_j$, $1 \le j \le m$. Then, $Q = fQ_1 \ldots Q_m$ holds for certain terms Q_j, $1 \le j \le m$, where $\bar{P}_j \xrightarrow[E]{} {}^{(k)}Q_j$, $1 \le j \le m$.

Note that $P \in \mathrm{sub}(\{M\} \cup R_E)$ holds by the assumption of this lemma and so $\bar{P}_j$. Hence, the induction hypotheses insure that $\bar{P}_j \xrightarrow[E(M)]{} {}^*Q_j$, $1 \le j \le m$. It follows that $P \xrightarrow[E(M)]{} {}^*Q$, as claimed.

Thus, the following case 2 remains.

Case 2: $\varepsilon = u_i$ for some i, $1 \le i \le k$.

In this case, $P \xrightarrow[E]{} {}^{i-1}P_i$ and $P_i = \sigma(\alpha)$ for some substitution σ and $\alpha \in L_E$. Moreover, $P_{i+1} = \beta$ holds for some $\beta \in R_E$ where $\alpha \to \beta \in E$. By Lemma 2, there exists a term $P_i' \in \Delta_E(P)$ such that $P \xrightarrow[E]{} {}^{(i-1)} P_i'$ and $P_i' = \sigma'(\alpha)$ for some substitution σ'. Note that $P_i' \in \Delta_E(M)$ holds, since $\Delta_E(P) \subseteq \Delta_E(M)$ by Lemma 1 and the assumption of this lemma (i.e., $P \in \mathrm{sub}(\{M\} \cup R_E)$). Thus, it is ensured that

$$P_i' \to P_{i+1} \in E(M)$$

Obviously, $P \xrightarrow[E(M)]{} {}^*P_i'$ holds by the induction hypothesis, so that $P \xrightarrow[E(M)]{} {}^*P_{i+1}$ holds. Since $P_{i+1} = \beta \in R_E \subseteq \mathrm{sub}(\{M\} \cup R_E)$, the induction hypothesis also insures that $P_{i+1} \xrightarrow[E(M)]{} {}^*P_{k+1}$. Hence, it follows that $P \xrightarrow[E(M)]{} {}^*P_{k+1} = Q$, as claimed. □

In the above lemma, let $P = M$. Then, we have $M \xrightarrow[E]{} {}^*Q$ iff $M \xrightarrow[E(M)]{} {}^*Q$. Thus, the reachability problem for quasi-ground TRS's is reducible to that for ground TRS's. Since the latter problem is decidable ([8]), we have the following main theorem.

Theorem 1. The reachability problem for quasi-ground TRS's is decidable. □

4. Time complexity

In the previous section we have shown that the reachability problem for quasi-ground systems is reducible to that for ground systems. Thus, we can use an algorithm solving the latter problem to decide the former problem. Togashi-Noguchi [8] have given an algorithm of deciding, for a given ground system Eg and two terms M and N, whether $M \xrightarrow[Eg]{} {}^*N$. This algorithm operates in a polynomial time of n where n is the sum of sizes of the given terms (i.e., $n = |M| + |N| + \mathrm{size}(Eg)$ where $\mathrm{size}(Eg) = \Sigma_{\alpha \to \beta \in Eg}(|\alpha| + |b|)$). (For the readers unfamiliar to [8], an outline of the algorithm is given in Appendix.) Henceforth, we call this algorithm Algorithm A. Using this Algorithm A, we can give an algorithm B of deciding the reachability problem for quasi-ground systems as follows:

Algorithm B

Input: a quasi-ground system E and two terms M, N.

Output: the same output value as that of algorithm A taking as input the corresponding ground system $E(M)$ and the two terms M, N.

The correctness of Algorithm B is ensured by Lemma 3.

We now discuss time-complexity of Algorithm B. Note that Algorithm B operates in a polynomial time of $k = \mathrm{size}(E(M)) + |M| + |N|$, since Algorithm A operates in the same time. It will be natural to define the input size of Algorithm B by $\mathrm{size}(E) + |M| + |N|$. So, let $n = \mathrm{size}(E) + |M| + |N|$. Generally, k may not be bounded by any polynomial size of n. Note that, for a term α in L_E, if $\|O_X(\alpha)\| = i$, then the number of terms of form $\sigma(\alpha)$ where σ: $X \to \mathrm{sub}(\{M\} \cup R_E)$ is bounded by the order of n^i. Thus, $\|\Delta_E(M)\|$ is bounded by the order of n^{i+1} if MAX $\{\|O_X(\alpha)\| \mid \alpha \in L_E\} \le i$, so that k $(= \mathrm{size}(E(M)) + |M| + |N|)$ is bounded by the same order of n^{i+1}.

By the above arguments, we have the following theorem.

Theorem 2. Let $n = \mathrm{size}(E) + |M| + |N|$ where E is a quasi-ground system and M and N are terms. Let $i = \mathrm{Max}\{\|O_X(\alpha)\| \mid \alpha \in L_E\}$. Then, whether or not $M \xrightarrow[E]{} {}^*N$ can be checked in time proportional to n^{ci} for some constant $c > 0$. □

Note. In many applications, it is not so restrictive to assume that, for every left-hand-side α in L_E, $\|O_X(\alpha)\|$ is bounded by a fixed constant. Under this assumption, the reachability problem for quasi-ground TRS's is decidable in a polynomial time of the input size.

5. Concluding Remarks

In this article, we have shown that the reachability problem for quasi-ground TRS's is decidable. This result can be compared with the undecidability of the same problem for general TRS's (also left-linear TRS's). From a given left-linear TRS E we can construct a quasi-ground TRS E' as follows: rules of E' are obtained from rules of E by replacing variables occurring in the right-hand-side terms by Ω which is the least information symbol (in the sense of [7, 2]) and regarded as a new operation symbol with arity 0, i.e., $\alpha \to \beta \in E$ iff $\alpha \to \sigma(\beta) \in E'$ where σ: $X \to \{\Omega\}$. Each reduction sequence of E' can be considered as a (rough) approximation of the corresponding reduction sequence of E. Thus, if we are content with such approximation, then we can use the decidability result of this paper for reachability, while there is no algorithm of deciding reachability for general TRS's. In fact, the result of this paper can be used to obtain a sufficient (and decidable) condition insuring call-by-need computations in left-linear TRS's defined in [2] (see [5]).

Using the result of this paper, we can show that the Church-Rosser Property for quasi-ground TRS's is decidable (see [6]). This result can also be compared with the undecidability of the same problem for (left-linear) TRS's.

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Appendix

A dag (directed acyclic graph) is used as the data structure for terms, *e.g.*, a set of terms $\{f(g, g), h(f(g, g)), f(g, e)\}$ is represented as Fig. 1. Here, $e, f, g, h \in F$ and d_i's are used as node names. Each node d_i is considered to represent the subterm whose root node is d_i, *e.g.*, d_1 represents $f(g, g)$. We will use $Tm(d_i)$ to denote the subterm which node d_i represents. (In the above example, $Tm(d_1)=f(g, g)$.) Note that two distinct nodes in the dag represent different terms. We will use $Lb(d_i)$ to denote the operation symbol assigned to node d_i, *e.g.*, $Lb(d_1)=f$. Let $Ch(d_i)$ be the sequence of children of d_i, *e.g.*, $Ch(d_1)=d_2d_2$ and $Ch(d_3)=d_2d_4$. We use $Ch(d_i, j)$ to denote the j-th child of d_i, *e.g.*, $Ch(d_1, 1)=d_2$ and $Ch(d_3, 2)=d_4$. Note that if $Ch(d_i)=e_1 \ldots e_n$, then $Tm(d_i)=Lb(d_i)\,(Tm(e_1), \ldots, Tm(e_n))$ holds.

We are now ready to explain an algorithm of deciding reachability for given terms. To decide, for a given ground system E_g and terms M and N, where $M \xrightarrow[Eg]{} {}^*N$ or not, our algorithm uses a dag D representing the set of terms $T_D=\{M, N\} \cup L_{Eg} \cup R_{Eg}$ where L_{Eg} and R_{Eg} are the sets of left-hand-side terms and right-hand-side terms of rules E_g, respectively. Let N_D be the set of nodes of this dag D. The following algorithm A computes the subset X of $N_D \times N_D$ such that $(d, d') \in X$ iff $Tm(d) \xrightarrow[Eg]{} {}^*Tm(d')$ for $d, d' \in N_D$, in order to decide whether $M \xrightarrow[Eg]{} {}^*N$.

[Algorithm A]

(1) Let $X=\text{Rule} \cup I$ where
$I=\{(d, d) \mid d \in N_D\}$ and
$\text{Rule}=\{(d, d') \in N_D \times N_D \mid T_m(d) \rightarrow Tm(d') \in Eg\}$

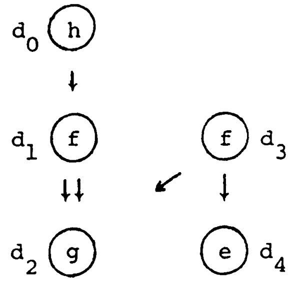

Fig. 1. A dag representing terms

(2) Let $X=\text{Closure}\,(X)$ where
Closure (X) is the reflexive-transitive closure of X.

(3) While there exist $d, d' \in N_D$ such that
$(d, d') \notin X$ and Con $(d, d')=$True {where Con $(d, d')=$True iff $Lb(d)=Lb(d')$ and, for $Ch(d)=d_1 \ldots d_n$ and $Ch(d')=d'_1 \ldots d'_n$, (d_i, d'_i) belongs to X, $1 \le i \le n$}
do let $X=\text{Closure}\,(X \cup \{(d, d')\})$

(4) If $(d, d') \in X$ such that $Tm(d)=M$ and $Tm(d')=N$
then output ('True: Reachable')
else output ('False')

Intuitively, Algorithm A first assigns pairs $(d, d') \in N_D \times N_D$ to X such that either $d=d'$ holds or $Tm(d) \rightarrow Tm(d')$ is a rule in Eg. Next it computes the reflexive-transitive closure of X and then the congruence closure of X. Let $d, d' \in N_D$ where $Lb(d)=Lb(d')$, and let $Ch(d)=d_1 \ldots d_n$ and $Ch(d')=d'_1 \ldots d'_n$ for some nodes $d_i, d'_i \in N_D$, $1 \le i \le n$. Then, the pair (d, d') is said to be congruent if $(d_i, d'_i) \in X$ for all i, $1 \le i \le n$. Note that $Tm(d) \rightarrow^* Tm(d')$ if (d, d') is congruent and $(d_i, d'_i) \in X$ implies $Tm(d_i) \rightarrow^* Tm(d'_i)$, $1 \le i \le n$. Whenever such a congruent pair (d, d') is discovered, Algorithm A adds the pair (d, d') to X and computes the transitive closure of $X \cup \{(d, d')\}$.

The fundamental idea of Algorithm A is the same as those in [8, 3].

Now, we can give a simpler proof for the correctness of Algorithm A than that in [8].

Lemma. Let X_f be the final value of X by the execution of Algorithm A (*i.e.*, X_f is the value in the line (4)). Then, for any $d, d' \in N_D$,

$$Tm(d) \xrightarrow[Eg]{} {}^*Tm(d') \Leftrightarrow (d, d') \in X_f$$

Proof. ($\Leftarrow$). The proof is obvious, since we can easily prove this if-part by induction on the number of execution steps of Algorithm A.

($\Rightarrow$). For $d, d' \in N_D$, let $P=Tm(d)$, $Q=Tm(d')$ and $P \xrightarrow[Eg]{} {}^kQ$ for some $k \ge 0$. Let $|P|=n$. We will prove this only-if-part by induction on (k, n). Here, we define a noetherian ordering $>$ on (k, n) as follows: $(k, n)>(k', n')$ iff $(k>k') \vee (k=k' \wedge n>n')$. (This proof is similar to that of Lemma 3.)

Basis. The case of $k=0$ is obvious. Let $k=1$ and $P \xrightarrow{u} Q$ for some $u \in O(P)$. Obviously there exist nodes

e, $e' \in N_D$ such that $Tm(e)=P/u$ and $Tm(e')=Q/u$, since d, $d' \in N_D$. In the line (1) of Algorithm A, (e, e') is added to X, so that by the executions of the line (3), (d, d') is eventually added to X. Thus, $(d, d') \in X_f$, as claimed.

Induction Step: The case of $k>1$.

Let $P=P_1 \xrightarrow{u_1} P_2 \xrightarrow{u_2} \ldots P_k \xrightarrow{u_k} P_{k+1}=Q$. In the case where $|P|=1$, $P \to P_2$ is a rule in E, so that $P_2 \in R_{Eg}$. Thus, there exists a node $e \in N_D$ such that $Tm(e)=P_2$. Obviously, $(d, e) \in X_f$ holds by the execution of the line (1) of Algorithm A. Since $Tm(e) \to^{k-1} Tm(d')$ and $Tm(e) \in R_{Eg}$, the induction hypothesis insures that $(e, d') \in X_f$. Since X_f is closed under transitivity, it follows that $(d, d') \in X_f$, as claimed.

In the case where $|P|>1$, there are two cases depending on whether $\varepsilon = u_i$ for some i, $1 \le i \le k$.

Case 1: $\varepsilon \ne u_i$ for any i, $1 \le i \le k$.

Let $P=f\bar{P}_1 \ldots \bar{P}_m$ for some operation $f \in F$ and terms $\bar{P}_j$, $1 \le j \le m$. Then, $Q=fQ_1 \ldots Q_m$ holds for certain terms Q_j, $1 \le j \le m$, where $\bar{P}_j \xrightarrow[Eg]{} {}^{(k)} Q_j$, $1 \le j \le m$. Obviously, there are nodes e_j and e_j' such that $Tm(e_j)=\bar{P}_j$ and $Tm(e_j')=Q_j$, $1 \le j \le m$. Thus, the induction hypotheses insure that $(e_j, e_j') \in X_f$, $1 \le j \le m$. Hence, in the line (3) of algorithm A, Con (d, d')=True, so that (d, d') must be added to X in the line (3) (if (d, d') is not still added to X). Thus, $(d, d') \in X_f$, as claimed.

Thus, the following case 2 remains:

Case 2: $\varepsilon = u_i$ for some i, $1 \le i \le k$.

In this case, $P \xrightarrow[Eg]{} {}^{i-1} P_i$ and $P_i \in L_{Eg}$. Obviously, there exists a node e in N_D such that $Tm(e)=P_i$. Hence, the induction hypothesis insures that $(d, e) \in X_f$. Note that $P_i \to P_{i+1}$ is a rule in Eg. Thus, for the node e' such that $T_m(e')=P_{i+1}$, $(e, e') \in X_f$ holds. Moreover, since $P_{i+1} = Tm(e') \to^{(k-1)} P_{k+1} = Q = Tm(d')$, the induction hypothesis insures that $(e', d') \in X_f$. Since X_f is closed under transitivity, it follows that $(d, d') \in X_f$, as claimed. □

The above lemma insures the correctness of Algorithm A. We can show that some implementation of Algorithm A operates in a polynomial time of n where $n=|M|+|N|+\text{size}(Eg)$ (see [8]). Thus, the reachability problem for ground TRS's is polynomially decidable.

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