Research Contribution

On the Attainable Order of Convergence for Some Multipoint Iteration Functions

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In this paper, we deal with a class of multipoint iterative formulas that find new approximations to a zero of a function $f(x)$. First of all, we show that the attainable order of convergence is equal to 7 for a class of formulas that require two evaluations of $f(x)$ and two of $f'(x)$ per iteration. Furthermore, we show that the attainable order of convergence is equal to 4 for a class of formulas that require one evaluation of $f(x)$ and two of $f'(x)$ per iteration and that the attainable order of convergence is equal to 4 for a class of formulas that require two evaluations of $f(x)$ and one of $f'(x)$ per iteration.

1. Introduction

We consider numerical iterative formulas for the computation of the solution of the nonlinear scalar equation

$$f(x)=0 \qquad (1.1)$$

where f is a given function.

It follows from Theorem 2-4 [1, pp. 27-28] that the multipoint iterative method derived from the composition of two Newton methods has the fourth order of convergence for all simple zeros of Eq. (1.1). In a previous paper [2], we showed some fifth-order multipoint iterative formulas containing one parameter and requiring two evaluations of $f(x)$ and two of $f'(x)$ per iteration.

In this paper, we consider a class of iteration functions of the type

$$\phi(x)=x-hR(X, Y) \qquad (1.2)$$

where $R(\xi, \eta)$ is a function of ξ and η,

$$h=\frac{f(x)}{f'(x)}, \quad X=\frac{f(x+\alpha h)}{f(x)}, \quad Y=\frac{f'(x+\beta h+\gamma hx)}{f'(x)}$$

and α, β, and γ are parameters.

We show that the attainable order of convergence for Eq. (1.2) is equal to 7 by suitable choices of the parameters and the function $R(\xi, \eta)$.

2. Derivation of Iteration Functions

In this section, we suppose $f(x)$ and $R(\xi, \eta)$ are as smooth as necessary in x, ξ and η.

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Expanding $f(x+\alpha h)$ and $f'(x+\beta h+\gamma hX)$ respectively, we obtain

$$f(x+\alpha h)=f(x)+\sum_{i=1}^{7}\frac{\alpha^i h^i}{i!}f^{(i)}(x)+O(h^8)$$

and

$$f'(x+\beta h+\gamma hX)=f'(x)+\sum_{i=1}^{6}\frac{z^i}{i!}f^{(i+1)}(x)+O(h^7)$$

where $z=\beta h+\gamma hX$. Here, putting $A_i\equiv A_i(x)=f^{(i)}(x)/i!f'(x)$, $i=2(1)7$, we obtain

$$X=1+\alpha+\sum_{i=2}^{7}\alpha^i A_i h^{i-1}+O(h^7),$$

$$Y=1+\sum_{i=2}^{7}iA_i z^{i-1}+O(h^7)$$

and

$$z=\delta h+\gamma\sum_{i=2}^{6}\alpha^i A_i h^i+O(h^7)$$

where $\delta=\beta+\gamma(1+\alpha)$.

Hence we obtain

$$Y=1+\sum_{i=2}^{6}i\delta^{i-1}A_i h^{i-1}+\gamma H_1+\gamma^2 H_2+O(h^6)$$

where

$$\begin{aligned}H_1=&2\alpha^2A_2^2h^2+(2\alpha^3+6\delta\alpha^2)A_2A_3h^3+6\delta\alpha^3A_3^2h^4\\&+(2\alpha^4+12\delta^2\alpha^2)A_2A_4h^4+(6\delta\alpha^4+12\delta^2\alpha^3)A_3A_4h^5\\&+(2\alpha^5+20\delta^3\alpha^2)A_2A_5h^5\end{aligned}$$

and

$$H_2=3\alpha^4A_2^2A_3h^4+6\alpha^5A_2A_3^2h^5+12\delta\alpha^4A_2^2A_4h^5.$$

Putting

$$H=\sum_{i=2}^{7}\alpha^i A_i h^{i-1},\quad K=\sum_{i=2}^{6} i\delta^{i-1}A_i h^{i-1}+\gamma H_1+\gamma^2 H_2$$

and expanding $R(X, Y)$, we obtain

$$R(X, Y)=\sum_{n=0}^{6}\frac{1}{n!}\left(H\frac{\partial}{\partial\xi}+K\frac{\partial}{\partial\eta}\right)^n R(\varepsilon, 1)+O(h^6)$$

where $H\,\partial/\partial\xi+K\,\partial/\partial\eta$ is the differential operator and $\varepsilon=1+\alpha$. Hence we obtain

$$\begin{aligned}
hR(X, Y)=\;&hR+\left(\sum_{i=2}^{6}\alpha^i A_i h^i\right)R_{1,0}+\left(\sum_{i=2}^{6} i\delta^{i-1}A_i h^i+\gamma hH_1+\gamma^2 hH_2\right)R_{0,1}+\frac{1}{2}(\alpha^4A_2^2h^3+2\alpha^5A_2A_3h^4+\alpha^6A_3^2h^5\\
&+2\alpha^6A_2A_4h^5+2\alpha^7A_3A_4h^6+2\alpha^7A_2A_5h^6)R_{2,0}+[2\delta\alpha^2A_2^2h^3+(2\delta\alpha^3+3\delta^2\alpha^2)A_2A_3h^4+3\delta^2\alpha^3A_3^2h^5\\
&+(2\delta\alpha^4+4\delta^3\alpha^2)A_2A_4h^5+(3\delta^2\alpha^4+4\delta^3\alpha^3)A_3A_4h^6+(2\delta\alpha^5+5\delta^4\alpha^2)A_2A_5h^6+\gamma\{2\alpha^4A_2^3h^4\\
&+(4\alpha^5+6\delta\alpha^4)A_2^2A_3h^5+(2\alpha^6+12\delta\alpha^5)A_2A_3^2h^6+(4\alpha^6+12\delta^2\alpha^4)A_2^2A_4h^6\}+3\gamma^2\alpha^6A_2^3A_3h^6]R_{1,1}\\
&+\frac{1}{2}\{4\delta^2A_2^2h^3+12\delta^3A_2A_3h^4+9\delta^4A_3^2h^5+16\delta^4A_2A_4h^5+24\delta^5A_3A_4h^6+20\delta^5A_2A_5h^6\\
&+\gamma\{8\delta\alpha^2A_2^3h^4+(8\delta\alpha^3+36\delta^2\alpha^2)A_2^2A_3h^5+O(h^6)\}+4\gamma^2\{\alpha^4A_2^4h^5+O(h^6)\}R_{0,2}\\
&+\frac{1}{6}(\alpha^6A_2^3h^4+3\alpha^7A_2^2A_3h^5+3\alpha^8A_2A_3^2h^6+3\alpha^8A_2^2A_4h^6)R_{3,0}+\frac{1}{2}[2\delta\alpha^4A_2^3h^4+(4\delta\alpha^5+3\delta^2\alpha^4)A_2^2A_3h^5\\
&+(2\delta\alpha^6+6\delta^2\alpha^5)A_2A_3^2h^6+(4\delta\alpha^6+4\delta^3\alpha^4)A_2^2A_4h^6+\gamma\{2\alpha^6A_2^4h^5+(6\alpha^7+6\delta\alpha^6)A_2^3A_3h^6\}]R_{2,1}\\
&+\frac{1}{2}[4\delta^2\alpha^2A_2^3h^4+(4\delta^2\alpha^3+12\delta^3\alpha^2)A_2^2A_3h^5+(12\delta^3\alpha^3+9\delta^4\alpha^2)A_2A_3^2h^6+(4\delta^2\alpha^4+16\delta^4\alpha^2)A_2^2A_4h^6\\
&+\gamma\{8\delta\alpha^4A_2^4h^5+(16\delta\alpha^5+36\delta^2\alpha^4)A_2^3A_3h^6\}+4\gamma^2\alpha^6A_2^5h^6]R_{1,2}+\frac{1}{6}[8\delta^3A_2^3h^4+36\delta^4A_2^2A_3h^5\\
&+54\delta^5A_2A_3^2h^6+48\delta^5A_2^2A_4h^6+\gamma\{24\delta^2\alpha^2A_2^4h^5+(24\delta^2\alpha^3+144\delta^3\alpha^2)A_2^3A_3h^6\}+24\gamma^2\delta\alpha^4A_2^5h^6]R_{0,3}\\
&+\frac{1}{24}(\alpha^8A_2^4h^5+4\alpha^9A_2^3A_3h^6)R_{4,0}+\frac{1}{6}\{2\delta\alpha^6A_2^4h^5+(6\delta\alpha^7+3\delta^2\alpha^6)A_2^3A_3h^6+2\gamma\alpha^8A_2^5h^6\}R_{3,1}\\
&+\frac{1}{4}\{4\delta^2\alpha^4A_2^4h^5+(8\delta^2\alpha^5+12\delta^3\alpha^4)A_2^3A_3h^6+8\gamma\delta\alpha^6A_2^5h^6\}R_{2,2}+\frac{1}{6}\{8\delta^3\alpha^2A_2^4h^5+(8\delta^3\alpha^3+36\delta^4\alpha^2)A_2^3A_3h^6\\
&+24\gamma\delta^2\alpha^4A_2^5h^6\}R_{1,3}+\frac{1}{24}(16\delta^4A_2^4h^5+96\delta^5A_2^3A_3h^6+64\gamma\delta^3\alpha^2A_2^5h^6)R_{0,4}+\frac{1}{120}\alpha^{10}A_2^5h^6R_{5,0}\\
&+\frac{1}{12}\delta\alpha^8A_2^5h^6R_{4,1}+\frac{1}{3}\delta^2\alpha^6A_2^5h^6R_{3,2}+\frac{2}{3}\delta^3\alpha^4A_2^5h^6R_{2,3}+\frac{2}{3}\delta^4\alpha^2A_2^5h^6R_{1,4}+\frac{4}{15}\delta^5A_2^5h^6R_{0,5}+O(h^7)
\end{aligned}$$

where

$$R\equiv R(\varepsilon, 1),\quad R_{i,0}\equiv\frac{\partial^i R(\varepsilon, 1)}{\partial\xi^i},\quad R_{0,j}\equiv\frac{\partial^j R(\varepsilon, 1)}{\partial\eta^j},$$

and

$$R_{i,j}\equiv\frac{\partial^{i+j}R(\varepsilon, 1)}{\partial\xi^i\partial\eta^j}.$$

Since the basic sequence [1, pp. 78–84] is given by

$$\begin{aligned}
E_7=\;&x-h-A_2h^2-2A_2^2h^3+A_3h^3-5A_2^3h^4+5A_2A_3h^4-A_4h^4-(14A_2^4-21A_2^2A_3+3A_3^2+6A_2A_4-A_5)h^5\\
&-(42A_2^5-84A_2^3A_3+28A_2A_3^2+28A_2^2A_4-7A_3A_4-7A_2A_5+A_6)h^6
\end{aligned}$$

we obtain

$$\begin{aligned}
\phi(x)-E_7=\;&(1-R)h+\{1-(\alpha^2R_{1,0}+2\delta R_{0,1})\}A_2h^2+\left\{2-\left(\frac{1}{2}\alpha^4R_{2,0}+2\delta\alpha^2R_{1,1}+2\delta^2R_{0,2}+2\gamma\alpha^2R_{0,1}\right)\right\}A_2^2h^3\\
&+\{-1-(\alpha^3R_{1,0}+3\delta^2R_{0,1})\}A_3h^3+\left\{5-\left(\frac{1}{6}\alpha^6R_{3,0}+\delta\alpha^4R_{2,1}+2\delta^2\alpha^2R_{1,2}\right.\right.\\
&\left.\left.+\frac{4}{3}\delta^3R_{0,3}+2\gamma\alpha^4R_{1,1}+C_1R_{0,2}\right)\right\}A_2^3h^4+[-5-\{\alpha^5R_{2,0}+(2\delta\alpha^3+3\delta^2\alpha^2)R_{1,1}+6\delta^3R_{0,2}
\end{aligned}$$

$$+\gamma(2\alpha^3+6\delta\alpha^2)R_{0,1}\}]A_2A_3h^4+\{1-(\alpha^4R_{1,0}+4\delta^3R_{0,1})\}A_4h^4+\left[14-\left\{\frac{1}{24}\alpha^8R_{4,0}+\frac{1}{3}\delta\alpha^6R_{3,1}+\delta^2\alpha^4R_{2,2}\right.\right.$$

$$\left.\left.+\frac{4}{3}\delta^3\alpha^2R_{1,3}+\frac{2}{3}\delta^4R_{0,4}+\gamma(\alpha^6R_{2,1}+4\delta\alpha^4R_{1,2}+4\delta^2\alpha^2R_{0,3})+C_2R_{0,2}\right\}\right]A_2^4h^5$$

$$+\left[-21-\left\{\frac{1}{2}\alpha^7R_{3,0}+\frac{1}{2}(4\delta\alpha^5+3\delta^2\alpha^4)R_{2,1}+(2\delta^2\alpha^3+6\delta^3\alpha^2)R_{1,2}+6\delta^4R_{0,3}\right.\right.$$

$$\left.\left.+\gamma(4\alpha^5+6\delta\alpha^4)R_{1,1}+B_1R_{0,1}+C_3R_{0,2}\right\}\right]A_2^3A_3h^5+\left\{3-\left(\frac{1}{2}\alpha^6R_{2,0}+3\delta^2\alpha^3R_{1,1}+\frac{9}{2}\delta^4R_{0,2}+B_2R_{0,1}\right)\right\}A_3^2h^5$$

$$+[6-\{\alpha^6R_{2,0}+(2\delta\alpha^4+4\delta^3\alpha^2)R_{1,1}+8\delta^4R_{0,2}+\gamma(2\alpha^4+12\delta^2\alpha^2)R_{0,1}\}]A_2A_4h^5+\{-1-(\alpha^5R_{1,0}+5\delta^4R_{0,1})\}A_5h^5$$

$$+\left[42-\left\{\frac{1}{120}\alpha^{10}R_{5,0}+\frac{1}{12}\delta\alpha^8R_{4,1}+\frac{1}{3}\delta^2\alpha^6R_{3,2}+\frac{2}{3}\delta^3\alpha^4R_{2,3}+\frac{2}{3}\delta^4\alpha^2R_{1,4}+\frac{4}{15}\delta^5R_{0,5}\right.\right.$$

$$\left.\left.+\gamma\left(\frac{1}{3}\alpha^8R_{3,1}+2\delta\alpha^6R_{2,2}+4\delta^2\alpha^4R_{1,3}+\frac{8}{3}\delta^3\alpha^2R_{0,4}\right)+\gamma^2(2\alpha^6R_{1,2}+4\delta\alpha^4R_{0,3})\right\}\right]A_2^5h^6$$

$$+\left[-84-\left\{\frac{1}{6}\alpha^9R_{4,0}+\frac{1}{2}(2\delta\alpha^7+\delta^2\alpha^6)R_{3,1}+(2\delta^2\alpha^5+3\delta^3\alpha^4)R_{2,2}+\frac{1}{3}(4\delta^3\alpha^3+18\delta^4\alpha^2)R_{1,3}\right.\right.$$

$$\left.\left.+4\delta^5R_{0,4}+\gamma(3\alpha^7+3\delta\alpha^6)R_{2,1}+\gamma(8\delta\alpha^5+18\delta^2\alpha^4)R_{1,2}+\gamma(4\delta^2\alpha^3+24\delta^3\alpha^2)R_{0,3}+3\gamma^2\alpha^6R_{1,1}+C_4R_{0,2}\right\}\right]A_2^3A_3h^6$$

$$+\left[28-\left\{\frac{1}{2}\alpha^8R_{3,0}+(\delta\alpha^6+3\delta^2\alpha^5)R_{2,1}+\frac{1}{2}(12\delta^3\alpha^3+9\delta^4\alpha^2)R_{1,2}+9\delta^5R_{0,3}+\gamma(2\alpha^6+12\delta\alpha^5)R_{1,1}\right.\right.$$

$$\left.\left.+B_3R_{0,1}+C_5R_{0,2}\right\}\right]A_2A_3^2h^6+\left[28-\left\{\frac{1}{2}\alpha^8R_{3,0}+(2\delta\alpha^6+2\delta^3\alpha^4)R_{2,1}+(2\delta^2\alpha^4+8\delta^4\alpha^2)R_{1,2}\right.\right.$$

$$\left.\left.+8\delta^5R_{0,3}+\gamma(4\alpha^6+12\delta^2\alpha^4)R_{1,1}+B_4R_{0,1}+C_6R_{0,2}\right\}\right]A_2^2A_4h^6+[-7-\{\alpha^7R_{2,0}+(3\delta^2\alpha^4+4\delta^3\alpha^3)R_{1,1}$$

$$+12\delta^5R_{0,2}+B_5R_{0,1}\}]A_3A_4h^6+[-7-\{\alpha^7R_{2,0}+(2\delta\alpha^5+5\delta^4\alpha^2)R_{1,1}+10\delta^5R_{0,2}+B_6R_{0,1}\}]A_2A_5h^6$$

$$+\{1-(\alpha^6R_{1,0}+6\delta^5R_{0,1})\}A_6h^6+O(h^7)$$

where B_1, B_2, B_3, B_4, B_5, and B_6 denote the coefficients of $R_{0,1}A_2^2A_3h^4$, $R_{0,1}A_2^3h^4$, $R_{0,1}A_2A_3^2h^5$, $R_{0,1}A_2^2A_4h^5$, $R_{0,1}A_3A_4h^5$, and $R_{0,1}A_2A_5h^5$ in $hR(X, Y)$, respectively, and C_1, C_2, C_3, C_4, C_5, and C_6 denote the coefficients of $R_{0,2}A_2^3h^3$, $R_{0,2}A_2^4h^4$, $R_{0,2}A_2^2A_3h^4$, $R_{0,2}A_2^3A_3h^5$, $R_{0,2}A_2A_3^2h^5$, and $R_{0,2}A_2^2A_4h^5$ in $hR(X, Y)$, respectively.

Hence it follows from Theorem 5-2 [1, pp. 86–87] that for (1.2) to be iteration functions of order 7, the following system of equations must be satisfied:

h: $R=1$ ⓪

A_2h^2: $\alpha^2R_{1,0}+2\delta R_{0,1}=1$ ①

$A_2^2h^3$: $\frac{1}{2}\alpha^4R_{2,0}+2\delta\alpha^2R_{1,1}+2\gamma\alpha^2R_{0,1}+2\delta^2R_{0,2}=2$ ②

A_3h^3: $\alpha^3R_{1,0}+3\delta^2R_{0,1}=-1$ ①$^{(1)}$

$A_2^3h^4$: $\frac{1}{6}\alpha^6R_{3,0}+\delta\alpha^4R_{2,1}+2\delta^2\alpha^2R_{1,2}+\frac{4}{3}\delta^3R_{0,3}+2\gamma\alpha^4R_{1,1}+C_1R_{0,2}=5$ ③

$A_2A_3h^4$: $\alpha^5R_{2,0}+(2\delta\alpha^3+3\delta^2\alpha^2)R_{1,1}+6\delta^3R_{0,2}+\gamma(2\alpha^3+6\delta\alpha^2)R_{0,1}=-5$ ②$^{(1)}$

A_4h^4: $\alpha^4R_{1,0}+4\delta^3R_{0,1}=1$ ①$^{(2)}$

$A_2^4h^5$: $\frac{1}{24}\alpha^8R_{4,0}+\frac{1}{3}\delta\alpha^6R_{3,1}+\delta^2\alpha^4R_{2,2}+\frac{4}{3}\delta^3\alpha^2R_{1,3}+\frac{2}{3}\delta^4R_{0,4}$

$+\gamma(\alpha^6R_{2,1}+4\delta\alpha^4R_{1,2}+4\delta^2\alpha^2R_{0,3})+C_2R_{0,2}=14$ ④

$A_2^3A_3h^5$: $\frac{1}{2}\alpha^7R_{3,0}+\frac{1}{2}(4\delta\alpha^5+3\delta^2\alpha^4)R_{2,1}+(2\delta^2\alpha^3+6\delta^3\alpha^2)R_{1,2}+6\delta^4R_{0,3}+\gamma(4\alpha^5+6\delta\alpha^4)R_{1,1}+B_1R_{0,1}+C_3R_{0,2}=-21$ ③$^{(1)}$

$A_3^2h^5$: $\frac{1}{2}\alpha^6R_{2,0}+3\delta^2\alpha^3R_{1,1}+\frac{9}{2}\delta^4R_{0,2}+B_2R_{0,1}=3$ ②$^{(2)}$

$A_2A_4h^5$: $\alpha^6R_{2,0}+(2\delta\alpha^4+4\delta^3\alpha^2)R_{1,1}+8\delta^4R_{0,2}+\gamma(2\alpha^4+12\delta^2\alpha^2)R_{0,1}=6$ ②$^{(3)}$

$$A_5h^5:\ \alpha^5R_{1,0}+5\delta^4R_{0,1}=-1 \qquad ①^{(3)}$$

$$A_2^5h^6:\ \frac{1}{120}\alpha^{10}R_{5,0}+\frac{1}{12}\delta\alpha^8R_{4,1}+\frac{1}{3}\delta^2\alpha^6R_{3,2}+\frac{2}{3}\delta^3\alpha^4R_{2,3}+\frac{2}{3}\delta^4\alpha^2R_{1,4}$$
$$+\frac{4}{15}\delta^5R_{0,5}+\gamma\left(\frac{1}{3}\alpha^8R_{3,1}+2\delta\alpha^6R_{2,2}+4\delta^2\alpha^4R_{1,3}+\frac{8}{3}\delta^3\alpha^2R_{0,4}\right)+\gamma^2(2\alpha^6R_{1,2}+4\delta\alpha^4R_{0,3})=42 \qquad ⑤$$

$$A_2^3A_3h^6:\ \frac{1}{6}\alpha^9R_{4,0}+\frac{1}{2}(2\delta\alpha^7+\delta^2\alpha^6)R_{3,1}+(2\delta^2\alpha^5+3\delta^3\alpha^4)R_{2,2}+\frac{1}{3}(4\delta^3\alpha^3+18\delta^4\alpha^2)R_{1,3}$$
$$+4\delta^5R_{0,4}+\gamma(3\alpha^7+3\delta\alpha^6)R_{2,1}+\gamma(8\delta\alpha^5+18\delta^2\alpha^4)R_{1,2}$$
$$+\gamma(4\delta^2\alpha^3+24\delta^3\alpha^2)R_{0,3}+3\gamma^2\alpha^6R_{1,1}+C_4R_{0,2}=-84 \qquad ④^{(1)}$$

$$A_2A_3^2h^6:\ \frac{1}{2}\alpha^8R_{3,0}+(\delta\alpha^6+3\delta^2\alpha^5)R_{2,1}+\frac{1}{2}(12\delta^3\alpha^3+9\delta^4\alpha^2)R_{1,2}+9\delta^5R_{0,3}$$
$$+\gamma(2\alpha^6+12\delta\alpha^5)R_{1,1}+B_3R_{0,1}+C_5R_{0,2}=28 \qquad ③^{(2)}$$

$$A_2^2A_4h^6:\ \frac{1}{2}\alpha^8R_{3,0}+(2\delta\alpha^6+2\delta^3\alpha^4)R_{2,1}+(2\delta^2\alpha^4+8\delta^4\alpha^2)R_{1,2}+8\delta^5R_{0,3}+\gamma(4\alpha^6+12\delta^2\alpha^4)R_{1,1}+B_4R_{0,1}+C_6R_{0,2}=28 \qquad ③^{(3)}$$

$$A_3A_4h^6:\ \alpha^7R_{2,0}+(3\delta^2\alpha^4+4\delta^3\alpha^3)R_{1,1}+12\delta^5R_{0,2}+B_5R_{0,1}=-7 \qquad ②^{(4)}$$

$$A_2A_5h^6:\ \alpha^7R_{2,0}+(2\delta\alpha^5+5\delta^4\alpha^2)R_{1,1}+10\delta^5R_{0,2}+B_6R_{0,1}=-7 \qquad ②^{(5)}$$

$$A_6h^6:\ \alpha^6R_{1,0}+6\delta^5R_{0,1}=1 \qquad ①^{(4)}$$

Then, from equations ①, ①$^{(1)}$ and ①$^{(2)}$ it follows that

$$\begin{vmatrix} \alpha^2 & 2\delta & -1 \\ \alpha^3 & 3\delta^2 & 1 \\ \alpha^4 & 4\delta^3 & -1 \end{vmatrix}=0$$

Thus $\alpha^2(\alpha+1)\delta\{4\delta^2+3\delta-(3\delta+2)\alpha\}=0$, and hence

$$\delta=0 \text{ or } \alpha=-1 \text{ or } 4\delta^2+3\delta=(3\delta+2)\alpha.$$

Case 1: $\delta=0$. It follows from systems ① and ①$^{(1)}$ and from $\delta=\beta+\gamma(1+\alpha)$ that $\alpha=-1$ and $\beta=0$. Furthermore, it is easy to show that systems ②$^{(1)}$ and ②$^{(3)}$ cannot be satisfied if $\alpha=-1$ and $\beta=0$.

Case 2: $\alpha=-1$. It follows from systems ①, ①$^{(1)}$, ①$^{(2)}$, ①$^{(3)}$, and ①$^{(4)}$ and from $\delta=\beta+\gamma(1+\alpha)$ that $R_{1,0}=1$, $R_{0,1}=0$, and $\delta=\beta$. Furthermore, systems ②, ②$^{(1)}$, ②$^{(2)}$ and ②$^{(3)}$ can be represented in the following form:

$$(\mathrm{A})\quad \begin{cases} \frac{1}{2}R_{2,0}+2\beta R_{1,1}+2\beta^2R_{0,2}=2 \\ -R_{2,0}+(3\beta^2-2\beta)R_{1,1}+6\beta^3R_{0,2}=-5 \\ \frac{1}{2}R_{2,0}-3\beta^2R_{1,1}+\frac{9}{2}\beta^4R_{0,2}=3 \\ R_{2,0}+(4\beta^3+2\beta)R_{1,1}+8\beta^4R_{0,2}=6. \end{cases}$$

Then, it follows from the system (A) that

$$\begin{vmatrix} \frac{1}{2} & 2\beta & 2\beta^2 & -2 \\ -1 & 3\beta^2-2\beta & 6\beta^3 & 5 \\ \frac{1}{2} & -3\beta^2 & \frac{9}{2}\beta^4 & -3 \\ 1 & 4\beta^3+2\beta & 8\beta^4 & -6 \end{vmatrix}=0.$$

Thus $\beta^3(\beta+1)(2\beta+1)(3\beta+2)^2=0$, and hence $\beta=-1$ or $\beta=-\frac{1}{2}$ or $\beta=-\frac{2}{3}$.

Case 2(i): $\beta=-1$. It follows from systems (A), ②$^{(4)}$ and ②$^{(5)}$ that $R_{2,0}=R_{0,2}=0$, $R_{1,1}=-1$. Next, systems ③, ③$^{(1)}$, ③$^{(2)}$, and ③$^{(3)}$ can be represented in the following form:

$$\begin{cases} \frac{1}{6}R_{3,0}-R_{2,1}+2R_{1,2}-\frac{4}{3}R_{0,3}-2\gamma=5 \\ -\frac{1}{2}R_{3,0}+\frac{7}{2}R_{2,1}-8R_{1,2}+6R_{0,3}+10\gamma=-21 \\ \frac{1}{2}R_{3,0}-4R_{2,1}+\frac{21}{2}R_{1,2}-9R_{0,3}-14\gamma=28 \\ \frac{1}{2}R_{3,0}-4R_{2,1}+10R_{1,2}-8R_{0,3}-16\gamma=28. \end{cases}$$

From the above system, it follows that

$$\gamma=-\frac{1}{2},\ R_{3,0}=8R_{0,3},\ R_{2,1}=4R_{0,3},\ R_{1,2}=2R_{0,3}+2.$$

Furthermore, systems ④, ④$^{(1)}$, and ⑤ can be represented in the following form:

$$\frac{1}{24}R_{4,0}-\frac{1}{3}R_{3,1}+R_{2,2}-\frac{4}{3}R_{1,3}+\frac{2}{3}R_{0,4}=10 \qquad (4)$$

$$-\frac{1}{6}R_{4,0}+\frac{3}{2}R_{3,1}-5R_{2,2}+\frac{22}{3}R_{1,3}-4R_{0,4}=-\frac{229}{4} \qquad (4)^{(1)}$$

$$\frac{1}{120}R_{5,0}-\frac{1}{12}R_{4,1}+\frac{1}{3}R_{3,2}-\frac{2}{3}R_{2,3}$$
$$+\frac{2}{3}R_{1,4}-\frac{4}{15}R_{0,5}=\frac{95}{4}. \qquad (5)$$

Case 2(ii): $\beta=-\frac{1}{2}$. It is easy to show that systems (A) and ②$^{(5)}$ cannot be satisfied.

Case 2(iii): $\beta=-\frac{2}{3}$. It is easy to show that systems ② and ②$^{(1)}$ cannot be satisfied.

Case 3: $4\delta^2+3\delta=(3\delta+2)\alpha$ (1)

The solution of systems ①, ①$^{(1)}$ and ①$^{(2)}$ is given by

$$R_{1,0}=\frac{3\delta+2}{\alpha^2(3\delta-2\alpha)},\quad R_{0,1}=-\frac{\alpha+1}{\delta(3\delta-2\alpha)}.$$

Substituting them into ①$^{(3)}$, we obtain

$$\alpha^3(3\delta+2)-5\delta^3(\alpha+1)=2\alpha-3\delta. \tag{2}$$

Next, it follows from (1) and (2) that

$$(\alpha+1)\delta\{5\delta^2+4\delta-(4\delta+3)\alpha\}=0.$$

Hence

$$\alpha=-1 \quad\text{or}\quad 5\delta^2+4\delta=(4\delta+3)\alpha. \tag{3}$$

Furthermore, eliminating α from (1) and (3), we obtain

$$\delta(\delta+1)^2=0.$$

Thus $\delta=-1$ and hence $\alpha=\beta=-1$. Therefore, from the above discussion, we can conclude that Case 3 is equivalent to Case 2 (i).

Next, we calculate the asymptotic error constant of $\phi(x)$ for Case 2(i). Then, since $R=1$, $R_{1,0}=1$, $R_{0,1}=R_{2,0}=R_{0,2}=0$ and $R_{1,1}=-1$, we obtain

$$\begin{aligned}R(X,Y)=&1+H-HK+\frac{1}{6}H^3R_{3,0}+\frac{1}{2}H^2KR_{2,1}\\&+\frac{1}{2}HK^2R_{1,2}+\frac{1}{6}K^3R_{0,3}+\frac{1}{24}H^4R_{4,0}\\&+\frac{1}{6}H^3KR_{3,1}+\frac{1}{4}H^2K^2R_{2,2}+\frac{1}{6}HK^3R_{1,3}\\&+\frac{1}{24}K^4R_{0,4}+\frac{1}{120}H^5R_{5,0}+\frac{1}{24}H^4KR_{4,1}\\&+\frac{1}{12}H^3K^2R_{3,2}+\frac{1}{12}H^2K^3R_{2,3}+\frac{1}{24}HK^4R_{1,4}\\&+\frac{1}{120}K^5R_{0,5}+\frac{1}{720}H^6R_{6,0}+\frac{1}{120}H^5KR_{5,1}\\&+\frac{1}{48}H^4K^2R_{4,2}+\frac{1}{36}H^3K^3R_{3,3}\\&+\frac{1}{48}H^2K^4R_{2,4}+\frac{1}{120}HK^5R_{1,5}\\&+\frac{1}{720}K^6R_{0,6}+O(h^7).\end{aligned}$$

Now, since

$$H=A_2h-A_3h^2+A_4h^3-A_5h^4+A_6h^5-A_7h^6,$$

$$\begin{aligned}K=&-2A_2h-A_2^2h^2+3A_3h^2+4A_2A_3h^3-4A_4h^3\\&+\frac{3}{4}A_2^2A_3h^4-3A_3^2h^4-7A_2A_4h^4+5A_5h^4\\&-\frac{3}{2}A_2A_3^2h^5-3A_2^2A_4h^5+9A_3A_4h^5\\&+11A_2A_5h^5-6A_6h^5,\end{aligned}$$

$$R_{3,0}=8R_{0,3},\quad R_{2,1}=4R_{0,3}$$

and $R_{1,2}=2R_{0,3}+2$, the coefficient of h^6 of $R(X,Y)$ is given by

$$\begin{aligned}&\left(-\frac{1}{6}R_{0,3}+\frac{1}{4}R_{2,2}-R_{1,3}+R_{0,4}-\frac{1}{24}R_{4,1}+\frac{1}{3}R_{3,2}-R_{2,3}+\frac{4}{3}R_{1,4}\right.\\&\quad\left.-\frac{2}{3}R_{0,5}+\frac{1}{720}R_{6,0}-\frac{1}{60}R_{5,1}+\frac{1}{12}R_{4,2}-\frac{2}{9}R_{3,3}+\frac{1}{3}R_{2,4}-\frac{4}{15}R_{1,5}+\frac{4}{45}R_{0,6}\right)A_2^6\\&\quad+\left(-12+\frac{1}{2}R_{0,3}+\frac{7}{6}R_{3,1}-\frac{15}{2}R_{2,2}+16R_{1,3}-\frac{34}{3}R_{0,4}-\frac{1}{24}R_{5,0}+\frac{11}{24}R_{4,1}-2R_{3,2}+\frac{13}{3}R_{2,3}\right.\\&\quad\left.-\frac{14}{3}R_{1,4}+2R_{0,5}\right)A_2^4A_3+\left(\frac{241}{4}-\frac{1}{2}R_{0,3}+\frac{1}{4}R_{4,0}-\frac{5}{2}R_{3,1}+\frac{37}{4}R_{2,2}-15R_{1,3}+9R_{0,4}\right)A_2^2A_3^2\\&\quad+\left(43+\frac{1}{6}R_{4,0}-\frac{5}{3}R_{3,1}+6R_{2,2}-\frac{28}{3}R_{1,3}+\frac{16}{3}R_{0,4}\right)A_2^3A_4+\left(-12+\frac{1}{6}R_{0,3}\right)A_3^3-72A_2A_3A_4\\&\quad-36A_2^2A_5+4A_4^2+8A_3A_5+8A_2A_6-A_7.\end{aligned}$$

Using Theorem 5-1 [1, p.83], we obtain

$$Y_7=132A_2^6-330A_2^4A_3+180A_2^2A_3^2+120A_2^3A_4-12A_3^3-72A_2A_3A_4-36A_2^2A_5+4A_4^2+8A_3A_5+8A_2A_6-A_7.$$

Then, since the basic sequence is defined by [1, p. 83],

$$E_8=E_7-Y_7h^7$$

$$\begin{aligned}\phi(x)-E_8&=\phi(x)-E_7+Y_7h^7\\&=\left\{\left(132+\frac{1}{6}R_{0,3}-\frac{1}{4}R_{2,2}+R_{1,3}-R_{0,4}+\frac{1}{24}R_{4,1}-\frac{1}{3}R_{3,2}+R_{2,3}-\frac{4}{3}R_{1,4}+\frac{2}{3}R_{0,5}-\frac{1}{720}R_{6,0}\right.\right.\end{aligned}$$

$$+\frac{1}{60}R_{5,1}-\frac{1}{12}R_{4,2}+\frac{2}{9}R_{3,3}-\frac{1}{3}R_{2,4}+\frac{4}{15}R_{1,5}-\frac{4}{45}R_{0,6}\Big)A_2^6+\Big(-318-\frac{1}{2}R_{0,3}-\frac{7}{6}R_{3,1}+\frac{15}{2}R_{2,2}-16R_{1,3}$$
$$+\frac{34}{3}R_{0,4}+\frac{1}{24}R_{5,0}-\frac{11}{24}R_{4,1}+2R_{3,2}-\frac{13}{3}R_{2,3}+\frac{14}{3}R_{1,4}-2R_{0,5}\Big)A_2^4A_3+\Big(\frac{479}{4}+\frac{1}{2}R_{0,3}-\frac{1}{4}R_{4,0}$$
$$+\frac{5}{2}R_{3,1}-\frac{37}{4}R_{2,2}+15R_{1,3}-9R_{0,4}\Big)A_2^2A_3^2+\Big(77-\frac{1}{6}R_{4,0}+\frac{5}{3}R_{3,1}-6R_{2,2}+\frac{28}{3}R_{1,3}-\frac{16}{3}R_{0,4}\Big)A_2^3A_4$$
$$-\frac{1}{6}R_{0,3}A_3^3\Big\}h^7+O(h^8).$$

Hence, it follows from Theorem 5-2 [1, pp. 86-87] that the asymptotic error constant of $\phi(x)$, C is given by

$$C=\lim_{x\to\zeta}\frac{\phi(x)-E_8}{h^7}$$
$$=\Big(132+\frac{1}{6}R_{0,3}-\frac{1}{4}R_{2,2}+R_{1,3}-R_{0,4}+\frac{1}{24}R_{4,1}-\frac{1}{3}R_{3,2}+R_{2,3}-\frac{4}{3}R_{1,4}+\frac{2}{3}R_{0,5}-\frac{1}{720}R_{6,0}+\frac{1}{60}R_{5,1}-\frac{1}{12}R_{4,2}$$
$$+\frac{2}{9}R_{3,3}-\frac{1}{3}R_{2,4}+\frac{4}{15}R_{1,5}-\frac{4}{45}R_{0,6}\Big)\tilde{A}_2^6+\Big(-318-\frac{1}{2}R_{0,3}-\frac{7}{6}R_{3,1}+\frac{15}{2}R_{2,2}-16R_{1,3}+\frac{34}{3}R_{0,4}$$
$$+\frac{1}{24}R_{5,0}-\frac{11}{24}R_{4,1}+2R_{3,2}-\frac{13}{3}R_{2,3}+\frac{14}{3}R_{1,4}-2R_{0,5}\Big)\tilde{A}_2^4\tilde{A}_3+\Big(\frac{479}{4}+\frac{1}{2}R_{0,3}-\frac{1}{4}R_{4,0}+\frac{5}{2}R_{3,1}$$
$$-\frac{37}{4}R_{2,2}+15R_{1,3}-9R_{0,4}\Big)\tilde{A}_2^2\tilde{A}_3^2+\Big(77-\frac{1}{6}R_{4,0}+\frac{5}{3}R_{3,1}-6R_{2,2}+\frac{28}{3}R_{1,3}-\frac{16}{3}R_{0,4}\Big)\tilde{A}_2^3\tilde{A}_4-\frac{1}{6}R_{0,3}\tilde{A}_3^3 \quad (6)$$

where ζ is a simple zero of Eq. (1.1), $\tilde{A}_j\equiv A_j(\zeta)$, $R_{i,0}\equiv R_{i,0}(0,1)$, $R_{0,j}\equiv R_{0,j}(0,1)$, and $R_{i,j}\equiv R_{i,j}(0,1)$.

If $R_{0,3}=0$ and the coefficients of both $\tilde{A}_2^2\tilde{A}_3^2$ and $\tilde{A}_2^3\tilde{A}_4$ are equal to zero in Eq. (6), then

$$\text{(B)}\quad\begin{cases}\frac{1}{4}R_{4,0}-\frac{5}{2}R_{3,1}+\frac{37}{4}R_{2,2}-15R_{1,3}+9R_{0,4}=\frac{479}{4}\\[4pt] \frac{1}{6}R_{4,0}-\frac{5}{3}R_{3,1}+6R_{2,2}-\frac{28}{3}R_{1,3}+\frac{16}{3}R_{0,4}=77.\end{cases}$$

In this case, systems (4), $(4)^{(1)}$ and (B) have no solution.

Finally, we have shown that the attainable order of convergence for Eq. (1.2) is equal to 7 for all simple zeros of Eq. (1.1).

Remarks. We consider a class of iteration functions of the type

$$\psi(x)=x-hG(Y) \quad (1.3)$$

where $Y=f'(x+\beta h)/f'(x)$ (β: parameter) and $G(\eta)$ is a function of η. The attainable order of convergence for Eq. (1.3) is equal to 4 for all simple zeros of Eq. (1.1), iff $\beta=-\frac{2}{3}$, $G(1)=1$, $G'(1)=-\frac{3}{4}$ and $G''(1)=\frac{9}{4}$. Furthermore, from Theorem 5-2 [1, pp. 86-87] it follows that the asymptotic error constant C of $\psi(x)$ is given by

$$C=\lim_{x\to\zeta}\frac{\psi(x)-E_5}{h^4}$$
$$=\Big\{5+\frac{32}{81}G^{(3)}(1)\Big\}\tilde{A}_2^3-\tilde{A}_2\tilde{A}_3+\frac{1}{9}\tilde{A}_4.$$

On the other hand, in the case of Jarratt's methods [3],

$$G(Y)=a_1+\frac{a_2}{Y}+\frac{1}{b_1+b_2Y}.$$

Next, we consider a class of iteration functions of the type

$$\lambda(x)=x-hF(X) \quad (1.4)$$

where $X=f(x+\alpha h)/f(x)$ (α: parameter) and $F(\xi)$ is a function of ξ. Now, the attainable order of convergence for Eq. (1.4) is equal to 4 for all simple zeros of Eq. (1.1), iff $\alpha=-1$, $F(0)=1$, $F'(0)=1$ and $F''(0)=4$. Furthermore, the asymptotic error constant C of $\lambda(x)$ is given by

$$C=\lim_{x\to\zeta}\frac{\lambda(x)-E_5}{h^4}$$
$$=\Big\{5-\frac{1}{6}F^{(3)}(0)\Big\}\tilde{A}_2^3-\tilde{A}_2\tilde{A}_3.$$

3. Some Examples

Example 1. In Eq. (1.2), we consider the following case

$$R(X,Y)=G(Y)+XF(Y)$$

where both $G(\eta)$ and $F(\eta)$ are functions of η. Then,

$$R_{0,j}=G^{(j)}(1),\ R_{1,j}=F^{(j)}(1)$$

and

$$R_{i,j}=0\quad(i\geqq 2,\ j\geqq 0).$$

Furthermore, if $G^{(6)}(1)=0$ and the coefficients of both

Table 1 $\zeta=-1.0$.

n	x_n	Bounds for the absolute errors
0	-1.5	
1	$-0.132087560745924616116706878445344Q+01$	
2	$-0.117372979682132013873607239385296Q+01$	1.74×10^{-1}
3	$-0.106842904801136927396876980542564Q+01$	6.84×10^{-2}
4	$-0.101403681171360624544230395346633Q+01$	1.40×10^{-2}
5	$-0.100070769061142463730424428424212Q+01$	7.08×10^{-4}
6	$-0.100000189039822930019255327437268Q+01$	1.90×10^{-6}
7	$-0.100000000001352522197924590748050Q+01$	1.35×10^{-11}
8	$-0.100000000000000000000006923564502IQ+01$	6.92×10^{-22}
9	$-0.100000000000000000000000000000000Q+01$	

n	z_n	Bounds for the absolute errors
0	-1.5	
1	$-0.114539434235757457702713522744105Q+01$	1.45×10^{-1}
2	$-0.100319881363274600834052917774214Q+01$	3.20×10^{-3}
3	$-0.100000000000001719421538083956766Q+01$	1.72×10^{-14}
4	$-0.100000000000000000000000000000000Q+01$	

n	s_n	Bounds for the absolute errors
0	-1.5	
1	$-0.118825442400055725840981414964488Q+01$	1.88×10^{-1}
2	$-0.102017595019595908034088409438180Q+01$	2.02×10^{-2}
3	$-0.100000202803819611038792163791163Q+01$	2.03×10^{-6}
4	$-0.999999999999999999999562652786257Q+00$	4.37×10^{-22}
5	$-0.100000000000000000000000000000000Q+01$	

Table 2 $\zeta=4+\sqrt{2}=5.414213562373095048801688724209$.

n	x_n	Bounds for the absolute errors
0	6.0	
1	$0.573735968867812674689459052720007Q+01$	
2	$0.554897547060844250557214071079366Q+01$	1.35×10^{-1}
3	$0.544613291375023679805973057388123Q+01$	3.19×10^{-2}
4	$0.541644529870381977830833891481897Q+01$	2.23×10^{-3}
5	$0.541422530732583566397289534999706Q+01$	1.17×10^{-5}
6	$0.541421356270025564778081210899843Q+01$	3.27×10^{-10}
7	$0.541421356237309504905554836635516Q+01$	2.54×10^{-19}
8	$0.541421356237309504880168872420966Q+01$	

n	z_n	Bounds for the absolute errors
0	6.0	
1	$0.551420693738235151365394234638961Q+01$	1.00×10^{-1}
2	$0.541428110132569738802665826642525Q+01$	6.75×10^{-5}
3	$0.541421356237309504880168872392133Q+01$	2.88×10^{-28}
4	$0.541421356237309504880168872420966Q+01$	

n	s_n	Bounds for the absolute errors
0	6.0	
1	$0.556427518303437182139037589868769Q+01$	1.50×10^{-1}
2	$0.541760483932039923194480509794585Q+01$	3.39×10^{-3}
3	$0.541421356178928095123321886031207Q+01$	5.84×10^{-10}
4	$0.541421356237309504880168872420964Q+01$	

$\tilde{A}_2^6$ and $\tilde{A}_2^4\tilde{A}_3$ are equal to zero in Eq. (6), then

$$G(1)=1,\ G'(1)=G''(1)=G^{(3)}(1)=0,\ G^{(4)}(1)=\frac{27}{4},$$

$$G^{(5)}(1)=-\frac{555}{8},\ F(1)=1,\ F'(1)=-1,\ F''(1)=2,$$

$$F^{(3)}(1)=-\frac{33}{8},\ F^{(4)}(1)=-\frac{63}{8}\quad\text{and}\quad F^{(5)}(1)=-\frac{7725}{32}.$$

In particular, let both $G(Y)$ and $F(Y)$ be polynomials. Then, Eq. (1.2) takes the form

$$\phi(x)=x-h\left\{1+\frac{9}{32}(Y-1)^4-\frac{37}{64}(Y-1)^5\right\}$$
$$-k\left\{1-(Y-1)+(Y-1)^2-\frac{11}{16}(Y-1)^3\right.$$
$$\left.+\frac{21}{64}(Y-1)^4-\frac{515}{256}(Y-1)^5\right\} \qquad (1.2)'$$

where

$$k=hX=\frac{f(x-h)}{f'(x)}\quad\text{and}\quad Y=\frac{f'\left(x-h-\frac{1}{2}k\right)}{f'(x)}.$$

On the other hand, the asymptotic error constant C is given by

$$C=-\frac{23}{8}\tilde{A}_2^2\tilde{A}_3^2+\frac{5}{2}\tilde{A}_2^3\tilde{A}_4.$$

Example 2. If the coefficient of $\tilde{A}_2^3$ of the asymptotic error constant of Eq. (1.3) is equal to zero, then

$$G^{(3)}(1)=-\frac{405}{32}.$$

In particular, let $G(Y)$ be a polynomial. Then, Eq. (1.3) takes the form

$$\psi(x)=x-h\left\{1-\frac{3}{4}(Y-1)+\frac{9}{8}(Y-1)^2-\frac{135}{64}(Y-1)^3\right\} \qquad (1.3)'$$

where $Y=\dfrac{f'\left(x-\frac{2}{3}h\right)}{f'(x)}$.

Furthermore, the asymptotic error constant C is given by

$$C=-\tilde{A}_2\tilde{A}_3+\frac{1}{9}\tilde{A}_4.$$

Example 3. We show numerical examples by the following three methods:

Newton's method: $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$

The iterative method for Eq. (1.2)′: $z_{n+1}=\phi(z_n)$

The iterative method for Eq. (1.3)′: $s_{n+1}=\psi(s_n)$ $(n=0, 1, \cdots)$.

Here, we consider the following equation:

$$f(x)\equiv x^{12}-13x^{11}+66x^{10}-185x^9+346x^8-497x^7+546x^6-439x^5+101x^4+470x^3-798x^2+664x-280=0.$$

All the roots of this equation are

$$\zeta=\pm 1,\ 4\pm\sqrt{2},\ 2\pm i,\ \frac{1\pm\sqrt{7}\,i}{2},\ \frac{-1\pm\sqrt{7}}{2},\ \frac{1\pm\sqrt{3}\,i}{2}.$$

The numerical examples by the three methods are given in Tables 1 and 2.

All calculations are made with quadruple precision.

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