Research Contribution

The Reachability and Joinability Problems for Right-Ground Term-Rewriting Systems

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A term rewriting system is said to be a right-ground system if no variable occurs on the right-hand side of a rewrite rule. This paper shows that both reachability and joinability are decidable for right-ground term rewriting systems.

1. Introduction

The reachability problem for term-rewriting systems (TRSs) is the problem of deciding, for a given TRS E and two terms M and N, whether M can be reduced to N by applying the rules of E (i.e., $M \rightarrow^* N$). It is known that this problem is undecidable for general TRSs, but decidable for ground TRSs [7] and for left-linear and right-ground TRSs [4]. (See Deruyver and Gilleron [9] for a related discussion.) Here, a TRS is left-linear if no variable occurs more than once on the left-hand side of a rule, and right-ground (resp., left-ground) if no variable occurs on the right-hand (resp., left-hand) side. *A* TRS is a ground system if it is both left-ground and right-ground.

In this paper, we extend the above result by showing that reachability is decidable for right-ground TRSs (that is, that the left-linearity restriction can be removed).

The joinability problem for TRSs is the problem of deciding, for a TRS and a finite set of terms $M_1, \cdots, M_k$ whether $M_1, \cdots, M_k$ can be reduced to some common term (i.e., $\downarrow\{M_1, \cdots, M_k\}$). This paper shows that joinability is also decidable for right-ground TRSs. This result is an extension of the results on joinability of left-linear and right-ground TRSs given by Oyamaguchi [5] and Danchet *et al.* [8].

Reachability and joinability for non-linear TRSs are closely related: if a term can be reduced to an instance of the non-linear left-hand side of some rule, then the subterms matching the occurences of a variable appearing more than once on the left-hand side must be joinable. Thus, to check reachability between two terms, we need to check the joinability of a set of other terms (since the reachability may be attained by using rules with non-linear left-hand sides). Conversely, to check joinability we obviously need to check the reachability of pairs of terms. (Note that $\downarrow\{M, N\}$ iff $\exists Q$: $M \rightarrow^* Q \wedge N \rightarrow^* Q$, so it is easy to show that the reachability problem is reducible to the joinability problem.)

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In this paper, we investigate this close relationship and show that each instance $\theta = (M \rightarrow^* N)$ of the reachability problem is determined by some other instances $\theta_1, \cdots, \theta_i$ of the reachability problem and instances $\theta_{i+1}, \cdots, \theta_n$ of the joinability problem. In other words, deciding whether θ is true (i.e., $M \rightarrow^* N$) is equivalent to the problem of deciding whether all $\theta_1, \cdots, \theta_n$ are true. A similar result holds for each instance of the joinability problem. Using these results, we show that for each instance θ of reachability (and also of joinability), the shortest length of reduction sequences ensuring reachability (and also joinability) is bounded by some fixed constant depending only on the sizes of TRS E and of M, N if $\theta = (M \rightarrow^* N)$ (and also of $M_1, \cdots, M_k$ if $\theta = \downarrow\{M_1, \cdots, M_k\}$). It follows that both reachability and joinability are decidable for right-ground TRSs.

Reachability and joinability are also closely related to the Church-Rosser property (i.e., confluence). For ground TRSs and for left-linear and right-ground TRSs, the decidability of the former problems (reachability and joinability) was used to show the decidability of the confluence problem [3, 5]. We therefore strongly conjecture that the results of this paper will be also used to prove the decidability of the Church-Rosser property for right-ground TRSs, which remains open.

Termination for right-ground TRSs was shown to be decidable by N. Dershowitz [1]. In view of these results, right-ground TRSs seem to have many good properties. However, some undecidable problems exist. The common ancestor problem is that of deciding, for a TRS and two terms M and N, whether there is a term that can be reduced to both M and N. This problem has been shown to be undecidable for right-ground TRSs [6], while it is decidable for ground TRSs [3] and also for left-linear and right-ground TRSs. The word problem (whether $M(\rightarrow \cup \leftarrow)^* N$ for two terms M and N) is also undecidable for right-ground TRSs. These results show a gap between right-ground TRSs and left-linear and right-ground TRSs (and also a gap between non-left-linear and left-linear TRSs).

The remaining part of this paper is organized as follows: Section 2 is devoted to the standard definitions used in this paper. The close relationship between reachability and joinability for right-ground TRSs is explained in Section 3. In Section 4, we show that both reachability and joinability are decidable for right-ground TRSs.

2. Preliminaries

We use ε to denote the empty string and ϕ to denote the empty set. For a set Y, we let $\|Y\|$ be the cardinality of Y. We use $\mathcal{N}$ to denote the set of positive integers and $\mathcal{N}_0$ to denote $\mathcal{N} \cup \{0\}$.

The following definitions and notations are similar to those in Huet [2] and Oyamaguchi [3]. Let X be a set of *variables* and let F be a finite set of *operation symbols* graded by an *arity* function a: $F \to \mathcal{N}_0$. Let T be the set of *terms* constructed from X and F.

For a term M in T, we use $\mathcal{O}(M)$ to denote the set of *occurrences* (positions) of M, and M/u to denote the subterm of M at occurrence u, and $M[u \leftarrow N]$ to denote the term obtained from M by replacing the subterm M/u by N. Let sub(M) be the set of subterms of M, that is, let sub$(M) = \{M/u \mid u \in \mathcal{O}(M)\}$. This definition is naturally extended to that for subsets of T: sub$(\Gamma) = \cup_{M \in \Gamma}$ sub(M) for $\Gamma \subseteq T$.

Example 1. Let M be a term $f(g, x)$ where $f, g \in F$ and $x \in X$. Then, $\mathcal{O}(M) = \{\varepsilon, 1, 2\}$, $M/1 = g$, $M/\varepsilon = M$, sub$(M) = \{M, g, x\}$ and $M[2 \leftarrow g] = f(g, g)$.

Let $\mathcal{O}_x(M)$ be the set of occurrences of variable $x \in X$ in M, that is, let $\mathcal{O}_x(M) = \{u \in \mathcal{O}(M) \mid M/u = x\}$. Let $\mathcal{O}_X(M) = \cup_{x \in X} \mathcal{O}_x(M)$, the set of variable occurrences. Let $V(M)$ be the set of variables occurring on M. We use $h(M)$ to denote the *height* of M and $|M|$ to denote the *size* of M. We use root(M) to denote the operation symbol of a term M at occurrence ε, that is, the top symbol.

Example 2. For the term $M = f(g, x)$ of Example 1, $\mathcal{O}_X(M) = \mathcal{O}_x(M) = \{2\}$, $V(M) = \{x\}$, $h(M) = 2$, $|M| = 3$ and root$(M) = f$.

The set of occurrences $\mathcal{O}(M)$, where $M \in T$, is partially ordered by prefix ordering: $u \le v$ iff $\exists w$: $uw = v$. If $u \le v$ and $u \ne v$, then $u < v$.

A term M is said to be *linear* if no variable occurs more than once in M, and a *ground* term if there is no variable occurring in M, that is, if $V(M) = \phi$.

A *rule* $\alpha \to \beta$ is a directed equation over terms where $\alpha \ne \beta$, $V(\beta) \subseteq V(\alpha)$ and $\alpha \notin X$. Rule $\alpha \to \beta$ is said to be a right-ground rule if β is a ground term.

A term rewriting sytem (TRS) is a finite set of rules $E = \{\alpha_i \to \beta_i \mid 1 \le i \le n\}$ for some $n > 0$, where $\alpha_i, \beta_i \in T$. A *substitution* is a mapping σ: $X \to T$, and σ is extended to a mapping from terms to terms: $\sigma(fM_1 \cdots M_m) = f\sigma(M_1) \cdots \sigma(M_m)$ for $f \in F$, where $m = a(f)$. A term M is reduced to N at occurrence u iff $M/u = \sigma(\alpha)$ and $N = M[u \leftarrow \sigma(\beta)]$ for some substitution σ and rule $\alpha \to \beta \in E$. In this case, M/u is called the *redex* and u called the redex occurrence of this reduction. We denote this reduction by $M \xrightarrow[E]{u} N$. In this notation, u and E may be omitted (i.e., $M \to N$) and $\to$ is regarded as a relation over T. Let $\to^+$ and $\to^*$ be the transitive closure and the reflexive-transitive closure of $\to$, respectively. Let $\to^0$ be the identity relation and let $\to^k = \to \cdot \to^{k-1}$ for $k > 0$. A term M is *reachable* from N iff $N \to^* M$.

A set on n terms $M_1, \cdots, M_n$ is *joinable*, denoted $\downarrow\{M_1, \cdots, M_n\}$, if there exists a common term N such that $M_i \to^* N$ for all M_i, $1 \le i \le n$. We use $\{M_1, \cdots, M_n\} \to^* N$ to denote $M_i \to^* N$ for all M_i, $1 \le i \le n$.

Definition 1. Let γ be a reduction sequence $M_0 \xrightarrow{u_1} M_1 \cdots \xrightarrow{u_k} M_k$. Then, γ is called a k-step reduction sequence and the length of γ is k, denoted $lg(\gamma) = k$. If there is no ε in $\{u_1, \cdots u_k\}$, then γ is said to be *top-invariant*. In this case, root(M_0) = root(M_k). We use γ: $M_0 \to^* M_k$ to show that γ is a reduction sequence from M_0 to M_k.

Notation. For a term M, let $Y = \{u_1, \cdots, u_{n-1}, u_n\} \subseteq \mathcal{O}(M)$, where $u_i \not\le u_j$ and $u_j \not\le u_i$ (i.e., u_i and u_j are disjoint) for $1 \le i < j \le n$. We use $M[u_1 \leftarrow N_1, \cdots, u_{n-1} \leftarrow N_{n-1}, u_n \leftarrow N_n]$ to denote $(M[u_1 \leftarrow N_1, \cdots, u_{n-1} \leftarrow N_{n-1}])[u_n \leftarrow N_n]$. We also use $M[u_i \leftarrow N_i, 1 \le i \le n]$ to denote this term.

Henceforth, we are dealing with a fixed right-ground system $E = \{\alpha_i \to \beta_i \mid 1 \le i \le n_0\}$ such that every $\alpha_i \to \beta_i$ is a right-ground rule, $1 \le i \le n_0$. Let $L_E = \{\alpha_i \mid 1 \le i \le n_0\}$ and $R_E = \{\beta_i \mid 1 \le i \le n_0\}$. (Note that R_E is the set of ground terms β_i.) Let a_E be the maximum number of occurrences of variables appearing in $\alpha_i \in L_E$, that is, let $a_E = \max\{\|\mathcal{O}_x(\alpha_i)\| \mid \alpha_i \in L_E, x \in X\}$. We define the size of system E by $\Sigma_{\alpha_i \to \beta_i \in E}(|\alpha_i| + |\beta_i|)$, denoted *size*($E$).

Definition 2. Let $\Delta_E(M) = \{\sigma(\alpha) \mid \alpha \in L_E$ and σ: $X \to$ sub$(\{M\} \cup R_E)\}$, that is, let each element of $\Delta_E(M)$ be a redex obtained by a mapping σ from X to sub$(\{M\} \cup R_E)$.

Notation. We denote an instance of the reachability problem by $(M, N)_R$, where $M, N \in T$, and an instance of the joinability problem by $\{M_1, \cdots, M_k\}_J$, where $M_1, \cdots, M_k \in T$. (We will use $M \to^* N$ to denote $(M, N)_R$ and $\downarrow\{M_1, \cdots, M_k\}$ to denote $\{M_1, \cdots, M_k\}_J$ when confusion does not occur.)

Definition 3. We denote the shortest length of reduction sequences from a term M to N by min$_R(M, N)$. That is,

$$\min_R(M, N) = \begin{cases} \min\{n \mid M \to^n N\} & \textit{if} \quad M \to^* N \\ \infty & \textit{otherwise} \end{cases}$$

We will use min $(M \to^* N)$ or min $((M, N)_R)$ to denote min$_R(M, N)$. A reduction sequence γ: $M \to^* N$ is shortest if $lg(\gamma)$ = min $(M \to^* N)$. Similarly, for a set of terms $M_1, \cdots, M_k$ (where $k \ge 1$), we define the minimum of the sums of the lengths of k reduction sequences ensuring the joinability by min$_J(\{M_1, \cdots, M_k\})$:

$$\min_J(\{M_1, \cdots, M_k\}) = \begin{cases} \min\{\sum_{i=1}^{k} \min(M_i \to^* N) \mid \{M_1, \cdots, M_k\} \to^* N\} \\ \qquad\qquad\qquad\qquad \textit{if} \downarrow\{M_1, \cdots, M_k\} \\ \infty \qquad\qquad\qquad\qquad \textit{otherwise} \end{cases}$$

Here, we allow $M_i = M_j$ for some i, j where $i \neq j$. Thus, the set $\{M_1, \cdots, M_k\}$ is assumed to be a multi-set. We will use $\min(\downarrow\{M_1, \cdots, M_k\})$ or $\min(\{M_1, \cdots, M_k\}_J)$ to denote $\min_J(\{M_1, \cdots, M_k\})$.

Definition 4. We use Set(R) and Set(J) to denote the sets of instances of reachability and joinability, respectively. That is,

$$\text{Set}(R) = \{(M, N)_R \mid M, N \in T \text{ and } \min(M \to^* N) < \infty\}$$

$$\text{Set}(J) = \{\{M_1, \cdots, M_k\}_J \mid k \geq 1,\ M_i \in T,\ 1 \leq i \leq k, \text{ and } \min(\downarrow\{M_1, \cdots, M_k\}) < \infty\}$$

Let Set(R, J) = Set(R) ∪ Set(J). If $(M, N)_R \in$ Set(R), then $(M, N)_R$ is said to be true. Similarly, $\theta = \{M_1, \cdots, M_k\}_J$ is true if $\theta \in$ Set(J).

3. Relation Between Reachability and Joinability

In this section, we will investigate the close relationship between reachability and joinability. We will show that each instance θ of the reachability problem (and also of the joinability problem) is determined by some other instances $\theta_1 \cdots, \theta_i$ of the reachability problem and $\theta_{i+1}, \cdots, \theta_n$ of the joinability problem, namely, that θ is true iff all $\theta_1, \cdots, \theta_n$ are true (i.e., $\min(\theta) < \infty$ iff $\min(\theta_1) < \infty, \cdots, \min(\theta_n) < \infty$). Using this result, we will define a replacement function Φ such that $\Phi(\theta) = \{\theta_1, \cdots, \theta_n\}$, that is, Φ: Set(R, J) $\to 2^{\text{Set}(R, J)}$. In other words, Φ defines the replacements of instances of reachability and of joinability. See Def. 4 for Set(R, J).

In the next section, this function Φ will be used to define a new reduction system (TRS) satisfying $\theta \Rightarrow \theta'$ iff $\theta' \in \Phi(\theta)$. Our main result, the decidability of reachability and of joinability, will be obtained by using the noetherian property of $\Rightarrow$.

Let Φ_R and Φ_J be subfunctions of Φ satisfying $\Phi_R(\theta) = \Phi(\theta)$ for $\theta \in$ Set(R) and $\Phi_J(\theta) = \Phi(\theta)$ for $\theta \in$ Set(J). That is, Φ_R (resp. Φ_J) is the same function as Φ when the domain is restricted to Set(R) (resp. Set(J)). We construct first Φ_R and then Φ_J.

To define the function Φ_R for reachability, we start with the following Lemma 1, which is a technical lemma for Lemma 2 and deals with a special case of reachability. Let γ be a reduction sequence from a term M to a redex $N = \sigma(\alpha)$ (where $\alpha \in L_E$ and $\sigma: X \to T$ is a mapping). Then, the reductions of γ can be classified into two parts, of which one corresponds to the construction process of the non-variable part of α, and the other corresponds to the construction processes of $\sigma(x)$'s for variable $x \in V(\alpha)$. Thus, we can rearrange the order of the reductions of γ so that the construction of $\sigma(x)$'s may follow the construction of the non-variable part of α. Hence, the following lemma holds.

Lemma 1. Let γ: $M = M_0 \xrightarrow{v_1} M_1 \cdots \xrightarrow{v_k} M_k = N$ be a k-step reduction sequence where $N = \sigma(\alpha)$ for some $\alpha \in L_E$ and mapping $\sigma: X \to T$. Let $\mathcal{O}_X(\alpha) = \{u_1, \cdots, u_n\}$. Then, there exists a k-step reduction sequence δ: $M \to^* Q \to^* N$ such that

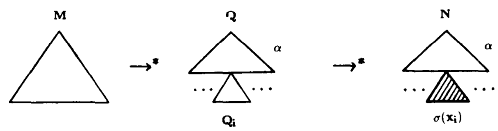


Fig. 1 Reduction sequence δ of Lemma 1.

(i) $Q = \alpha[u_1 \leftarrow Q_1, \cdots, u_n \leftarrow Q_n]$, where $Q_1, \cdots, Q_n \in \text{sub}(\{M\} \cup R_E)$, i.e., $Q \in \Delta_E(M)$, and

(ii) $Q_i \to^* N/u_i = \sigma(\alpha/u_i)$, $1 \leq i \leq n$

Proof. For each variable occurrence $u_i \in \mathcal{O}_X(\alpha)$, $1 \leq i \leq n$, if there is a reduction in γ properly including the occurrence u_i, that is, if $\exists v_j$: $v_j < u_i$, then let the last reduction of γ including u_i be the l_i-th reduction of γ, that is, let $l_i = \max\{j \mid v_j < u_i,\ 1 \leq j \leq k\}$. Then, $M_{l_i}/u_i \in \text{sub}(R_E)$ holds, since TRS E is right-ground. In this case, let

$$Q_i = M_{l_i}/u_i \in \text{sub}(R_E)$$

Otherwise (if $\forall v_j$: $v_j \not< u_i$), let $Q_i = M/u_i \in \text{sub}(M)$. Thus,

$$Q_i \in \text{sub}(\{M\} \cup R_E),\ 1 \leq i \leq n \tag{3.1}$$

By the definition of Q_i, obviously

$$Q_i \to^* N/u_i = \sigma(\alpha/u_i),\ 1 \leq i \leq n \tag{3.2}$$

holds if we choose the construction process of $\sigma(\alpha/u_i)$ from γ. Hence, by altering the order of the reductions of γ so that the construction of $\sigma(\alpha/u_i)$, $1 \leq i \leq n$, may follow the construction of the non-variable part of α, we obtain a k-step reduction sequence

$$\delta:\ M \to^* \alpha[u_1 \leftarrow Q_1, \cdots, u_n \leftarrow Q_n] \to^* N$$

where $N = \alpha[u_1 \leftarrow N/u_1, \cdots, u_n \leftarrow N/u_n]$ (see Figure 1). Let $Q = \alpha[u_1 \leftarrow Q_1, \cdots, u_n \leftarrow Q_n]$. Then, Q satisfies condition (i) of this lemma from Eq. (3.1) and condition (ii) holds from Eq. (3.2), so this lemma holds. □

We can use Lemma 1 to obtain the following Lemma 2, which is the key lemma for obtaining the replacement function Φ_R. It shows how to compute $\min(M \to^* N)$ (see Def. 3) by using the minimum values $\min(\theta)$'s of other instances θ's of reachability and of joinability when $M \to^* N$.

Lemma 2. Let γ: $M = M_0 \to M_1 \cdots \to M_k = N$ be a reduction sequence from a term M to N, where $l_g(\gamma) = k > 0$ is shortest, that is, where $\min(M \to^* N) = k$. Then, the following conditions (i) and (ii) hold.

(i) If γ is top-invariant, then

(a) $\min(M \to^* N) = \sum_{i=1}^{n} \min(M_i \to^* N_i)$

where $M = fM_1 \cdots M_n$ and $N = fN_1 \cdots N_n$ for some $f \in F$ and $M_i, N_i \in T$, $1 \leq i \leq n$.

(ii) If γ is not top-invariant, that is, if $\exists i (1 \leq i < k)$: $M_i = \sigma(\alpha)$ and $M_{i+1} = \beta$, where $\alpha \to \beta \in E$ and $\sigma: X \to T$ is a mapping, then there exists a term $Q \in \Delta_E(M)$ satisfying the following conditions (b) and (c):

(b) $\min(M \to^* N) = \min(M \to^* Q) + \min(Q \to^* \beta) + \min(\beta \to^* N)$

(c) $\min(Q \to^* \beta) = 1 + \sum_{x \in V(\alpha)} \min(\downarrow\{Q/u \mid u \in \mathcal{O}_x(\alpha)\})$

(Here, $Q=\alpha[u\leftarrow Q/u,\ u\in \mathcal{O}_X(\alpha)]$ holds.)

Proof.

(i) The proof is obvious.

(ii) Since the length of γ is shortest,

$$\min(M\to^* N)=\min(M\to^* M_i)+\min(M_i\to M_{i+1}) + \min(M_{i+1}\to^* N) \quad (3.3)$$

holds. Consider the subsequence γ_1: $M=M_0\to M_1\cdots\to M_i=\sigma(\alpha)$ of γ. Then, by Lemma 1, there exists a reduction sequence δ_1: $M\to^* Q\to^* M_i$ having the same i steps such that

(*I*) $Q=\alpha[u_j\leftarrow Q_j,\ 1\le j\le n]\in \Delta_E(M)$

where $\{u_1,\cdots,u_n\}=\mathcal{O}_X(\alpha)$ and $Q_j\in \mathrm{sub}(\{M\}\cup R_E)$, $1\le j\le n$, and

(*II*) $Q_j\to^* M_i/u_j=\sigma(\alpha/u_j)$, $1\le j\le n$

Note that γ_1 is shortest, since γ is shortest. Thus, δ_1 is also shortest, so

$$\min(M\to^* M_i)=\min(M\to^* Q)+\min(Q\to^* M_i) \quad (3.4)$$

From Eqs. (3.3) and (3.4), we have

$$\min(M\to^* N)=\min(M\to^* Q)+\min(Q\to^* M_i) + \min(M_i\to M_{i+1})+\min(M_{i+1}\to^* N) \quad (3.5)$$

Note that

$$\min(Q\to^* M_{i+1})=\min(Q\to^* M_i)+\min(M_i\to M_{i+1}) \quad (3.6)$$

since obviously the left-hand side $\le$ the right-hand side, and if the left-hand side $<$ the right-hand side, then we would have a reduction sequence from M to N of length $<\min(M\to^* N)$ from Eq. (3.5), a contradiction. Hence

$$\min(M\to^* N)=\min(M\to^* Q)+\min(Q\to^* M_{i+1}) + \min(M_{i+1}\to^* N)$$

holds, from Eqs. (3.5) and (3.6). Thus, condition (b) of this lemma holds, since $M_{i+1}=\beta$.

To show that condition (c) of this lemma holds, let

$$l=\sum_{x\in V(\alpha)} \min(\downarrow\{Q/u \mid u\in \mathcal{O}_x(\alpha)\}) \quad (3.7)$$

Then,

$$\min(Q\to^* M_i)\ge l \quad (3.8)$$

holds, since the subsequence δ' from Q to $M_i=\sigma(\alpha)$ of δ_1 is shortest and ensures joinability $\downarrow\{Q/u \mid u\in \mathcal{O}_x(\alpha)\}$ for all $x\in V(\alpha)$ by the above condition (*II*). Thus, from Eqs. (3.6) and (3.8) and since $\min(M_i\to M_{i+1})=1$,

$$\min(Q\to^* M_{i+1})\ge l+1$$

Conversely, from Eq. (3.7), there exists an l-step reduction sequence from Q to $\sigma'(\alpha)$ for some mapping σ': $X\to T$. Hence, we have

$$\min(Q\to^* M_{i+1})\le \min(Q\to^* \sigma'(\alpha))+1\le l+1$$

(Note that $M_{i+1}=\beta$ and $\alpha\to\beta\in E$.) Thus, $\min(Q\to^* M_{i+1})=l+1$, that is, condition (*c*) of this lemma holds. □

Using lemma 2, we can now define the replacement function Φ_R: $\mathrm{Set}(R)\to 2^{\mathrm{Set}(R,J)}$ as follows:

Replacement Function Φ_R.

Let $(M, N)_R\in \mathrm{Set}(R)$, that is, let $\min(M\to^* N)<\infty$: N be reachable from M. Then, $\Phi_R((M,N)_R)$ is defined iff $M\ne N$. Let $M\ne N$. If there exists a shortest reduction sequence γ: $M\to^* N$ such that γ is top-invariant, then

$$\Phi_R((M,N)_R)=\{(M_i,N_i)_R \mid 1\le i\le n,\ M=fM_1\cdots M_n,\ N=fN_1\cdots N_n \text{ for some } f\in F\}$$

where condition (*a*) of Lemma 2 holds. Otherwise,

$$\Phi_R((M,N)_R)=\{(M,Q)_R,(\beta,N)_R\} \cup (\cup_{x\in V(\alpha)}\{\{Q/u \mid u\in \mathcal{O}_x(\alpha)\}_J\})$$

where $\alpha\to\beta\in E$ and Q satisfy conditions (*b*) and (*c*) of Lemma 2. (Here, $Q=\alpha[u\leftarrow Q/u,\ u\in \mathcal{O}_X(\alpha)]\in \Delta_E(M)$ where $Q/u\in \mathrm{sub}(\{M\}\cup R_E)$.)

Then, the following Lemma 3 holds for Φ_R.

Lemma 3. Let $\theta=(M,N)_R\in \mathrm{Set}(R)$, that is, let $\min(\theta)<\infty$. Then, the following conditions (i) and (ii) hold:

(i) If $\min(\theta)>0$, then $\Phi_R(\theta)$ is defined and $\Phi_R(\theta)\ne\phi$. Further,

$$\min(\theta)\le 1+\sum_{\theta'\in\Phi_R(\theta)}\min(\theta')$$

$$\|\Phi_R(\theta)\|\le l_0$$

where $l_0=\max\{l_1,l_2\}$, $l_1=\max\{a(f) \mid f\in F\}$ and $l_2=2+\max\{\|V(\alpha)\| \mid \alpha\in L_E\}$.

(ii) If $\min(\theta)=0$, then $\Phi_R(\theta)$ is undefined.

Proof: Obvious from Lemma 2 and the definition of Φ_R. □

Note that we do not explain how to inplement Φ_R. However, our arguments require only the existence of the function Φ_R, which is ensured by Lemma 2.

Next, to define the other replacement function Φ_J for joinability, we need the following Lemma 4.

Lemma 4. Let $\min(\downarrow\{M_1,\cdots,M_k\})<\infty$, that is, let $\{M_1,\cdots,M_k\}$ be joinable where $k>1$ and $M_i\ne M_j$ for some i, j $(1\le i<j\le k)$. Let the k shortest reduction sequences to some common term N be γ_1: $M_1\to^* N,\cdots,\gamma_k$: $M_k\to^* N$, where $\min(\downarrow\{M_1,\cdots,M_k\})=\Sigma_{i=1}^k lg(\gamma_i)$.

If all γ_i is top-invariant, $1\le i\le k$, then

(d) $\min(\downarrow\{M_1,\cdots,M_k\})=\Sigma_{j=1}^n \min(\downarrow\{M_1/j,\cdots,M_k/j\})$ where n is the arity of $\mathrm{root}(M_1)=\cdots=\mathrm{root}(M_k)$.

If some γ_i is not top-invariant, $1\le i\le k$, then there exists $\beta\in R_E$ such that

(e) $\min(\downarrow\{M_1,\cdots,M_k\})$
$=\min(M_i\to^*\beta)+\min(\downarrow\{M_1,\cdots,M_{i-1},\beta,M_{i+1},\cdots,M_k\})$

where $\min(\downarrow\{M_1,\cdots,M_k\})\ge\min(M_i\to^*\beta)>0$ (so that $\min(\downarrow\{M_1,\cdots,M_k\})>\min(\downarrow\{M_1,\cdots,M_{i-1},\beta,M_{i+1},\cdots,M_k\})$).

Proof. If all γ_i are top-invariant, $1\le i\le k$, then obviously (d) holds. If some γ_i is not top-invariant, $1\le i\le k$, then some $\beta\in R_E$ appears in γ_i, that is, γ_i:

$M_i \to^+ \beta \to^* N$, since TRS E is right-ground. Hence,

$$\min(\downarrow\{M_1, \cdots, M_k\}) = \min(M_i \to^* \beta) + \min(\downarrow\{M_1, \cdots, M_{i-1}, \beta, M_{i+1}, \cdots, M_k\})$$

where $\min(M_i \to^* \beta) > 0$. Thus, condition (e) holds. □

Using Lemma 4, we now define the replacement function Φ_J: $\text{Set}(J) \to 2^{\text{Set}(R,J)}$ as follows:

Replacement Function Φ_J.

Let $\{M_1, \cdots, M_k\}_J \in \text{Set}(J)$, that is, let $\min(\downarrow\{M_1, \cdots, M_k\}) < \infty$: $\{M_1, \cdots, M_k\}$ be joinable. Then, $\Phi_J(\{M_1, \cdots, M_k\}_J)$ is defined iff $M_i \neq M_j$ for some i, j where $1 \leq i < j \leq k$. If there exist shortest reduction sequences γ_1: $M_1 \to^* N, \cdots, \gamma_k$: $M_k \to^* N$ (to some common term N) such that $\min(\downarrow\{M_1, \cdots, M_k\}) = \sum_{i=1}^{k} lg(\gamma_i)$ and all γ_i's are top-invariant, $1 \leq i \leq k$, then

$$\Phi_J(\{M_1, \cdots, M_k\}_J) = \{\{M_1/j, \cdots, M_k/j\}_J \mid 1 \leq j \leq n, \ n \text{ is the arity of root}(M_1)\}$$

where condition (d) of Lemma 4 holds. Otherwise,

$$\Phi_J(\{M_1, \cdots, M_k\}_J) = \{(M_i, \beta)_R\} \cup \{\{M_1, \cdots, M_{i-1}, \beta, M_{i+1}, \cdots, M_k\}_J\}$$

for some i $(1 \leq i \leq k)$, where $\beta \in R_E$ satisfies condition (e) of Lemma 4.

Then, the following Lemma 5 holds for Φ_J.

Lemma 5. Let $\theta = \{M_1, \cdots, M_k\}_J \in \text{Set}(J)$, that is, let $\min(\theta) < \infty$. Then, the following conditions (i) and (ii) hold:

(i) If $\min(\theta) > 0$, then $\Phi_J(\theta)$ is defined and $\Phi_J(\theta) \neq \phi$. Further,

$$\min(\theta) = \sum_{\theta' \in \Phi_J(\theta)} \min(\theta')$$

$$\|\Phi_J(\theta)\| \leq \max\{l_1, 2\} \leq l_0$$

where l_1 and l_0 are the constants defined in Lemma 3.

(ii) If $\min(\theta) = 0$, then $\Phi_J(\theta)$ is undefined.

Proof: Obvious from Lemma 4 and the definition of Φ_J. □

4. Decision Procedure

In the previous section, we introduced the replacement function Φ: $\text{Set}(R, J) \to 2^{\text{Set}(R,J)}$ such that

$$\Phi(\theta) = \begin{cases} \Phi_R(\theta) \ \textit{if} \ \theta \in \text{Set}(R) \\ \Phi_J(\theta) \ \textit{if} \ \theta \in \text{Set}(J) \end{cases}$$

In this section, by using this function Φ, we introduce a reduction system (i.e., a ground TRS) $\Rightarrow$ over $\text{Set}(R, J)$ as follows:

$$\theta_1 \Longrightarrow \theta_2 \quad \text{iff} \quad \theta_2 \in \Phi(\theta_1)$$

We first show that this system $\Rightarrow$ is noetherian, that is, that no infinite sequences of $\Rightarrow$ exist. Next, we use this noetherian property of $\Rightarrow$ to show that both reachability and joinability are decidable for right-ground TRSs.

For this purpose, we define the sizes of the instances of reachability and of joinability as follows:

Definition 5. For $\theta \in \text{Set}(R, J)$, size($\theta$) in $\mathcal{N}_0 \times \mathcal{N}_0$ is defined as follows:

$$\text{size}((M, N)_R) = (\min(M \to^* N), |M|)$$

$$\text{size}(\{M_1, \cdots, M_k\}_J) = (\min(\downarrow\{M_1, \cdots, M_k\}), |M_1| + \cdots + |M_k|)$$

Definition 6. An ordering $>$ on $\mathcal{N}_0 \times \mathcal{N}_0$ is defined by $(n, k) > (n', k') \Leftrightarrow (n > n') \vee (n = n' \wedge k > k')$

Note that the ordering $>$ is noetherian [2-4]. We are now ready to show that the above system $\Rightarrow$ is noetherian, that is, that for any $\theta \in \text{Set}(R, J)$, if $\theta' \in \Phi(\theta)$, then size(θ) $>$ size(θ') holds. This means that all sequences of replacements $\Rightarrow$ (induced by Φ) from any instance of reachability or of joinability will eventually be reduced to the elements of form $(M, M)_R$ or $\{M, \cdots, M\}_J$. (Note that $\Phi((M, M)_R)$ and $\Phi(\{M, \cdots, M\}_J)$ are undefined by the definitions of Φ_R and of Φ_J, so further replacements are impossible.)

Lemma 6. Let θ, $\theta' \in \text{Set}(R, J)$ where $\theta' \in \Phi(\theta)$. Then, size(θ) $>$ size(θ') holds.

Proof. either (i) $\theta \in \text{Set}(R)$ or (ii) $\theta \in \text{Set}(J)$ holds.

(i) The case where $\theta = (M, N)_R \in \text{Set}(R)$:
If $\theta' = (M/i, N/i)_R \in \Phi(\theta)$ for some $i \in \mathcal{N}$, then by condition (a) of Lemma 2,

$$\min(\theta) \geq \min(\theta') \quad \text{and} \quad |M| > |M/i|$$

hold, so size(θ) $>$ size(θ'), as claimed.

The remaining cases are $\theta' = (M, Q)_R$, $(\beta, N)_R$ and $\{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J$ where $x \in V(\alpha)$. Here, $\alpha \to \beta \in E$ and Q satisfy conditions (b) and (c) of Lemma 2. Conditions (b) and (c) ensure that

$$\min(M \to^* N) > \min(M \to^* Q),$$

$$\min(M \to^* N) > \min(\beta \to^* N),$$

$$\min(M \to^* N) > \min(\downarrow\{Q/u \mid u \in \mathcal{N}_x(\alpha)\})$$

Thus, $\min(\theta) > \min(\theta')$ holds, so size(θ) $>$ size(θ').

(ii) The case where $\theta = \{M_1, \cdots, M_k\}_J \in \text{Set}(J)$, where $k \geq 2$: If $\theta' = \{M_1/j, \cdots, M_k/j\}_J \in \Phi(\theta)$ for some $j \in \mathcal{N}$, then by condition (d) of lemma 4 we have

$$\min(\theta) \geq \min(\theta') \quad \text{and} \quad \sum_{i=1}^{k} |M_i| > \sum_{i=1}^{k} |M_i/j|$$

Thus, size(θ) $>$ size(θ'), as claimed. The remaining case is either $\theta' = (M_i, \beta)_R$ for some i $(1 \leq i \leq k)$ or $\theta' = \{M_1, \cdots, M_{i-1}, \beta, M_{i+1}, \cdots, M_k\}_J$ where condition (e) of Lemma 4 holds. Condition (e) ensures that

$$\min(\downarrow\{M_1, \cdots, M_k\}) \geq \min(M_i \to^* \beta) \qquad (4.1)$$

$$\min(\downarrow\{M_1, \cdots, M_k\}) > \min(\downarrow\{M_1, \cdots, M_{i-1}, \beta, M_{i+1}, \cdots, M_k\}) \qquad (4.2)$$

If $\theta' = (M_i, \beta)_R$, then size(θ) $>$ size(θ'), from Eq. (4.1) and since $\sum_{j=1}^{k} |M_j| > |M_i|$. If $\theta' = \{M_1, \cdots, M_{i-1}, \beta, M_{i+1}, \cdots, M_k\}_J$ then obviously size(θ) $>$ size(θ'), from Eq. (4.2). Thus, this lemma holds. □

From Lemma 6, $\Rightarrow$ is noetherian. (Note that $\theta \Rightarrow \theta'$ iff $\theta' \in \Phi(\theta)$.) We now explain what this noetherian property means. Let $\theta \in \mathrm{Set}(R, J)$, where $0 < \min(\theta) < \infty$. Then, $\Phi(\theta)$ is defined and $\Phi(\theta) \neq \phi$, from the definition of Φ. (Note that $\min(\theta)=0$ iff $\theta=(M, M)_R$ or $\theta=\{M, \cdots, M\}_J$ for some term M.) Note that $\min(\theta') < \infty$ holds for any $\theta' \in \Phi(\theta)$, since $\min(\theta') \leq \min(\theta)$ holds because $\mathrm{size}(\theta') < \mathrm{size}(\theta)$. If $\min(\theta') > 0$, then $\Phi(\theta')$ is also defined and $\Phi(\theta') \neq \phi$. Thus, any sequence of $\Rightarrow$ from θ eventually goes to an element $\theta'' \in \mathrm{Set}(R, J)$ such that $\min(\theta'')=0$, since $\Rightarrow$ is noetherian.

Next, we will give an upper bound on the maximum length of the possible sequences of $\Rightarrow$ from a given $\theta \in \mathrm{Set}(R, J)$. To show this, we need the following definitions.

Definition 7. For $(M, N)_R \in \mathrm{Set}(R)$, we define $S_R((M, N)_R)$ and $S_J((M, N)_R)$ as follows:

$$S_R((M, N)_R)=\{(P, P')_R \mid P \in \mathrm{sub}(\{M\} \cup R_E),\ P' \in \mathrm{sub}(\Delta_E(M) \cup \{N\} \cup R_E)\}$$

$$S_J((M, N)_R)=\{\{Q_1, \cdots, Q_k\}_J \mid 0<k \leq a_E,\ Q_i \in \mathrm{sub}(\{M\} \cup R_E),\ 1 \leq i \leq k\}$$

(Here, $a_E=\max\{\|\mathcal{O}_x(\alpha)\| \mid \alpha \in L_E, x \in X\}$. See Def. 2 for $\Delta_E(M)$.) Note that both $S_R((M, N)_R)$ and $S_J((M, N)_R)$ are finite sets.

Definition 8. Let $\Gamma_J \in \mathrm{Set}(J)$, where $\Gamma=\{M_1, \cdots, M_k\}$. Then we define $S_R(\Gamma_J)$ and $S_J(\Gamma_J)$ as follows:

$$S_R(\Gamma_J)=\{(P, Q)_R \mid P \in \mathrm{sub}(\Gamma \cup R_E)\},\ Q \in \mathrm{sub}(\cup_{M_i \in \Gamma} \Delta_E(M_i) \cup R_E)\}$$

$$S_J(\Gamma_J)=\{\{Q_1, \cdots, Q_n\}_J \mid 0<n \leq \max(a_E, k),\ Q_i \in \mathrm{sub}(\Gamma \cup R_E),\ 1 \leq i \leq n\}$$

Note that both $S_R(\Gamma_J)$ and $S_J(\Gamma_J)$ are finite sets.

We now prove the following Lemma 7, which says that for any $\theta \in \mathrm{Set}(R, J)$, if θ' is reachable from θ (i.e., if $\theta \Rightarrow^* \theta'$), then $\theta' \in S_R(\theta) \cup S_J(\theta)$ holds. (It follows that the length of any sequence from θ to θ' is bounded by $\|S_R(\theta) \cup S_J(\theta)\|$ for each $\theta \in \mathrm{Set}(R, J)$, since $\Rightarrow$ is noetherian.)

Lemma 7.

(i) Let $\theta=(M, N)_R \in \mathrm{Set}(R)$. If θ' is reachable from θ (i.e., if $\theta \Rightarrow^* \theta'$), then $\theta' \in S_R(\theta) \cup S_J(\theta)$ holds.

(ii) Let $\Gamma_J \in \mathrm{Set}(J)$ where $\Gamma=\{M_1, \cdots, M_k\}$ for some $M_i \in T$, $1 \leq i \leq k$. If θ' is reachable from Γ_J (i.e., if $\Gamma_J \Rightarrow^* \theta'$), then $\theta' \in S_R(\Gamma_J) \cup S_J(\Gamma_J)$ holds.

Proof. The proof is straightforward (see Appendix).

From this lemma, the maximum length of the possible sequences of $\Rightarrow$ from a given $\theta \in \mathrm{Set}(R, J)$ is bounded by a fixed constant, namely, $\|S_R(\theta) \cup S_J(\theta)\|$, because the noetherian property of $\Rightarrow$ ensures that for any reduction sequence of $\Rightarrow$ from θ, each element in $S_R(\theta) \cup S_J(\theta)$ appears at most once. Thus, we have the following corollary.

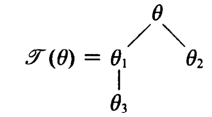
Corollary 1. Let $\theta \in \mathrm{Set}(R, J)$. Then, if $\theta \Rightarrow^n \theta'$ for some $\theta' \in \mathrm{Set}(R, J)$, then $n < m_\theta$ holds. Here, $m_\theta = \|S_R(\theta) \cup S_J(\theta)\|$. □

We are now ready to give an upper bound of $\min(\theta)$ for a given $\theta \in \mathrm{Set}(R, J)$. We define a tree $\mathcal{T}(\theta)$ constructed from θ as follows:

(i) The root of $\mathcal{T}(\theta)$ is labelled θ.

(ii) Each node in $\mathcal{T}(\theta)$ is labelled with some θ' in $S_R(\theta) \cup S_J(\theta)$ and has children with labels $\theta'_1, \cdots, \theta'_k$ iff $\Phi(\theta')$ is defined and $\Phi(\theta')=\{\theta'_1, \cdots, \theta'_k\}$.

For example, if $\Phi(\theta)=\{\theta_1, \theta_2\}$, $\Phi(\theta_1)=\{\theta_3\}$ and $\min(\theta_2)=\min(\theta_3)=0$, then $\mathcal{T}(\theta)$ is given as follows:



(Note that every leaf of $\mathcal{T}(\theta)$ has a label θ' such that $\min(\theta')=0$.) We call $\mathcal{T}(\theta)$ the min-tree of θ, which satisfies the following Lemma 8.

Lemma 8. Let $\theta \in \mathrm{Set}(R, J)$ and let $\mathcal{T}(\theta)$ be the min-tree of θ. Then, the following conditions (i)-(iii) hold.

(i) The height of $\mathcal{T}(\theta) \leq m_\theta$, where m_θ is the constant of Corollary 1.

(ii) Each node in $\mathcal{T}(\theta)$ has at most l_0 children, where l_0 is the constant of Lemmas 3 and 5 (i.e., $\|\Phi(\theta)\| \leq l_0$).

(iii) If a node in $\mathcal{T}(\theta)$ whose label is θ' has height k $(1 \leq k \leq m_\theta)$, then $\min(\theta') \leq \sum_{i=0}^{k-1} l_0^i$

Proof. The proofs of conditions (i) and (ii) are obvious from Corollary 1 and Lemmas 3 and 5.

The proof of condition (iii). We prove this by induction on k. If $k=1$, then the node with θ' is a leaf node, so that $\min(\theta')=0$ by the definition of $\mathcal{T}(\theta)$. Thus, condition (iii) holds.

Consider the case where $k>1$. Let $\Phi(\theta')=\{\theta'_1, \cdots, \theta'_n\} \neq \phi$. Then, $n \leq l_0$ holds, from the condition (ii). By the induction hypothesis,

$$\min(\theta'_j) \leq \sum_{i=0}^{k-2} l_0^i, \quad 1 \leq j \leq n, \tag{4.3}$$

holds (since the node whose label is θ'_j has height $k-1$). From Lemmas 3. (i) and 5. (i), we have

$$\begin{aligned}
\min(\theta') &\leq 1+\sum_{j=1}^{n} \min(\theta'_j) \\
&\leq 1+n \cdot \sum_{i=0}^{k-2} l_0^i \qquad \text{(from (4.3))} \\
&\leq 1+\sum_{i=1}^{k-1} l_0^i \qquad \text{(from (ii))} \\
&\leq \sum_{i=0}^{k-1} l_0^i
\end{aligned}$$

Thus, condition (iii) holds. □

Lemma 8 ensures that $\min(\theta) \leq \Sigma_{i=0}^{m_\theta-1} l_0^i (=l_\theta)$ holds for $\theta \in \mathrm{Set}(R, J)$. Hence, we can compute an upper bound of $\min(\theta)$, since it is obvious that m_θ and l_0 are computable. Thus, it follows that whether $\theta \in \mathrm{Set}(R, J)$, that is, whether $\min(\theta) < \infty$ (i.e., whether θ is true), is decidable both for $\theta=(M, N)_R$ and for $\theta=\{M_1, \cdots,$

$M_k\}_J$. (Note that we can check whether $M \rightarrow^l N$ for $l \leq l_\theta$ when $\theta = (M, N)_R$ and also check for $\theta = \{M_1, \cdots, M_k\}_J$.)

Therefore, we obtain the following main result.

Theorem 1.

(i) The reachability problem for right-ground TRSs is decidable.

(ii) The joinability problem for right-ground TRSs is decidable. □

As a corollary, the word problem is decidable for confluent and right-ground TRSs, since the word problem can be reduced to the joinability problem for confluent TRSs. This result is compared with the undecidability of the word problem for right-ground TRSs [6].

Corollary 2. The word problem is decidable for confluent and right-ground TRSs. □

We have shown that both reachability and joinability are decidable for right-ground TRSs. However, we have not explicitly shown how to construct such an algorithm. We now explain how to do so.

To obtain our main result, Theorem 1, we have shown that for any $\theta \in \mathrm{Set}(R, J)$, that is, $\min(\theta) < \infty$, we can construct the min-tree $\mathscr{T}(\theta)$ of θ by using the replacement function Φ. (Here, the height of $\mathscr{T}(\theta) < m_\theta$, from Lemma 8.) Note that the existence of Φ is ensured by Lemmas 2 and 4. Therefore, our algorithm can be constructed by using these results. That is, when θ is given as an input, this algorithm first guesses a candidate of Φ, attempts to construct $\mathscr{T}(\theta)$ from θ by using this candidate of Φ and then produces as output 'True: $\min(\theta) < \infty$' iff $\mathscr{T}(\theta)$ satisfies the required condition. More precisely, this algorithm, called Proc, is given as follows:

Proc(θ){where $\theta = (M, N)_R$ or $\theta = \{M_1, \cdots, M_n\}_J$};

(1) **if** $\min(\theta) = 0$ **then** return ('True: $\min(\theta) < \infty$');

(2) **for** each candidate Φ_{ca} of Φ **do**

begin

(2.1) construct $\mathscr{T}(\theta)$ where if the height of $\mathscr{T}(\theta)$ exceeds m_θ, then stop this construction (that is, this candidate Φ_{ca} has failed to construct $\mathscr{T}(\theta)$);

(2.2) **if** the construction of $\mathscr{T}(\theta)$ succeeds, **then** return ('True: $\min(\theta) < \infty$')

end;

(3) return ('False');

Here, we require that if $\mathscr{T}(\theta)$ is constructed, then every leaf of $\mathscr{T}(\theta)$ has a label θ' such that $\min(\theta') = 0$. Further, any candidate Φ_{ca} of Φ is required to satisfy the following conditions (A) and (B):

(A) The domain of Φ_{ca} is restricted to $S_R(\theta) \cup S_J(\theta)$ for a given input θ. (This sufficiency is ensured by Lemma 7.)

(B) For each $\theta' \in S_R(\theta) \cup S_J(\theta)$, the value $\Phi_{ca}(\theta')$ must be chosen from the possible values as $\Phi(\theta')$. For example, if $\theta' = (P, P')_R$, then $\Phi_{ca}(\theta')$ must be either $\{(P/i, P'/i)_R \mid 1 \leq i \leq m$, m is the arity of root$(P)\}$ or $\{(P, Q)_R, (\beta, P')_R\} \cup (\cup_{x \in V(\alpha)} \{\{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J\})$ for some $\alpha \rightarrow \beta \in E$ and $Q \in \Delta_E(P)$.

Note that $\|S_R(\theta) \cup S_J(\theta)\|$ is finite, from Definitions 7 and 8. Thus, the number of possible candidates Φ_{ca} of Φ is also finite. Hence, Proc(θ) always terminates. The proof of the correctness of Proc is easily obtained, and is therefore omitted. (Note that the correctness of line (2.1) of Proc is verified by Lemma 8, which states that the height of $\mathscr{T}(\theta) \leq m_\theta$, so we can conclude that if the height of $\mathscr{T}(\theta)$ exceeds m_θ in line (2.1), then the guess of Φ is incorrect. The correctness of line (3) is obvious, because if all possible candidates of Φ fail to construct $\mathscr{T}(\theta)$, then θ is false, that is, $\min(\theta) = \infty$, since if $\min(\theta) < \infty$, then the existence of Φ is ensured by Lemmas 2 and 4.)

Appendix:

The proof of condition (i) of Lemma 7:

Let $\theta \Rightarrow^k \theta'$, where $\theta = (M, N)_R$. We prove $\theta' \in S_R(\theta) \cup S_J(\theta)$ by induction on k. When $k=0$ the proof is trivial. We therefore consider the case in which $k>0$. Then, there esists θ'' such that $\theta \Rightarrow^{k-1} \theta'' \Rightarrow \theta'$. By the induction hypothesis, $\theta'' \in S_R(\theta) \cup S_J(\theta)$ holds (that is, $\theta'' \in S_R(\theta)$ or $\theta'' \in S_J(\theta)$).

(a) The case where $\theta'' \in S_R(\theta)$, that is, $\theta'' = (P, P')_R$ for some P, P' where

$$P \in \mathrm{sub}(\{M\} \cup R_E) \quad \text{(A.1)}$$

$$P' \in \mathrm{sub}(\Delta_E(M) \cup \{N\} \cup R_E) \quad \text{(A.2)}$$

Since $\theta'' \Rightarrow \theta'$, the definition of $\Rightarrow$ ensures that $\theta' \in \Phi(\theta'')$. Thus, from the definition of Φ (i.e., of Φ_R), either $\theta' = (P/i, P'/i)_R$ for some $i \in \mathscr{N}$ or $\theta' \in \{(P, Q)_R, (\beta, P')_R, \{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J\}$, where

$$\alpha \rightarrow \beta \in E \quad \text{(A.3)}$$

$$Q = \alpha[u \leftarrow Q/u, u \in \mathcal{O}_x(\alpha)] \in \Delta_E(P) \quad \text{(A.4)}$$

$$Q/u \in \mathrm{sub}(\{P\} \cup R_E) \quad \text{(A.5)}$$

If $\theta' = (P/i, P'/i)_R$, then obviously $\theta' \in S_R(\theta)$ holds, since $P/i \in \mathrm{sub}(P)$ and $P'/i \in \mathrm{sub}(P')$. (Note that sub is idempotent: $\mathrm{sub}(\mathrm{sub}(U)) = \mathrm{sub}(U)$ for any $U \subseteq T$.)

If $\theta' = (\beta, P')_R$, where $\beta \in R_E$ from Eq. (A.3), then obviously $\theta' \in S_R(\theta)$ holds, from Eq. (A.2).

If $\theta' = \{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J$, then from Eqs. (A.5) and (A.1), we have

$$Q/u \in \mathrm{sub}(\{P\} \cup R_E) \subseteq \mathrm{sub}(\{M\} \cup R_E) \quad \text{(A.6)}$$

Thus, $\theta' \in S_J(\theta)$ holds.

The remaining case is $\theta' = (P, Q)_R$. In this case, since $\Delta_E(P) \subseteq \Delta_E(M)$ holds from Definition 2 of Δ_E and Eq. (A.6), we have

$$Q \in \Delta_E(P) \subseteq \Delta_E(M)$$

from Eq. (A.4). Thus, $\theta' = (P, Q)_R \in S_R(\theta)$ holds.

(b) The case where $\theta'' = S_J(\theta)$, that is, $\theta'' = \{Q_1, \cdots, Q_n\}_J$ for some Q_i, $1 \leq i \leq n$, where

$$Q_i \in \mathrm{sub}(\{M\} \cup R_E) \quad \text{(A.7)}$$

By $\theta'' \Rightarrow \theta'$, the definition of $\Rightarrow$ ensures that $\theta' \in \Phi(\theta'')$. Thus, from the definition of Φ, either $\theta' = \{Q_1/j, \cdots,$

$Q_n/j\}_J$ for some $j \in \mathcal{N}$ or $\theta' \in \{(Q_i, \beta)_R, \{Q_1, \cdots, Q_{i-1}, \beta, Q_{i+1}, \cdots, Q_n\}_J\}$ for some i $(1 \leq i \leq n)$ and $\beta \in R_E$.

If $\theta' = \{Q_1/j, \cdots, Q_n/j\}_J$, then obviously $\theta' \in S_J(\theta)$ holds, since $Q_i/j \in \mathrm{sub}(\{M\} \cup R_E)$ from Eq. (A.7), $1 \leq i \leq n$.

If $\theta' = (Q_i, \beta)_R$, then obviously $\theta' \in S_R(\theta)$ holds, from Eq. (A.7) and since $\beta \in R_E$.

If $\theta' = \{Q_1, \cdots, Q_{i-1}, \beta, Q_{i+1}, \cdots, Q_n\}_J$, then $\theta' \in S_J(\theta)$ also holds, from Eq. (A.7) and since $\beta \in R_E$. Thus, in either case, $\theta' \in S_R(\theta) \cup S_J(\theta)$.

Hence, if $\theta \Rightarrow^k \theta'$, then $\theta' \in S_R(\theta) \cup S_J(\theta)$. Thus, (i) of Lemma 7 holds. □

The proof of condition (ii) of Lemma 7.

Let $\theta = \Gamma_J$, where $\Gamma = \{M_1, \cdots, M_k\}$, and let $\theta \Rightarrow^n \theta'$. We prove $\theta' \in S_R(\theta) \cup S_J(\theta)$ by induction on n. When $n=0$, the proof is trivial. We therefore consider the case in which $n>0$. Then, there exists θ'' such that $\theta \Rightarrow^{n-1} \theta'' \Rightarrow \theta'$. By the induction hypothesis, $\theta'' \in S_R(\theta) \cup S_J(\theta)$ holds (i.e., $\theta'' \in S_R(\theta)$ or $\theta'' \in S_J(\theta)$).

The case in which $\theta'' \in S_R(\theta)$, that is $\theta'' = (P, P')_R$ for some P, P', where

$$P \in \mathrm{sub}(\Gamma \cup R_E) \tag{A.8}$$

$$P' \in \mathrm{sub}(\cup_{M_i \in \Gamma} \Delta_E(M_i) \cup R_E) \tag{A.9}$$

Since $\theta'' \Rightarrow \theta'$, the definition of $\Rightarrow$ ensures that $\theta' \in \Phi(\theta'')$. Thus, from the definition of Φ, θ' satisfies one of the following conditions (c) and (d):

(c) $\theta' = (P/i, P'/i)_R$ for some $i \in \mathcal{N}$

(d) $\theta' \in \{(P, Q)_R, (\beta, P')_R, \{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J\}$,

where

$$\alpha \rightarrow \beta \in E \tag{A.10}$$

$$Q = \alpha[u \leftarrow Q/u, u \in \mathcal{O}_x(\alpha)] \in \Delta_E(P) \tag{A.11}$$

$$Q/u \in \mathrm{sub}(\{P\} \cup R_E) \tag{A.12}$$

If θ' satisfies condition (c), then obviously $\theta' \in S_R(\theta)$ holds, since $P/i \in \mathrm{sub}(P)$ and $P'/i \in \mathrm{sub}(P')$.

Let us therefore, consider the case where condition (d) holds.

If $\theta' = (\beta, P')_R$, then obviously $\theta' \in S_R(\theta)$ holds, since $\beta \in R_E$, from Eq. (A.10).

If $\theta' = \{Q/u \mid u \in \mathcal{O}_x(\alpha)\}_J$, then from Eqs. (A.8) and (A.12) we have

$$Q/u \in \mathrm{sub}(\{P\} \cup R_E) \subseteq \mathrm{sub}(\{M_j\} \cup R_E) \quad \text{for some } M_j \in \Gamma \tag{A.13}$$

Thus, $\theta' \in S_J(\theta)$ holds.

The remaining case is $\theta' = (P, Q)_R$. In this case, since $\Delta_E(P) \subseteq \Delta_E(M_j)$ holds from the definition of Δ_E and from Eq. (A.13), we have

$$Q \in \Delta_E(P) \subseteq \Delta_E(M_j)$$

from Eq. (A.11). Thus, $\theta' = (P, Q)_R \in S_R(\theta)$ holds.

The case where $\theta'' \in S_J(\theta)$, that is, $\theta'' = \{Q_1, \cdots, Q_m\}_J$ for some Q_i, $1 \leq i \leq m$, where

$$Q_i \in \mathrm{sub}(\Gamma \cup R_E) \tag{A.14}$$

Since $\theta'' \Rightarrow \theta'$, the definition of $\Rightarrow$ ensures that $\theta' \in \Phi(\theta'')$. Thus, from the definition of Φ, either $\theta' = \{Q_1/j, \cdots, Q_m/j\}_J$ for some $j \in \mathcal{N}$ or $\theta' \in \{(Q_i, \beta)_R, \{Q_1, \cdots, Q_{i-1}, \beta, Q_{i+1}, \cdots, Q_m\}_J\}$ for some i $(1 \leq i \leq m)$ and $\beta \in R_E$.

If $\theta' = \{Q_1/j, \cdots, Q_m/j\}_J$, then obviously $\theta' \in S_J(\theta)$ holds, since $Q_i/j \in \mathrm{sub}(\Gamma \cup R_E)$ from Eq. (A.14), $1 \leq i \leq m$.

If $\theta' = (Q_i, \beta)_R$, then obviously $\theta \in S_R(\theta)$ holds, from Eq. (A.14) and since $\beta \in R_E$.

If $\theta' = \{Q_1, \cdots, Q_{i-1}, \beta, Q_{i+1}, \cdots, Q_m\}_J$, then $\theta' \in S_J(\theta)$ also holds from Eq. (A.14) and since $\beta \in R_E$. Thus, in either case, $\theta' \in S_R(\theta) \cup S_J(\theta)$.

Hence, if $\theta \Rightarrow^k \theta'$ then $\theta' \in S_R(\theta) \cup S_J(\theta)$. Thus, (ii) of Lemma 7 holds. □

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