Regular Paper

On the Global Convergence of Some Iterative Formulas

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We showed two types of third order iterative formulas containing two parameters. Let $f(x)$ be a polynomial with only real zeros or an entire function of a certain type with only real zeros. Then we established that the one type of the above-mentioned iterative formulas converges globally and monotonically to the zeros of the $f(x)$. The purpose of this paper is to show that the other type also converges globally and monotonically to the zeros of the $f(x)$.

1. Introduction

Ostrowski[1] and Traub[2] have shown various types of iterative formulas for the computation of the numerical solutions of the nonlinear scalar equation

$$f(x)=0 \tag{1.1}$$

where $f(x)$ is a real function of the real variable x. In Ref. 3)~6), we also have shown various types of iterative formulas.

Let $f(x)$ be a polynomial of exact degree $r>1$ with only real zeros given by the following form:

$$f(x)=\prod_{k=1}^{r}(x-a_k). \tag{1.2}$$

Then, it has been shown that Ostrowski's method (Ref. 1), pp. 110-115), Laguerre's method (Ref. 1), pp. 353-362), and Hansen and Patrick's methods[7] converge globally and monotonically to the zeros of $f(x)$.

Next, let $f(x)$ be given by the following form:

$$f(x)=x^p \exp(a+bx-cx^2) \cdot\prod_k\left(1-\frac{x}{a_k}\right)e^{x/a_k} \tag{1.3}$$

where p is a non-negative integer, a, b, c are real with $c\geqq 0$, and a_k are real. If the number of the a_k is infinite, then $\Sigma a_k^{-2}<\infty$. Furthermore, if the number of the a_k is finite, then we require that there be at least one a_k for $p\geqq 1$, and at least two for $p=0$.

Then, it has been shown that Ostrowski's method (Ref. 1), pp. 124-126), Hally's method[8], and Hansen and Patrick's methods[7] converge globally and monotonically to the zeros of $f(x)$.

In Ref. 6), we considered the following type of iterative formulas:

$$x_{n+1}=\phi(x_n),\ n=0,1,\cdots \tag{1.4}$$

where $\phi(x)=x-hR(X)$, $h\equiv h(x)=\dfrac{f(x)}{f'(x)}$, $X\equiv X(x)=h\dfrac{f''(x)}{f'(x)}$,

and $R(t)$ is a function of t. Then, it was shown that the order of convergence for Eq. (1.4) is equal to 3 for all simple zeros of Eq. (1.1), iff $R(0)=1$ and $R'(0)=1/2$.

Furthermore, we gave two examples of $R(X)$ in the form:

Ex. 1 $$R(X)=\frac{(\theta+\frac{1}{2})X+1}{\beta X^2+\theta X+1}$$ (β, θ: parameters),

Ex. 2 $$R(X)=\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}$$ (a, b: parameters).

In Ref. 6), we established that iterative formulas for $R(X)$ in Ex. 1 converge globally and monotonically to the zeros of $f(x)$ given by both the form (1.2) and the form (1.3). In 2 and 3, we will consider the monotonic global convergence of iterative formulas for $R(X)$ of Ex. 2.

2. Monotonicity of Convergence

We will consider the iterative methods for $R(X)$ of Ex. 2 that form the sequence x_n by the iterative rule:

$$x_{n+1}=x_n-K(x_n),\ n=0,1,\cdots \tag{2.1}$$

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where

$$K(x)=h\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}\,(a+\sqrt{b})X}}.$$

At first, let $f(x)$ be given by the form (1.3). Then, taking the logarithmic derivative of (1.3) and differentiating it, we have

$$\frac{f'(x)}{f(x)}=\frac{p}{x}+b-2cx+\sum_k\left(\frac{1}{x-\alpha_k}+\frac{1}{\alpha_k}\right), \quad (2.2)$$

$$-\left(\frac{f'(x)}{f(x)}\right)'=\frac{\{f'(x)\}^2-f(x)f''(x)}{\{f(x)\}^2}$$
$$=\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}. \quad (2.3)$$

Let the distinct zeros of $f(x)$ be ordered consecutively so that $\zeta_0<\zeta_1$. Then, since the right hand side of (2.3) is positive, it follows that $\frac{f'(x)}{f(x)}$ is monotonically decreasing in the open interval (ζ_0, ζ_1).

Furthermore, since $\lim_{x\to\zeta_0+0}\frac{f'(x)}{f(x)}=+\infty$, and $\lim_{x\to\zeta_1-0}\frac{f'(x)}{f(x)}=-\infty$, it follows that $f'(x)$ has exactly one zero ζ_0' such that $\zeta_0<\zeta_0'<\zeta_1$. Then, for $\forall x\in(\zeta_0, \zeta_0')\cup(\zeta_0', \zeta_1)$, we can define the associated zero $\alpha(x)$ of $f(x)$ to be $\alpha(x)=\zeta_0$ if $\zeta_0<x<\zeta_0'$ and $\alpha(x)=\zeta_1$ if $\zeta_0'<x<\zeta_1$.

Furthermore $\frac{f'(x)}{f(x)}$ and $x-\alpha(x)$ have the same sign. It now follows from (2.3) that

$$1-X=h^2\left\{\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}\right\}$$
$$>0 \quad (\zeta_0<x<\zeta_1). \quad (2.4)$$

It follows from (2.4) that for any real x such that $f(x)f'(x)\neq0$, we have

$$1-X-\frac{h^2}{\{x-\alpha(x)\}^2}$$
$$=h^2\left[\frac{p}{x^2}+2c+\sum_k\frac{1}{(x-\alpha_k)^2}-\frac{1}{\{x-\alpha(x)\}^2}\right]>0. \quad (2.5)$$

Hence, it follows from (2.5) that

$$|x-\alpha(x)|>\frac{|h|}{\sqrt{1-X}}. \quad (2.6)$$

We will need later the following lemma:

Lemma 1. If $-\sqrt{b}<a\leqq0$, then we have

$$0<\frac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}\,(a+\sqrt{b})X}}$$
$$\leqq\frac{1}{\sqrt{1-X}} \quad (X<1). \quad (2.7)$$

Proof. From the assumption, we have $a+\sqrt{b}>0$.

It then follows that for $X<1$,

$$b-\sqrt{b}\,(a+\sqrt{b})X>b-\sqrt{b}\,(a+\sqrt{b})$$
$$=-a\sqrt{b}\geqq0.$$

Putting $\Phi(X)=a+\sqrt{b-\sqrt{b}\,(a+\sqrt{b})X}$, we have $\Phi(X)>\Phi(1)$ $(X<1)$.

On the other hand, since $\Phi(1)=a+\sqrt{-a\sqrt{b}}>a+|a|=0$,

$$\Phi(X)>0 \quad (X<1). \quad (2.8)$$

In order to prove that we have (2.7), it suffices to prove that the following inequality holds:

$$(a+\sqrt{b})\sqrt{1-X}\leqq\Phi(X), \quad (X<1) \quad (2.9)$$

In order to prove that (2.9) holds, we put

$$g(X)=\Phi(X)-(a+\sqrt{b})\sqrt{1-X}.$$

Then,

$$g'(X)=\frac{-\sqrt{b}\,(a+\sqrt{b})}{2\sqrt{b-\sqrt{b}\,(a+\sqrt{b})X}}+\frac{a+\sqrt{b}}{2\sqrt{1-X}}$$
$$=\frac{1}{2}(a+\sqrt{b})\left(\frac{1}{\sqrt{1-X}}-\frac{1}{\sqrt{1-X-\frac{a}{\sqrt{b}}X}}\right).$$

Since $\frac{a}{\sqrt{b}}\leqq0$, we have $g'(X)\leqq0$ if $X<0$ and $g'(X)\geqq0$ if $X\geqq0$. In addition, since $g(0)=0$, we have $g(X)\geqq0$.

Hence (2.9) is proved. Consequently, from (2.8) and (2.9), we have (2.7).

Thus Lemma 1 is completely proved.

It now follows from (2.6) and Lemma 1 that

$$|x-\alpha(x)|>|K(x)| \quad (2.10)$$

Then, on the monotonic convergence of the rule (2.1), we have:

Theorem 1. Let $f(x)$ be given by the form (1.3). Then, if $-\sqrt{b}<a\leqq0$ and if we take the real starting value in (2.1) x_0 such that $f'(x_0)f(x_0)\neq0$, and x_0 is neither less nor greater than all zeros of $f(x)$, the sequence x_n in (2.1) converge monotonically to $\alpha(x_0)$.

Proof. Assume that we take the starting value x_0 such that $\zeta_0'<x_0<\zeta_1$.

Then, we have $\alpha(x_0)=\zeta_1$, $f(x_0)/f'(x_0)<0$, and $K(x_0)<0$. Next, applying (2.1) and (2.10), we have

$$x_0<x_1<\alpha(x_0),$$

and by the definition of $\alpha(x)$, $\alpha(x_0)\equiv\alpha(x_1)$.

By repetition of the same argument, we have

$$x_0 < x_1 < x_2 < \cdots < \alpha(x_0).$$

Hence, it follows that the sequence x_n converge monotonically to a certain limit α:

$$x_n \uparrow \alpha \leqq \alpha(x_0).$$

Therefore, it follows from (2.1) that

$$\lim_{x_n \to \alpha} K(x_n) = 0.$$

Furthermore, if $\alpha < \alpha(x_0)$, then from (2.4), we have $-\infty < \lim_{x_n \to \alpha} X(x_n) < 1$.

Therefore it follows from Lemma 1 and $\lim_{x_n \to \alpha} h(x_n) < 0$ that we have

$$\lim_{x_n \to \alpha} K(x_n) < 0.$$

We have our contradiction. Consequently, we have $\alpha \equiv \alpha(x_0)$.

In the same way, taking x_0 such that $\zeta_0 < x_0 < \zeta_0'$, we can prove that the sequence x_n converge monotonically to $\alpha(x_0)$. Then, Theorem 1 is completely proved.

Next, let $f(x)$ be given by the form (1.2). Also in this case, by repetition of the same discussion as the above, we can show the monotonic convergence of the rule (2.1). We have:

Theorem 2. Let $f(x)$ be given by the form (1.2). Then, if $-\sqrt{b} < a \leqq 0$ and if we take the real starting value x_0 in (2.1) such that $f'(x_0)f(x_0) \neq 0$, the sequence x_n in (2.1) converge monotonically to $\alpha(x_0)$ (see Ref. 1), pp. 110-113).

Thus, it was shown that the iterative methods (2.1) have the monotonic global convergence under the same assumptions as those on Ostrowski's method for the starting value x_0.

Remarks. For $b=1$, (2.1) coinsides with Hansen and Patrick's methods.

Furthermore, for $a=0$ and $b=0$, (2.1) coincides with Ostrowski's method.

3. Modification for Multiple Zeros

In Ref. 6), for the case where α is a zero of $f(x)$ of multiplicity $m>1$, we considered the order of convergence for Eq. (1.4). Then, we showed that if $R\left(1-\frac{1}{m}\right) \neq m$, then the convergence of x_n to α is only linear. Also in this case, it was shown that we can still have cubic convergence to α by modifying the rule (1.4) in the form:

$$x_{n+1} = \phi_m(x_n), \quad n=0, 1, 2, \cdots \qquad (3.1)$$

where $\phi_m(x) = x - mhR(1-m+mX)$.

Furthermore, the asymptotic error constant of $\phi_m(x)$, C_m was given by

$$C_m = \lim_{x \to \alpha} \frac{\phi_m(x) - \alpha}{(x-\alpha)^3} = \frac{1}{m^2(m+1)^2}\left\{\frac{3}{2} + \frac{m}{2} - 2R''(0)\right\}\left\{\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right\}^2 - \frac{1}{m(m+1)(m+2)} \cdot \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \qquad (3.2)$$

In our case, it follows from Lemma 1 that

$$R\left(1-\frac{1}{m}\right) \leqq \frac{1}{\sqrt{1-\left(1-\frac{1}{m}\right)}} = \sqrt{m} < m.$$

Therefore, the convergence of x_n in (2.1) to α is only linear.

Modifying (2.1) by (3.1), we have

$$x_{n+1} = x_n - mh_nR(1-m+mX_n), \quad n=0, 1, 2, \cdots \qquad (3.3)$$

where $h_n \equiv h(x_n)$, $X_n(x_n) \equiv X_n$, and $R(X) = \dfrac{a+\sqrt{b}}{a+\sqrt{b}-\sqrt{b}(a+\sqrt{b})X}$.

Then, from (3.2), the asymptotic error constant C_m is given by

$$C_m = \frac{1}{m^2(m+1)^2}\left(\frac{m}{2} - \frac{a}{2\sqrt{b}}\right)\left\{\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right\}^2 - \frac{1}{m(m+1)(m+2)} \cdot \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \qquad (3.4)$$

Then, we have:

Theorem 3. Let $f(x)$ be given by the form (1.3). Then, if under the conditions of Theorem 1 the muliplicity of $\alpha = \alpha(x_0)$ is m, the sequence x_n in (3.3) converge monotonically to $\alpha(x_0)$.

Furthermore, we have:

Theorem 4. Let $f(x)$ be given by the form (1.2). Then, if under the conditions of Theorem 2 the multiplicity of $\alpha = \alpha(x_0)$ is m, the sequence x_n in (3.3) converge monotonically to $\alpha(x_0)$.

In the case where α is a zero of $f(x)$ of multiplicity $m>1$, we have

$$1 - X - \frac{mh^2}{(x-\alpha)^2} = h^2\left\{\frac{P}{x^2} + 2c + \sum_k \frac{1}{(x-\alpha_k)^2} - \frac{m}{(x-\alpha)^2}\right\} > 0. \qquad (3.5)$$

Therefore, from (3.5), we have

$$|x-\alpha(x)|>\frac{\sqrt{m}\,|h|}{\sqrt{1-X}}. \qquad (3.6)$$

Furthermore, since $1-m+mX<1$, using Lemma 1, we have

$$\frac{\sqrt{m}}{\sqrt{1-X}}\geq mR(1-m+mX). \qquad (3.7)$$

Hence it follows from (3.6) and (3.7) that

$$|X-\alpha(x)|>m|h|R(1-m+mX). \qquad (3.8)$$

Finally, those Theorems can be shown in the way similar to the proof of Theorem 1 and Theorem 2.

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