A Graph Model for Probabilistic Computation

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We construct the reflexive graph model **F**ω for probabilistic computation corresponding to Scott's graph model **P**ω and characterize the computable elements in **F**ω in terms of Scott's language LAMBDA.

1. Introduction

Scott [2] constructed the graph model Pω for deterministic computation and characterized the computable functions in terms of the language LAMBDA. We shall construct the corresponding model **F**ω for probabilistic computation. Our model will be used in Denotational semantics for probabilistic programs, and will be a basis for the $*$-finite theory of probabilistic computation.

Probabilistic algorithms are extended algorithms which incorporate random input data and random choices. Let ω be the set of natural numbers and **P**ω the power set of ω. Then probabilistic algorithms can be considered as operators mapping a probability measure on a σ-field **P**ω defined on the space ω to another probability measure on **P**ω. In our construction, **F**ω is a subset of the power set **P**ω and an element of **F**ω represents a measure on **P**ω which can be thought of as a point function from ω to the extended reals. By this identification, finite sets in **F**ω represent measures with finite supports in ω which form a basis of the space **M**ω of measures on ω with respect to the canonical ordering. Also, **M**ω and **F**ω have another ordering $\sqsubseteq$ defined as follows: $\mu \sqsubseteq \nu$ if and only if Supp $(\mu) \subset$ Supp (ν) and $\nu|_{\mathrm{Supp}(\mu)} = \mu$. Now, for each partial computable measurable function f: $\omega \to \omega$, we can define the operator T_f: $\mathbf{M}\omega \to \mathbf{M}\omega$ which maps μ to $\mu \circ f_0^{-1}$ where f_0 is a strict extension of f defined as $f_0(x) = f(x)$ if $f(x)$ is defined and $f_0(x) = \perp$ otherwise. Also, we let $\mu(x) = 0$ if $f_0(x) = \perp$. Then T_f is continuous with respect to the Scott topology induced by the canonical ordering on **M**ω. Moreover, if $\mu \in \mathbf{F}\omega$ is finite, then $T_f(\mu) \in \mathbf{F}\omega$ and is finite. Let us denote the set of all continuous operators of this type by $[\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$. Define the order $\sqsubseteq$ on $[\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$ by $T_f \sqsubseteq T_g$ if and only if g is a functional extension of f. Then we can isomorphically embed $([\![\mathbf{M}\omega \to \mathbf{M}\omega]\!], \sqsubseteq)$ into $(\mathbf{F}\omega, \sqsubseteq)$. Also, we can characterize the computable operators using the language LAMBDA of D. Scott.

An interesting fact is that our definition is quite natural, faithfully reflecting the structure of practical spaces of measures compared with the **P**ω due to Scott. Although most of the calculus of retracts in Scott's **P**ω can be carried out in our domain, some of the corresponding properties becomes trivial since our domain is not a lattice but an algebraic ccp.

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2. Scott Topology

We shall define the Scott topology in terms of conditionally complete posets and review some definitions and theorems on the Scott topology. For the Scott topology, see Barendregt [1].

Let us first make some notational conventions. If $(D, \leq)$ is a partially ordered set, then $\leq$-sup X denotes the supremum of a subset X of D with respect to the order $\leq$. In the case no confusion arises, we simply write sup X instead of $\leq$-sup X.

Definition 2.1

A partially ordered set $(D, \leq)$ is called a conditionally complete poset (in short, ccp) if it satisfies:

(1) There is an element $\perp \in D$, called the bottom, such that $\perp \leq x$ for every $x \in D$;

(2) for every $X \subset D$, if X is a non-empty upwards directed set and is bounded from above in D, then it has the supremum in D.

Definition 2.2

(1) Let $(D, \leq)$ be a ccp. An element $x \in D$ is said to be compact if for every non-empty upwards directed subset $Z \subset D$ such that sup Z exists, $x \leq \sup Z$ implies $x \leq z$ for some $z \in Z$.

(2) A ccp $(D, \leq)$ is called algebraic if it satisfies:

(a) $\{y \leq x \mid y \text{ is compact}\}$ is upwards directed and

(b) $x = \sup \{y \leq x \mid y \text{ is compact}\}$

for each $x \in D$.

Definition 2.3

Let $(D, \leq)$ be a ccp and $E \subset D$. E is called a basis of $(D, \leq)$ if $x = \sup \{e \mid e \in E \text{ and } e \leq x\}$ for every $x \in D$ and $\sup F \in E$ for every finite subset $F \subset E$ bounded from above.

If $(D, \leq)$ is an algebraic ccp, then the set of compact elements in D is a basis of $(D, \leq)$.

Definition 2.4

Let $(D, \leq)$ be a ccp. The Scott topology on D is defined as follows:

$U \subset D$ is open if

(1) $x \in U$ and $x \leq y$ implies $y \in U$, and

(2) for every non-empty upwards directed subset $X \subset D$, $\sup X \in U$ implies $X \cap U \neq \phi$.

Fact 2.1

For a function T from a ccp $(D, \leq)$ into a ccp $(D', \leq')$, the following are equivalent:

(1) T is continuous with respect to the Scott topology, i.e., $T^{-1}(U')$ is open for every open subset $U' \subset D'$.

(2) $T(\sup X) = \sup T(X) = \sup \{Tx \mid x \in X\}$ for every upwards directed subset $X \subset D$ with $\sup X$ in D.

If $(D, \leq)$ has a basis $E \subset D$, (1) and (2) are equivalent to the following condition (3):

(3) $T(\sup Y) = \sup T(Y)$ for every upwards directed subset $Y \subset E$ with $\sup Y$ in D.

If $(D, \leq)$ is an algebraic ccp, (1) and (2) are equivalent to the following condition (4):

(4) $Tx = \sup \{Ty \mid y \leq x \text{ and } y \in E\}$ for every $x \in D$ where E is a basis of $(D, \leq)$.

For proofs, we refer to [3].

Definition 2.5

Let $\mathbf{D} = (D, \leq)$, and $\mathbf{D}' = (D', \leq')$ be ccp's. Then the function space $[\mathbf{D} \to \mathbf{D}']$ is the set of all Scott continuous maps from $\mathbf{D}$ into $\mathbf{D}'$ endowed with the ordering defined by:

$T \leq S$ in $[\mathbf{D} \to \mathbf{D}']$ if and only if for each $x \in D$, $Tx \leq' Sx$.

Fact 2.2

Let $\mathbf{D}$ and $\mathbf{D}'$ be ccp's. Then $[\mathbf{D} \to \mathbf{D}']$ is a ccp in which the supremum $\sup F$ of an upwards directed $F \subset [\mathbf{D} \to \mathbf{D}']$ is given by $(\sup F)(x) = \sup'_{T \in F} T(x)$.

For proofs, we refer to [3].

3. The Graph Model

In this section, we shall define a model on a subset of $\mathbf{P}\omega$, the power set of natural numbers, in which we can interpret the behaviour of all probabilistic calculations.

Definition 3.1

(1) ω denotes the set of natural numbers.

(2) $\langle m, n \rangle = (m+n)(m+n+1)/2 + n$ for each $m, n \in \omega$.

By identifying set functions defined on $\mathbf{P}\omega$ with point functions on ω, we define first a space which contains all measures on ω.

Definition 3.2

(1) $\mathbf{M}\omega$ denotes the set of all functions from ω to the interval $[0, 1]$ of real numbers, i.e., $\mathbf{M}\omega = [0, 1]^{\omega}$.

(2) For each $\mu, \nu \in \mathbf{M}\omega$, $\mu \leq \nu$ if and only if $\mu(x) \leq \nu(x)$ in the set of real numbers for every $x \in \omega$. We call this order the canonical order.

(3) For each $\mu, \nu \in \mathbf{M}\omega$, $\mu \sqsubseteq \nu$ if and only if $\mathrm{Supp}(\mu) \subset \mathrm{Supp}(\nu)$ and $\nu|_{\mathrm{Supp}(\mu)} = \mu$, where $\mathrm{Supp}(\mu) = \{x \in \mathrm{dom}(\mu) \mid \mu(x) \neq 0\}$.

(4) For each $\mu \in \mathbf{M}\omega$, define the norm $\|\mu\|$ of μ by

$$\|\mu\| = \sum_{x \in \omega} \mu(x)$$

if it exists.

(5) $\mu \in \mathbf{M}\omega$ is called a subprobability measure if $\|\mu\| \leq 1$.

Each element $\mu \in \mathbf{M}\omega$ represents a measure on ω which can be viewed as a formal linear combination of point masses 1_x, $x \in \omega$ as follows:

$$\mu = \sum_{x \in \omega} p_x \cdot 1_x,$$

where $p_x = \mu(x)$ and

$$1_x(m) = \begin{cases} 1 & \text{if } m = x \\ 0 & \text{otherwise} \end{cases}.$$

Definition 3.3

Let $\mathbf{B}$ be the set of binary rationals between 0 and 1. Then $\mathbf{B}$ is an effectively given countable super order dense set in $[0, 1]$. i.e., for all $x \in [0, 1]$, $x = \sup \{y \in \mathbf{B} \mid y \leq x\}$. We will fix a coding b of $\mathbf{B} - \{0\}$ onto ω.

For each $p \in \mathbf{B}$ and $x \in \omega$, let

$$[x, p] = \langle x, b(p) \rangle.$$

Now, we define the probabilistic graph model $\mathbf{F}\omega$.

Definition 3.4

(1) A subset $A \subset \omega$ is called single valued if for each $x, y, y' \in \omega$, $\langle x, y \rangle, \langle x, y' \rangle \in A$ implies $y = y'$.

(2) We define the set $\mathbf{F}\omega$ by

$$\mathbf{F}\omega = \{A \subset \omega \mid A \text{ is single valued}\}.$$

In other words, $\mathbf{F}\omega$ is the subset of $\mathbf{P}\omega$ consists of elements each representing a single valued partial function.

We shall define two orders on $\mathbf{F}\omega$. Let $\mu, \nu \in \mathbf{F}\omega$.

(3) $\mu \sqsubseteq \nu$ if and only if $\mu \subset \nu$ as sets. In other words, the function represented by ν is a functional extension of the function represented by μ.

(4) $\mu \leq \nu$ if and only if for every $[x, p] \in \mu$, there is $[x, p'] \in \nu$ such that $p \leq p'$ in $\mathbf{B}$.

Definition 3.5

(1) For each $\mu \in \mathbf{F}\omega$, we define:

$\mathrm{Supp}(\mu) = \{x \mid \langle x, y \rangle \in \mu \text{ for some } y\}$,

$\mathrm{ran}(\mu) = \{y \mid \langle x, y \rangle \in \mu \text{ for some } x\}$,

and

$\mu | X = \{\langle x, y \rangle \in \mu \mid x \in X\}$ for each $X \subset \omega$.

(2) We define the finite function μ_n as follows:

$$\mu_n = \{\langle x_1, y_1 \rangle, \ldots, \langle x_k. y_k \rangle\}$$

if and only if $n=2^{\langle x_1, y_1\rangle}+\ldots+2^{\langle x_k, y_k\rangle}$. □

$(\mathbf{F}\omega, \leq)$ can be order isomorphically embedded into $(\mathbf{M}\omega, \leq)$ and $(\mathbf{F}\omega, \sqsubseteq)$ can be order isomorphically embedded into $(\mathbf{M}\omega, \sqsubseteq)$ by the same map as follows:

Let $\mu \in \mathbf{F}\omega$ and $\mu=\{\langle x, y_x\rangle\}_{x\in \mathrm{Supp}(\mu)}=\{[x, p_x]\}_{x\in \mathrm{Supp}(\mu)}$ where $p_x=b^{-1}(y_x)\in \mathbf{B}$. Since b is the coding of positive elements in $\mathbf{B}$, we have $p_x>0$ for each $x\in \mathrm{Supp}(\mu)$. Now, let

$$\bar{\mu}=\sum_{x\in \mathrm{Supp}(\mu)} p_x\cdot 1_x \in \mathbf{M}\omega.$$

Then, $\bar{\mu}$ is a $\mathbf{B}$-valued function and $\mathrm{Supp}(\bar{\mu})=\{x\in\omega \mid \bar{\mu}(x)\neq 0\}=\mathrm{Supp}(\mu)$. Therefore, for each μ, $\nu\in \mathbf{F}\omega$, $\mu\leq\nu$ if and only if $\bar{\mu}\leq\bar{\nu}$ in $\mathbf{M}\omega$ and $\mu\sqsubseteq\nu$ in $\mathbf{F}\omega$ if and only if $\bar{\mu}\sqsubseteq\bar{\nu}$ in $\mathbf{M}\omega$. Conversely, for each $\mathbf{B}$-valued element $\xi\in \mathbf{M}\omega$, there is $\mu\in\mathbf{F}\omega$ such that $\bar{\mu}=\xi$. So, we shall henceforth identify $\mu\in\mathbf{F}\omega$ with $\mathbf{B}$-valued element $\bar{\mu}\in\mathbf{M}\omega$. We illustrate these by the following proposition.

Proposition 3.1

Let $\mathbf{M_B}\omega$ be the set of all $\mathbf{B}$-valued element of $\mathbf{M}\omega$. Define the orders $\leq$ and $\sqsubseteq$ on $\mathbf{M_B}\omega$ by restricting the orders $\leq$ and $\sqsubseteq$ of $\mathbf{M}\omega$ to $\mathbf{M_B}\omega$ respectively. Then,

$$(\mathbf{F}\omega, \leq)\simeq(\mathbf{M_B}\omega, \leq) \text{ and } (\mathbf{F}\omega, \sqsubseteq)\simeq(\mathbf{M_B}\omega, \sqsubseteq).$$

In particular,

$$(\mathbf{F}\omega, \leq)\hookrightarrow(\mathbf{M}\omega, \leq), (\mathbf{F}\omega, \sqsubseteq)\hookrightarrow(\mathbf{M}\omega, \sqsubseteq)$$

and $\mathbf{F}\omega$ is order dense in $\mathbf{M}\omega$ with respect to $\leq$.

$\mathbf{F}\omega$ is the set of computable elements in $\mathbf{M}\omega$ and the elements of $\mathbf{M}\omega$ is weak computable in the sense that it can be represented as the supremum of a set of computable elements.

Proposition 3.2

$(\mathbf{M}\omega, \leq)$ is a ccp.

Proof. Lex $X\subset\mathbf{M}\omega$ be a non-empty upwards directed set bounded from above, say by $\mu_0\in\mathbf{M}\omega$. For each $m\in\omega$, consider the set $X_m=\{\mathrm{p}\mid \nu(m)=\mathrm{p}$ for some $\nu\in X\}$. Then X_m is a set of real numbers bounded by $\mu_0(m)$. So, $\sup X_m$ exists. Define $\mu\in\mathbf{M}\omega$ by $\mu(m)=\sup X_m$. Then, $\mu=\sup X$. □

Proposition 3.3

(1) $(\mathbf{F}\omega, \sqsubseteq)$ is an algebraic complete poset (cpo) and $\mu\in\mathbf{F}\omega$ is compact if and only if μ is finite.

(2) For each μ, $\nu\in\mathbf{F}\omega$, $\mu\sqsubseteq\nu$ implies $\mu\leq\nu$.

Proof.

(1) It is easy to see that $(\mathbf{F}\omega, \sqsubseteq)$ is a complete poset such that if $X\subset\mathbf{F}\omega$ is a non-empty upwards directed set, then $\sup X=\cup X$, and the set of finite elements (elements with finite support) is a basis. If $\mu\in\mathbf{F}\omega$ is infinite, then $X=\{\mu_n\mid\mu_n\subset\mu\}$ is upwards directed and $\mu=\sup X$. But, $\mu\notin X$. Conversely, suppose that μ is finite. Assume that $X\subset\mathbf{F}\omega$ is a non-empty upwards directed set such that $\mu\sqsubseteq\sup X=\cup X$. Then for each $x\in\mu$, $x\in\xi_x$ for some $\xi_x\in X$. Since X is upwards directed and μ is finite, there is $\mu'\in X$ such that $\xi_x\sqsubseteq\mu'$ for each $x\in\mu$. Therefore, $\mu\sqsubseteq\mu'$.

(2) This is obvious. □

Proposition 3.4

The set of finite elements in $\mathbf{F}\omega$ (i.e., the set of compact elements in $\mathbf{F}\omega$) is a basis for $(\mathbf{M}\omega, \leq)$.

Proof. Because the set $\mathbf{B}$ is super oder dense in the interval [0, 1] in the order topology of real numbers. □

Definition 3.6

For each partial map f: $\omega\to\omega$, we define an operator T_f: $\mathbf{M}\omega\to\mathbf{M}\omega$ as follows:

$T_f(\mu)=\mu\circ f^{-1}=\Sigma_{x\in\omega}\,\mu(x)\cdot 1_{f_0(x)}$ where f_0 is a strict function such that $f_0(x)=f(x)$ if $x\in\mathrm{dom}(f)$ and $f_0(x)=\perp$ if $x\notin\mathrm{dom}(f)$ and $1_\perp=0$ (the zero function). Thus, T_f is a linear operator such that $T_f(1_x)=1_{f(x)}$ is $x\in\mathrm{dom}(f)$ and $T_f(1_x)=0$ if $x\notin\mathrm{dom}(f)$. $T_f(\mu)$ is defined (i.e. $T_f(\mu)\in\mathbf{M}\omega$) if for each $y\in f(\mathrm{Supp}(\mu))$, $T_f(y)=\Sigma_{x\in f^{-1}(y)}\,\mu(x)\leq 1$. In particular, T_f is defined on μ's such that $\|\mu\|\leq 1$.

$[\![\mathbf{M}\omega\to\mathbf{M}\omega]\!]$ will denote the set of all T_f's.

For each T_f, $T_g\in[\![\mathbf{M}\omega\to\mathbf{M}\omega]\!]$, $T_f\sqsubseteq T_g$ if and only if g is a functional extension of f, i.e., $\mathrm{dom}(f)\subset\mathrm{dom}(g)$ and $g\mid\mathrm{dom}(f)=f$.

Proposition 3.5

$([\![\mathbf{M}\omega\to\mathbf{M}\omega]\!], \sqsubseteq)$ is an algebraic cpo where $T_f\in[\![\mathbf{M}\omega\to\mathbf{M}\omega]\!]$ is compact if and only if f is a finite function.

Proof. Same as Proposition 3.3 (1). □

Proposition 3.6

Let $T_f\in[\![\mathbf{M}\omega\to\mathbf{M}\omega]\!]$.

(1) If $\mu\in\mathbf{M}\omega$ and $\mu\in\mathrm{dom}(T_f)$, then for each $\mu_m\leq\mu$, $\mu_m\in\mathrm{dom}(T_f)$ and $T_f(\mu_m)=\mu_n\in\mathbf{F}\omega$ for some n.

(2) $\mathrm{dom}(T_f)$ is a ccp and T_f is Scott continuous with respect to $\leq$. In particular,

$$T_f(\mu)=\leq\text{-}\sup_{\mu_m\leq\mu} T_f(\mu_m).$$

Proof.

(1) Suppose that $\mu_m\leq\mu$ and $T_f(\mu)$ is defined. Then, for each $y\in f(\mathrm{Supp}(\mu_m))$, $\Sigma_{x\in f^{-1}(y)}\,\mu_m(x)$ is a binary rational and is less than $\Sigma_{x\in f^{-1}(y)}\,\mu(x)=T_f(y)\leq 1$. Hence, $T_f(\mu_m)$ is defined and $T_f(\mu_m)\in\mathbf{F}\omega$.

Also, it is obvious that $\mathrm{Supp}\,T_f(\mu_m)$ is finite.

(2) By the same argument as in (1), we can show that if $T_f(\mu)$ is defined, then $T_f(\mu')$ is defined and $T_f(\mu')\leq T_f(\mu)$ for each $\mu'\leq\mu$. Hence, $\mathrm{dom}(T_f)$ is ccp and T_f is monotone. Suppose that $X\subset\mathbf{M}\omega$ is upwards directed, $\sup X=\mu\in\mathbf{M}\omega$ and $T_f(\mu)=\nu\in\mathbf{M}\omega$. Then, $T_f(X)$ is upwards directed and bounded by ν. Let $m\in\omega$ and $\varepsilon>0$ be a real number. We shall show that $0\leq\nu(m)-T_f(\mu')(m)<\varepsilon$ for some $\mu'\in X$. As $\nu(m)=\Sigma_{n\in f^{-1}(m)}\,\mu(n)$, there is a finite set of natural numbers $Y\subset f^{-1}(m)$ such that $0\leq\nu(m)-\Sigma_{n\in Y}\,\mu(n)<\varepsilon$. Let k be the number of elements in Y and $\varepsilon'=(\varepsilon-(\nu(m)-\Sigma_{n\in Y}\,\mu(n)))/k$. For

each $n \in Y$, choose $\xi_n \in X$ such that $0 \leq \mu(n) - \xi_n(n) < \varepsilon'$. Choose $\mu' \in X$ such that $\xi_n \leq \mu'$ for every $n \in Y$. Then, we have

$$\begin{aligned} v(m) &\geq T_f(\mu')(m) \\ &= \sum_{n \in f^{-1}(m)} \mu'(n) \\ &\geq \sum_{n \in Y} \xi_n(n) \\ &> \sum_{n \in Y} (\mu(n) - \varepsilon') \\ &= \sum_{n \in Y} \mu(n) - k\varepsilon'. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &\leq v(m) - T_f(\mu')(m) \\ &< v(m) - \sum_{n \in Y} \mu(n) + k\varepsilon' \\ &= \varepsilon. \end{aligned}$$

Now let $I \subset \omega$ be an arbitrary large finite set and $\varepsilon > 0$. We can choose $\mu'' \in X$ such that $0 \leq \mu(m) - \mu''(m) < \varepsilon$ for every $m \in I$ since X is upwards directed. □

Since each $T \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$ is determined by its values on finite measures which belong to $\mathbf{F}\omega$, we can encode T as an element of $\mathbf{F}\omega$.

Definition 3.7

(1) For each $T \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$, let

graph $(T) = \{\langle m, n \rangle \mid T(\mu_m) = \mu_n,\ T(\{x\}) \neq \phi$ for every $x \in \mu_m\}$.

(2) For each $\xi \in \mathbf{F}\omega$, let

fun $(\xi)(\mu) = \leq\text{-sup}\ \{\mu_n \mid \mu_m \leq \mu$ and $\langle m, n \rangle \in \xi\}$.

If $T = T_f \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$, then **graph** $(T_f) = \{\langle m, n \rangle \mid T(\mu_m) = \mu_n,\ \mathrm{Supp}\ (\mu_m) \subset \mathrm{dom}\ (f)\}$.

Proposition 3.7

(1) **graph**: $([\![\mathbf{M}\omega \to \mathbf{M}\omega]\!], \sqsubseteq) \to (\mathbf{F}\omega, \sqsubseteq)$ is Scott continuous.

(2) For each $T \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$, **fun(graph** $(T)) = T$.

Proof. (1) First of all, we show that **graph** is monotone. Suppose that $T_f, T_g \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$, and $T_f \sqsubseteq T_g$. If $\langle m, n \rangle \in$ graph (T_f), then Supp $(\mu_m) \subset$ dom (f) and $T_f(\mu_m) = \mu_n$. As dom $(f) \subset$ dom (g) and $g \mid$dom $(f) = f$, $T_g(\mu_m) = \mu_n$ also. Hence, **graph** $(T_f) \sqsubseteq$ **graph** (T_g). Now, let $T_f \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$ and consider the set $F = \{T_g \sqsubseteq T_f \mid g$ is finite$\}$. Then, the set **graph** $(F) = \{$**graph** $(T) \mid T \in F\}$ is upwards directed and bounded by **graph** (T_f) in $(\mathrm{F}\omega, \sqsubseteq)$. Now, suppose that $\langle m, n \rangle \in$ **graph** (T_f). Then, $T_f(\mu_m) = \mu_n$ and hence, $T_{f \mid \mathrm{Supp}(\mu_m)}(\mu_m) = \mu_n$. Thus, $\langle m, n \rangle \in$ **graph** $(T_{f \mid \mathrm{Supp}(\mu_m)}) \in$ **graph** (f). So, we have **graph** $(T_f) =$ sup **graph** (F).

(2) Let $T_f \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$ and $\mu \in \mathrm{F}\omega$.

fun(graph $(T_f))(\mu)$

$$\begin{aligned} &= \leq\text{-sup}\ \{\mu_m \mid \mu_n \leq \mu \text{ and } \langle n, m \rangle \in \textbf{graph}\ (T_f)\} \\ &= \leq\text{-sup}\ \{T_f(\mu_n) \mid \mu_n \leq \mu \text{ and Supp}\ (\mu_n) \subset \text{dom}\ (f)\} \\ &= T_f(\mu). \quad \square \end{aligned}$$

Theorem 1

$([\![\mathbf{M}\omega \to \mathbf{M}\omega]\!], \sqsubseteq)$ is isomorphically embedded into $(\mathrm{F}\omega, \sqsubseteq)$ by **graph** and the inverse is given by **fun**.

Proof. By Proposition 3.7, **graph** preserves the order and **fun(graph** $(T)) = T$ for each $T \in [\![\mathbf{M}\omega \to \mathbf{M}\omega]\!]$. To finish the proof, assume that **graph** $(T_f) \sqsubseteq$ **graph** (T_g). We shall show that g is a function extension of f. Suppose that $x \in$ dom (f). Consider the singleton set $\{\langle x, y \rangle\} = \mu_m$. Then $T_f(\mu_m) = \{\langle f(x), y \rangle\}$ and thus $T_f(\mu_m) = \mu_n$ for some n. Hence, $\langle m, n \rangle \in$ **graph** $(T_f) \subset$ **graph** (T_g) and we have $T_g(\mu_m) = \mu_n$ also. Thus, $\{\langle g(x), y \rangle\} = T_g(\mu_m) = T_f(\mu_m) = \{\langle f(x), y \rangle\}$ and $g(x) = f(x)$. We can conclude that g is an extension of f. □

$(\mathbf{F}\omega, \leq)$ is the computable graph model (see Figure 1). Note that we cannot restrict our model $(\mathbf{F}\omega, \leq)$ to $(\mathbf{F}\omega, \sqsubseteq)$ since we need the order $\leq$ for the definition of **graph** and **fun**.

$$\begin{array}{ccc} & (\mathbf{F}\omega, \sqsubseteq) \hookrightarrow (\mathbf{F}\omega, \leq) \\ ([\![\mathbf{M}\omega \to \mathbf{M}\omega]\!], \sqsubseteq) \hookrightarrow (\mathbf{M}\omega, \sqsubseteq) \hookrightarrow (\mathbf{M}\omega, \leq) \end{array}$$

4. Probabilistic LAMBDA

Recall that the language LAMBDA has one primitive constant symbol 0, two unary function symbols $(x+1)$ and $(x-1)$, one binary function symbol $(u(x))$, and one ternary function symbol $(z \supset x, y)$, and has one variable binding operator $(\lambda x.\tau)$ [2]. The formation of the terms is defined in the obvious way. The semantics of LAMBDA in $(\mathbf{F}\omega, \leq)$ is defined as follows.

$$m[\![0]\!] = \{[0, 1]\}.$$

$$m[\![\eta + 1]\!] = \{\langle x+1, y \rangle \mid \langle x, y \rangle \in m[\![\eta]\!]\}.$$

$$m[\![\xi - 1]\!] = \{\langle x, y \rangle \mid \langle x+1, y \rangle \in m[\![\eta]\!]\}.$$

$$\begin{aligned} m[\![\xi \supset \eta, \theta]\!] = &\{\langle n, m \rangle \mid \phi \neq \mu_n = e_\xi(\mu_n), \\ &\quad \langle n, m \rangle \in m[\![\eta]\!]\} \\ &\cup \{\langle n, m \rangle \mid \phi \neq \mu_n = e_{\sim\xi}(\mu_n), \\ &\quad \langle n, m \rangle \in m[\![\theta]\!]\}. \end{aligned}$$

Here,

$$e_\xi = \lambda\mu \in \mathbf{F}\omega.\{[x, p] \in \mu \mid [x, 1] \in m[\![\xi]\!]\}$$

and

$$e_{\sim\xi} = \lambda\mu \in \mathbf{F}\omega.\{[x, p] \in \mu \mid x \notin \mathrm{Supp}\ m[\![\xi]\!]\}.$$

$$m[\![\eta(\mu)]\!] = \mathrm{fun}\ (m[\![\eta]\!])(m[\![\mu]\!]).$$

$$m[\![\lambda\mu.\tau]\!] = \{\langle n, m \rangle \mid m[\![\tau[\mu_n/\mu]]\!] = \mu_m\}.$$

In the definition of $m[\![\xi \supset \eta, \theta]\!]$, we omit the case $\phi = \mu_n$ because it is easier to see that the definition is well-defined. But it does not matter since the image of the empty set by a function must be the empty set (see Theorem 2).

Definition 4.1

An operator T: $\mathbf{M}\omega \to \mathbf{M}\omega$ is computable if there is a partial recursive function f such that $T(\mu)=\mu \circ f^{-1}$.

Every computable operator $\mathbf{M}\omega \rightarrow \mathbf{M}\omega$ belongs to $[\![\mathbf{M}\omega \rightarrow \mathbf{M}\omega]\!]$. Now, we can assert the following definability theorem in the same way as the proof by Scott.

Theorem 2 (LAMBDA Definability)

An operator $T \in [\![\mathbf{M}\omega \rightarrow \mathbf{M}\omega]\!]$ is computable if and only if **graph** (T) is LAMBDA-definable.

Further discussions on LAMBDA proceed in parallel with Scott [2].

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