離散物体の単体共有とその応用

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3次元空間内の物体の幾何情報を計算機が獲得するためには，計測手法だけではなく，計測された物体を計算機の内部で的確に表現することが必要である．計算機で幾何形状を操作するには2つの立場がある．1つは数値計算結果よりも位相構造を優先する手法である．もう1つは計算機内部では有限桁数の数値しか表現できないことを利用して，整数値だけを利用して幾何形状を表現・処理する手法である．第2の方法の基礎として，現在まではグラフ理論に基づく方法が主に研究されてきた．しかし，この方法では処理対象の幾何構造を十分に反映した表現や処理はなされてこなかった．そこで，本論文では，処理対象の幾何構造を考慮して，幾何形状を整数値だけで表示・処理するために，古典的な組合せ幾何学の成果をもとに格子点でのみ定義された幾何形状の操作のための離散組合せ幾何学を構築する．それを使って，計算機内における3次元空間の対象物である曲線や曲面，物体の新しい表現方法を提案する．

Discrete Combinatorial Geometry

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In computer vision, one of the ultimate purposes is the acquisition of geometric concepts of a 3-dimensional world from measured data. As an expression intermediate between measured raw data and geometric concepts, we need a representation of objects in computers because in a computer we can only manipulate finite-precision numbers. In this paper, we construct discrete combinatorial geometry from fundamental definitions of classical combinatorial geometry to manipulate geometric data using only finite-precision numbers. Then, using new definitions we propose a new representation of curves, surfaces and objects in computers. Furthermore, our new representation implies that the boundary of a surface is curves and the boundary of an object is surfaces.

1 Introduction

In computer vision, one of the ultimate purposes is the acquisition of geometric concepts of a 3-dimensional world from measured data. As an expression intermediate between measured raw data and geometric concepts, we need a representation of objects in computers. Computers express and calculate data using finite-precision arithmetic. Thus, we need a method to express geometric objects in a computer memory using only finite-precision data. Data expressed using finite-precision arithmetic are equivalent to data expressed using only integers. This leads to the conclusion that geometric data should be expressed mathematically on lattice points in a 3-dimensional space. We call such geometrical data produced from an object in 3-dimensional world a discrete object because a space of lattice points is a discrete space.

Some authors have studied the method to represent discrete objects. Kong and Rosenfeld[1], Voss[2] and Udupa[3] constructed a representation of a discrete object using the graph theory approach. A graph is related to the data structure of a discrete object in a computer memory, and nodes and vertices of a graph correspond to pointers and addresses in a computer, respectively. However, this approach did not consider topological and geometric structures of discrete objects. Next, Herman proposed a representation of a discrete object which possesses topological structures of an object[4]. A discrete object is represented by a set of voxels, unit cubes, whose centers are lattice points. Furthermore, a discrete surface, a boundary of a discrete object, is represented by a set of the faces of voxels which separate the object from its surroundings. Therefore, discrete surfaces are not sets of voxels. In 3-dimensional computer graphics, the marching cube method is used to construct a surface of an object by patching triangles whose points are determined by the locations of discrete points in a region [5]. However, this method sometimes generates unexpected holes on surfaces. Recently, J. Françon constructed a discrete combinatorial surface[6]. He applied combinatorial geometry in 3-dimensional Euclidean space to the definition of a combinatorial manifold in a space of lattice points.

In combinatorial geometry, simplex and complex are the fundamental elements for the definitions of geometric properties of objects. An n-dimensional simplex is an n-dimensional unit and an n-dimensional complex is formed by combining n-dimensional simplexes. Furthermore, an n-dimensional complex forms n-dimensional objects. Thus, a 1-dimensional curve is made up of simplexes of dimension one or lower and a 2-dimensional surface is made up of simplexes of dimension two or lower. Then, Françon defined a simplex under the condition that the points of a simplex must be lattice points. However, he proved only the existence of simplexes of lattice points, and did not derive any method for the construction of simplexes.

Our method to represent a discrete object is similar to Françon's method. We also apply combinatorial geometry[7]. However, our condition with respect to simplexes is stricter than that of Françon. Under our condition, every point of a simplex must be located in the neighborhood of other points of a simplex. Thus, the number of simplexes which our method defines is fixed, while it is infinite in Françon's method. Consequently, we can form a discrete object, surface and curve by a constructive method because we obtain the complete set of simplexes.

The fact that the boundary of an object is a surface and the boundary of a surface is a curve shows us the importance of unified representation of discrete objects, discrete surfaces and discrete curves. Then, this paper introduces a new representation of not only discrete objects but also curves and surfaces, in the following steps. In section 2, we give the definitions of a discrete space and neighborhoods in the discrete space. In section 3, we define discrete simplexes whose dimensions are from 0 to 3. In section 4, we introduce some definitions from discrete combinatorial topology for use in following sections. In section 5, we define a discrete curve. In section 6, we define a discrete surface and prove that the boundary of a discrete surface is a discrete curve. In section 7, we define a discrete object and prove that the boundary of a discrete object is a discrete surface.

2 Discrete Space

Let $\mathbf{Z}$ be the set of all integers. $\mathbf{Z}^3$ is called a discrete space. Here, $\mathbf{Z}^3$ is a subset of 3-dimensional Euclidean space $\mathbf{R}^3$. In $\mathbf{Z}^3$ neighborhoods are defined as follows:

Definition 1 (neighborhood) *Let $\boldsymbol{x} = (i, j, k)$ be a point in $\mathbf{Z}^3$. Three kinds of neighborhoods of $\boldsymbol{x}$ are defined by*

$$N_6(\boldsymbol{x}) = \{(p,q,r) \mid (i-p)^2 + (j-q)^2 + (k-r)^2 \leq 1, \quad p,q,r \in \mathbf{Z}\}\ , \tag{1}$$

$$N_{18}(\boldsymbol{x}) = \{(p,q,r) \mid (i-p)^2 + (j-q)^2 + (k-r)^2 \leq 2, \quad p,q,r \in \mathbf{Z}\} \tag{2}$$

and

$$N_{26}(\boldsymbol{x}) = \{(p,q,r) \mid (i-p)^2 + (j-q)^2 + (k-r)^2 \leq 3, \quad p,q,r \in \mathbf{Z}\}\ . \tag{3}$$

They are called 6-neighborhood, 18-neighborhood and 26-neighborhood, respectively.

3 Discrete Simplex

Although the theory of simplexes is generally defined in Euclidean space, we deal with topological properties in $\mathbf{Z}^3$. Thus, in this section, we define a simplex in $\mathbf{Z}^3$. In $\mathbf{Z}^3$, if we choose two vertices, one of which is located in the neighborhood of another, then the distance between them is minimum. This is because we choose vertices only on lattice points in $\mathbf{Z}^3$. The existence of the minimum distance implies the existence of minimum simplexes in $\mathbf{Z}^3$ whose vertices have the minimum distance. We call such minimum simplexes in $\mathbf{Z}^3$ discrete simplexes, and hereafter abbreviate discrete simplex to d-simplex. Simplexes have dimension from 0 to 3. We define d-simplexes in order of increasing dimension. If the dimension of a d-simplex is n, we denote it as n-d-simplex.

Definition 2 (0-d-simplex) *A 0-d-simplex is composed of one vertex $\boldsymbol{x}$ in $\mathbf{Z}^3$ and represented by $[\boldsymbol{x}]$.*

Definition 3 (1-d-simplex) *A 1-d-simplex is composed of two vertices $\boldsymbol{x}_0, \boldsymbol{x}_1$ in $\mathbf{Z}^3$ and represented by $[\boldsymbol{x}_0, \boldsymbol{x}_1]$, where $\boldsymbol{x}_0 \in N_m(\boldsymbol{x}_1)$ $(m = 6, 18, 26)$.*

Since there are three kinds of neighborhoods in $\mathbf{Z}^3$, three kinds of 1-d-simplexes exist corresponding to their neighborhoods. If we select 6-neighborhood as a neighborhood in $\mathbf{Z}^3$, we obtain one 1-d-simplex. If we select 18-neighborhood, we obtain two 1-d-simplexes. If we select 26-neighborhood, we obtain three 1-d-simplexes. Here, we regard two d-simplexes as the same d-simplex if we can transform one to the other by a congruence transformation.

Before defining 2-d-simplexes, we define a region which includes a vertex $\boldsymbol{x} = (i, j, k) \in \mathbf{Z}^3$ and other vertices as follows:

$$D(\boldsymbol{x}) = \{(i + \epsilon_1, j + \epsilon_2, k + \epsilon_3) \mid \epsilon_i = 0 \text{ or } 1\}. \quad (4)$$

Definition 4 (2-d-simplex) *A 2-d-simplex is composed of $n + 1$ vertices $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ $(n = 2$ or $3)$ in $\mathbf{Z}^3$ and represented by $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$. $n + 1$ vertices are determined by the following procedure 1.*

procedure 1

1. Fix a vertex $\boldsymbol{x}_0$.
2. Select $\boldsymbol{x}_1, \boldsymbol{x}_2$ in $D(\boldsymbol{x}_0)$ where
$$\boldsymbol{x}_1 \in N_m(\boldsymbol{x}_0)$$
and
$$\boldsymbol{x}_2 \in N_m(\boldsymbol{x}_0).$$
3. If $\boldsymbol{x}_2 \in N_m(\boldsymbol{x}_1)$, we obtain $[\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2]$, else, select $\boldsymbol{x}_3$ in $D(\boldsymbol{x}_0)$ where
$$\boldsymbol{x}_3 \notin N_m(\boldsymbol{x}_0)$$
and
$$\boldsymbol{x}_3 \in N_m(\boldsymbol{x}_1) \cap N_m(\boldsymbol{x}_2),$$
then we obtain $[\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]$.

In the procedure mentioned above, m is 6, 18 or 26 which represents the neighborhood in $\mathbf{Z}^3$.

A 2-d-simplex is a set of three or four vertices which are located in the neighborhood of each other. If we select 6-neighborhood as a neighborhood in $\mathbf{Z}^3$, we obtain one 2-d-simplex consisting of four vertices. If we select 18-neighborhood, we obtain one 2-d-simplex consisting of four vertices and two 2-d-simplexes consisting of three vertices. If we select 26-neighborhood, we obtain three 2-d-simplexes consisting of three vertices. Here, we regard two d-simplexes as the same d-simplex if we can transform one to another by a congruence transformation.

Definition 5 (3-d-simplex) *A 3-d-simplex is composed of $n + 1$ vertices $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ $(n = 3, 4$ or $8)$ in $\mathbf{Z}^3$ and determined by $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$. $n + 1$ vertices are decided by the following procedure 2.*

procedure 2

1. Fix a vertex $\boldsymbol{x}_0$.
2. Select a 2-d-simplex $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s]$ $(s = 2$ or $3)$ in $D(\boldsymbol{x}_0)$ which includes $\boldsymbol{x}_0$.
3. Determine a 2-d-simplex
$$[a(i)] = [\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \boldsymbol{y}_{i(0)}, \cdots, \boldsymbol{y}_{i(t)}] \quad (t = 0 \text{ or } 1)$$
in $D(\boldsymbol{x}_0)$ which includes two vertices $\boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ $(i = 0, 1, 2, \ldots, s)$ such that $\boldsymbol{x}_{i+1} = \boldsymbol{x}_0$. Two 2-d-simplexes must satisfy the relation $\boldsymbol{y}_{i(t)} = \boldsymbol{y}_{i+1(0)}$.
4. If $\boldsymbol{y}_{i(0)}(i = 0, 1, \ldots, s)$ indicates one vertex, we set $\boldsymbol{y}_{i(0)}(i = 0, 1, \ldots, s)$ to $\boldsymbol{y}$ and obtain $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s, \boldsymbol{y}]$, else, if we can define $[\boldsymbol{y}_{0(0)}, \boldsymbol{y}_{1(0)}, \cdots, \boldsymbol{y}_{s(0)}]$ as a 2-d-simplex, we obtain $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s, \boldsymbol{y}_{0(0)}, \boldsymbol{y}_{1(0)}, \cdots, \boldsymbol{y}_{s(0)}]$.

In the procedure mentioned above, m is 6, 18 or 26 which represents the neighborhood in $\mathbf{Z}^3$.

A 3-d-simplex is a set of four, five or eight vertices which are located in the neighborhood of each other. If we select 6-neighborhood as a neighborhood in $\mathbf{Z}^3$, we obtain one 2-d-simplex consisting of eight vertices. If we select 18-neighborhood, we obtain two 2-d-simplexes consisting of four vertices and one 2-d-simplex consisting of five vertices. If we select 26-neighborhood, we obtain five 2-d-simplexes consisting of four vertices. Here, we regard two d-simplexes as the same d-simplex if we can transform one to another by a congruence transformation.

Figure 1 shows n-d-simplexes($n = 0, 1, 2, 3$) of each neighborhood.

4 Discrete Combinatorial Geometry

In this section we define a complex and some other topological concepts in $\mathbf{Z}^3$. By embedding d-simplexes in $\mathbf{R}^3$, we extend properties of classical combinatorial topology to $\mathbf{Z}^3$.

Definition 6 (embedded d-simplex) *Setting r-d-simplex ($r = 0, 1, \ldots, 3$) to be $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$, we embed $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ in $\mathbf{Z}^3$ to $\mathbf{R}^3$. An embedded d-simplex of $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is defined by*

$$|\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n| = \{\boldsymbol{x} | \ \boldsymbol{x} = \sum_{i=0}^{r} \lambda_i \boldsymbol{x}_i, \ \sum_{i=0}^{r} \lambda_i = 1, \ \lambda_i > 0\} \ . \quad (5)$$

An embedded d-simplex $|\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n|$ is a convex hull of vertices, $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, without its boundary. The vertices of an embedded d-simplex are not linear, which is a significant difference between such a simplex and ordinary simplexes. For the definition of a discrete complex, the following definition is necessary.

Definition 7 (face)
Among $r + 1$ vertices $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of an r-d-simplex $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots ; \boldsymbol{x}_n]$, we choose $s+1 (s < n)$ vertices and construct an s-d-simplex consisting of these $s + 1$ vertices. We call this s-d-simplex the s-face of $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$. The set of all faces of $[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is denoted by

$$face([\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]) = \{ \ [\boldsymbol{x}_{i(0)}, \boldsymbol{x}_{i(1)}, \ldots, \boldsymbol{x}_{i(s)}] \ | \ \boldsymbol{x}_{i(k)} = \boldsymbol{x}_j \ (0 \geq k \geq s, \ 0 \geq j \geq r), \ s < r \ \}. \quad (6)$$

If we select 6-neighborhood as a neighborhood in $\mathbf{Z}^3$, It is clear that a 1-d-simplex has two 0-faces, a 2-d-simplex has four 0-faces and four 1-faces and a 3-d-simplex has six 2-faces, twelve 1-faces and eight 0-faces. If we select the other neighborhood, the number of faces depends on the neighborhood. Here, we define discrete complex, d-complex, and related concepts.

Definition 8 (d-complex) *A finite set $\mathbf{K}$ whose elements are d-simplexes in $\mathbf{Z}^3$ is called a d-complex, if the following conditions are satisfied:*

1. *If $[a] \in \mathbf{K}$, $face([a]) \in \mathbf{K}$.*
2. *If $[a_1], [a_2] \in \mathbf{K}$ and $|a_1| \cap |a_2| \neq \emptyset$, $[a_1] = [a_2]$.*

where $[a]$ and $|a|$ are a d-simplex and its embedded d-simplex, respectively. The dimension of $\mathbf{K}$ is equivalent to the maximum dimension of all d-simplexes which belong to $\mathbf{K}$.

Definition 9 (pure d-complex) *An r-dimensional d-complex $\mathbf{K}$ is pure if every s-d-simplex $[a]$ $(s < r)$ in $\mathbf{K}$ satisfies*

$$[a] \in face([b]), \quad (7)$$

where $[b]$ is one of the r-d-simplexes in $\mathbf{K}$.

Definition 10 (connected d-complex) *A d-complex $\mathbf{K}$ is connected if two arbitrary elements $[a], [b]$ define a path*

$$[a_1] = [a], \ [a_2], \ \ldots, \ [a_n] = [b], \quad (8)$$

where

$$[a_i] \in \mathbf{K}$$

and

$$[a_i] \cap [a_{i+1}] \neq \emptyset.$$

There are three kinds of neighborhoods in $\mathbf{Z}^3$, and the shapes of d-simplexes depend on these neighborhoods. Thus, the shape and the property of a d-complex also depend on these neighborhoods because a d-complex is formed by combining d-simplexes.

A d-complex is a combination of d-simplexes. In a pure n-dimensional d-complex, d-simplexes of dimension less than n do not exist unless they belong to the faces of some n-dimensional d-simplexes. In a connected d-complex, d-simplexes are connected with each other. These d-complexes depend on the arrangement of d-simplexes in the d-complexes. Figure 2 illustrates the process of generating 3-dimensional d-complexes. The neighborhood in figure 2 is 26-neighborhood.

We define some further topological properties in $\mathbf{Z}^3$ using the definition of a d-complex.

Definition 11 (star) *Let $\boldsymbol{x}$ be a vertex of a d-complex* $\mathbf{K}$. *A star of $\boldsymbol{x}$ with respect to* $\mathbf{K}$ *is defined by*

$$\sigma(\boldsymbol{x}) = \{[a] \mid \boldsymbol{x} \in [a],\ [a] \in \mathbf{K}\}. \tag{9}$$

If we want emphasize the set $\mathbf{K}$, *we describe a star of $\boldsymbol{x}$ with respect to* $\mathbf{K}$ *as* $\sigma(\boldsymbol{x} : \mathbf{K})$.

Definition 12 (outer star) *An outer star is defined by*

$$\lfloor \sigma(\boldsymbol{x}) \rfloor = \bigcup_{[a]\in\sigma(\boldsymbol{x})} face([a]) \setminus \sigma(\boldsymbol{x}) . \tag{10}$$

5 Discrete Curve

We define linear stars and semi-linear stars in $\mathbf{Z}^3$ using basic properties of combinatorial geometry. Furthermore, using these definitions we define discrete curves in $\mathbf{Z}^3$.

Definition 13 (linear star) *If*

$$\lfloor \sigma(\boldsymbol{x}) \rfloor = \{[\boldsymbol{x}_i] \mid i = 1, 2\}, \tag{11}$$

$\sigma(\boldsymbol{x})$ is linear.

Definition 14 (semi-linear star) *If*

$$\lfloor \sigma(\boldsymbol{x}) \rfloor = \{[\boldsymbol{x}_1]\}, \tag{12}$$

$\sigma(\boldsymbol{x})$ is semi-linear.

Figure 3 illustrates the points of which stars are linear and semi-linear. The neighborhood of figure 3 is 6-neighborhood. These definitions leads to the following definition of a 1-dimensional discrete curve.

Definition 15 (discrete curve) *Let* $\mathbf{K}$ *be a connected and pure 1-dimensional d-complex. If the star of every vertex in* $\mathbf{K}$ *is linear or semi-linear,* $\mathbf{K}$ *is a discrete curve.*

It is clear that the star of a intersection point in a curve is neither linear nor semi-linear. Thus, definition 15 implies that a discrete curve does not have any intersection points because the star of every point is linear or semi-linear. The vertices of which star is semi-linear correspond to end points of a discrete curve. Therefore, we can define a discrete curve without boundary.

Definition 16 (discrete closed curve) *Let* $\mathbf{K}$ *be a discrete curve. If a set of vertices in* $\mathbf{K}$ *whose stars are semi-linear is empty,* $\mathbf{K}$ *is a discrete closed curve.*

Similarly, we can also define the boundary of a discrete curve.

Definition 17 (Boundary of discrete curve) *Let* $\mathbf{K}$ *be a discrete curve. A boundary of* $\mathbf{K}$ *is a set of vertices whose stars are semi-linear.*

The following theorem is derived from the above definitions.

Theorem 1 *If a discrete curve* $\mathbf{K}$ *is not closed, a boundary of* $\mathbf{K}$ *is*

$$\partial\mathbf{K} = \{[\boldsymbol{x}_1], [\boldsymbol{x}_2]\}. \tag{13}$$

(proof) Since $\mathbf{K}$ is not closed, $\partial\mathbf{K}$ is not empty. Assume that $\partial\mathbf{K}$ is a set of at least three 0-d-simplexes. All elements of $\partial\mathbf{K}$ are connected with each other through the path of d-simplexes in $\mathbf{K}$ because $\mathbf{K}$ is connected. This implies that there exist some vertices in $\mathbf{K} \setminus \partial\mathbf{K}$ such that stars are not linear. However, this leads to contradiction because the star of every vertex in $\mathbf{K}$ is linear or semi-linear. Thus, $\partial\mathbf{K}$ is a set of two 0-d-simplexes. *(Q.E.D)*

6 Discrete Surface

We define a discrete surface using the properties of stars. Here, we define cyclic stars and semi-cyclic stars.

Definition 18 (cyclic star) *If* $\lfloor \sigma(\boldsymbol{x}) \rfloor$ *is a discrete closed curve,* $\sigma(\boldsymbol{x})$ *is cyclic.*

Definition 19 (semi-cyclic star) *If* $\lfloor \sigma(\boldsymbol{x}) \rfloor$ *is a discrete curve which does not contain a closed curve,* $\sigma(\boldsymbol{x})$ *is semi-cyclic.*

Figure 4 illustrates the points of which stars are cyclic or semi-cyclic. The neighborhood of figure 4 is 6-neighborhood. These definitions lead to the definition of a 2-dimensional discrete surface.

Definition 20 (discrete surface) *Let* $\mathbf{K}$ *be a connected and pure 2-dimensional d-complex. If the star of every vertex in* $\mathbf{K}$ *is cyclic or semi-cyclic,* $\mathbf{K}$ *is a discrete surface.*

The vertices whose stars are cyclic correspond to inner points of a discrete surface and the vertices whose stars are semi-cyclic correspond to boundary points. Therefore, a discrete surface does not intersect itself because the star of every point is cyclic or semi-cyclic. Thus, we can define a discrete surface without boundary. Moreover, we can also define the boundary of a discrete surface.

Definition 21 (discrete closed surface) *Let* $\mathbf{K}$ *be a discrete surface. If a set of vertices in* $\mathbf{K}$ *whose stars are semi-cyclic is empty,* $\mathbf{K}$ *is a discrete closed surface.*

Definition 22 (Boundary of discrete surface) *Let* $\mathbf{K}$ *be a discrete surface and* $\mathbf{H}$ *be a set of vertices in* $\mathbf{K}$ *whose stars are semi-cyclic. A boundary of* $\mathbf{K}$ *is*

$$\partial\mathbf{K} = \mathbf{B}_0 \cup \mathbf{B}_1, \tag{14}$$

where

$$\mathbf{B}_0 = \{[\boldsymbol{x}] \mid \boldsymbol{x} \in \mathbf{H}\} \tag{15}$$

and

$$\mathbf{B}_1 = \{[\boldsymbol{x},\boldsymbol{y}] \mid \boldsymbol{x} \in \mathbf{H},\ [\boldsymbol{y}] \in \partial(\lfloor\sigma(\boldsymbol{x})\rfloor)\}. \tag{16}$$

These definitions lead to the following theorem.

Theorem 2 *If a discrete surface* $\mathbf{K}$ *is not closed, a boundary of* $\mathbf{K}$*,* $\partial\mathbf{K}$*, consists of a finite number of discrete closed curves.*

(proof) In order to show that $\partial\mathbf{K}$ consists of a finite number of discrete closed curves, we show that the star of every vertex in $\partial\mathbf{K}$ is linear.

For a fixed vertex $\boldsymbol{x}$ in $\partial\mathbf{K}$, the d-simplexes which are contained in $\sigma(\boldsymbol{x} : \partial\mathbf{K})$ are classified into the following three types:

1. a 0-d-simplex $[\boldsymbol{x}]$.
2. two 2-d-simplexes $[\boldsymbol{x}, \boldsymbol{y}_i]$ $(i = 1, 2)$, where $[\boldsymbol{y}_i] \in \partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$. Now, $\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ is a discrete curve with boundary because $\sigma(\boldsymbol{x} : \mathbf{K})$ is semi-cyclic. Thus, $\partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ consists of two vertices.

Therefore,

$$\sigma(\boldsymbol{x} : \partial\mathbf{K}) = \{[\boldsymbol{x}]\} \cup \{[\boldsymbol{x}, \boldsymbol{y}_i] \mid i = 1, 2\} \tag{17}$$

and

$$\lfloor\sigma(\boldsymbol{x} : \partial\mathbf{K})\rfloor = \{[\boldsymbol{y}_1, \boldsymbol{y}_2]\}. \tag{18}$$

Namely, $\lfloor\sigma(\boldsymbol{x} : \partial\mathbf{K})\rfloor$ is linear. Thus, $\partial\mathbf{K}$ is a finite number of discrete closed curves. *(Q.E.D)*

7 Discrete Object

As a natural extension of the relation between surfaces and curves, here, we define the relation between objects and surfaces using properties of stars. First, we define spherical stars and semi-spherical stars.

Definition 23 (spherical star) *If* $\lfloor\sigma(\boldsymbol{x})\rfloor$ *is a discrete closed surface,* $\sigma(\boldsymbol{x})$ *is spherical.*

Definition 24 (semi-spherical star) *If* $\lfloor\sigma(\boldsymbol{x})\rfloor$ *is a discrete surface which does not contain a closed surface,* $\sigma(\boldsymbol{x})$ *is semi-spherical.*

Figure 5 illustrates the points of which stars are spherical or semi-spherical. The neighborhood of figure 5 is 26-neighborhood.

Definitions 23 and 24 lead to the definition of a 3-dimensional discrete object.

Definition 25 (discrete object) *Let* $\mathbf{K}$ *be a connected and pure 3-dimensional d-complex. If the star of every vertex in* $\mathbf{K}$ *is spherical or semi-spherical,* $\mathbf{K}$ *is a discrete object.*

A vertex whose star is spherical corresponds to the central point of a sphere. Therefore, such a vertex is located in the interior of an object. A vertex whose star is semi-spherical is located on the boundary of an object. Thus, all of the 3-dimensional d-complexes in figure 2 are not discrete objects. The d-complex in figure 6 is a discrete object. There do not exist discrete closed objects analogous to discrete closed surfaces. Thus, we obtain the following theorem.

Theorem 3 *Let* $\mathbf{K}$ *be a discrete object. A set of vertices in* $\mathbf{K}$ *whose stars are semi-spherical is never empty in* $\mathbf{Z}^3$.

(Proof) If the set of vertices in $\mathbf{K}$ whose stars are semi-spherical is empty, the star of every vertex in $\mathbf{K}$ must be spherical. If the star of every vertex in $\mathbf{K}$ is spherical, we must embed 3-d-simplexes in $\mathbf{Z}^3$ leaving no space between them. Now, a discrete object $\mathbf{K}$ is constructed from a finite number of 3-d-simplexes and their faces. Thus, we cannot continue the embedding process infinitely. Therefore, $\mathbf{K}$ possesses some vertices whose stars are semi-spherical. *(Q.E.D)*

A discrete object always has a boundary, which is defined as follows:

Definition 26 (Boundary of discrete object) *Let* $\mathbf{K}$ *be a discrete object and* $\mathbf{H}$ *be a set of vertices in* $\mathbf{K}$ *whose stars are semi-spherical. A boundary of* $\mathbf{K}$ *is*

$$\partial\mathbf{K} = \mathbf{B}_0 \cup \mathbf{B}_1 \cup \mathbf{B}_2, \tag{19}$$

where

$$\mathbf{B}_0 = \{[\boldsymbol{x}] \mid \boldsymbol{x} \in \mathbf{H}\}, \tag{20}$$

$$\mathbf{B}_1 = \{[\boldsymbol{x},\boldsymbol{y}] \mid \boldsymbol{x} \in \mathbf{H},\ [\boldsymbol{y}] \in \partial\lfloor\sigma(\boldsymbol{x})\rfloor\} \tag{21}$$

and

$$\mathbf{B}_2 = \{[\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n] \mid \boldsymbol{x}_0 \in \mathbf{H},\ [\boldsymbol{x}_i, \boldsymbol{x}_{i+1}] \in \partial\lfloor\sigma(\boldsymbol{x}_0)\rfloor\ (i = 1, 2, \ldots, n-1)\}. \tag{22}$$

In the above definition, $\mathbf{B}_0$, $\mathbf{B}_1$ and $\mathbf{B}_2$ indicate a set of 0-d-simplexes, 1-d-simplexes and 2-d-simplexes, respectively. The next theorem shows that these three sets, $\mathbf{B}_0$, $\mathbf{B}_1$ and $\mathbf{B}_2$, form discrete closed surfaces. It is an analogue of theorem 2. By replacing curves and surfaces in theorem 2 with surfaces and objects, respectively, we obtain the following theorem.

Theorem 4 *The boundary of a discrete object consists of a finite number of discrete closed surfaces.*

(proof) Let the boundary of a discrete object $\mathbf{K}$ be $\partial\mathbf{K}$. In order to show that $\partial\mathbf{K}$ consists of a finite number of discrete closed surfaces, we show that the star of every vertex in $\partial\mathbf{K}$ is cyclic.

For a fixed vertex $\boldsymbol{x}$ in $\partial\mathbf{K}$, the d-simplexes which are contained in $\sigma(\boldsymbol{x} : \partial\mathbf{K})$ are classified into the following three types:

1. a 0-d-simplex $[\boldsymbol{x}]$.
2. 1-d-simplexes $[\boldsymbol{x}, \boldsymbol{y}_i]$ $(i = 0, 1, 2, \ldots, n)$, where $[\boldsymbol{y}_i] \in \partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$. Now, $\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ is a discrete surface with boundary because $\sigma(\boldsymbol{x} : \mathbf{K})$ is semi-spherical. Thus, Theorem 2 implies that $\partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ is a discrete closed curve.
3. 2-d-simplexes $[\boldsymbol{x}, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}]$ $(i = 0, 1, \ldots, n)$ such that $\boldsymbol{y}_{n+1} = \boldsymbol{y}_0$, where $[\boldsymbol{y}_i, \boldsymbol{y}_{i+1}] \in \partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ and $\partial\lfloor\sigma(\boldsymbol{x} : \mathbf{K})\rfloor$ is a discrete curve.

Therefore,

$$\sigma(\boldsymbol{x} : \partial\mathbf{K}) = \{[\boldsymbol{x}]\} \cup \{[\boldsymbol{x}, \boldsymbol{y}_i] \mid i = 0, 1, \ldots, n\} \cup \{[\boldsymbol{x}, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}] \mid i = 0, 1, \ldots, n, \boldsymbol{y}_{n+1} = \boldsymbol{y}_0\} \quad (23)$$

and

$$\lfloor\sigma(\boldsymbol{x} : \partial\mathbf{K})\rfloor = \{[\boldsymbol{y}_i], [\boldsymbol{y}_i, \boldsymbol{y}_{i+1}] \mid i = 0, 1, \ldots, n, \boldsymbol{y}_{n+1} = \boldsymbol{y}_0\}. \quad (24)$$

$\sigma(\boldsymbol{x} : \partial\mathbf{K})$ is cyclic because $\lfloor\sigma(\boldsymbol{x} : \partial\mathbf{K})\rfloor$ is linear. Thus, $\partial\mathbf{K}$ is a finite number of discrete closed surfaces. *(Q.E.D)*

8 Conclusions

We proposed a new representation of curves, surfaces and objects in a discrete space using combinatorial geometry. We introduced the method of constructing simplexes and complexes in a discrete space, practically, although Françon proved the existence of simplexes and complexes in a discrete space. A finite number of complexes generate discrete curves, surfaces and objects. Therefore, we defined discrete curves, surfaces and objects by a constructive method. Furthermore, we proved that the boundary of a discrete surface consists of a finite number of discrete closed curves and that the boundary of a discrete object consists of a finite number of discrete closed surfaces. For the application of these representations to geometric data processing, in forthcoming papers, we will present algorithms which extract curves, surfaces and objects from raw data. Discrete objects, surfaces and curves which are formed by our method are classified into three types which depend on the neighborhood in $\mathbf{Z}^3$. Then, we obtain geometric parameters, for example, Euler numbers of objects and curvatures of curves or surfaces, of objects in the three-dimensional world from raw data for each neighborhood.

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References

1: Kong T. Y. , Rosenfeld A. , Digital Topology: Introduction and Survey, Computer Vision, Graphics, and Image Processing, 48, 357-393, 1989.

2: Voss K. , *Discrete Images, Objects, and Functions in* $\mathbf{Z}^3$, Algorithms and Combinatorics 11, Springer-Verlag, 1987.

3: Udupa J. K. , it Multidimensional Digital Boundaries, CVGIP: Graphical Models and Image Processing, Vol. 56, No. 4, July, pp. 311-323, 1994.

4: Herman G. T. , Discrete Multidimensional Jordan Surfaces, CVGIP: Graphical Model and Image Processing, Vol. 54, No. 6, November, pp. 507-515, 1992.

5: Lorensen W. E. , Cline H. E. , Marching Cube: A High-Resolution 3D Surface Construction Algorithm, Computer Graphics (SIGGRAPH '87), Vol. 21, No. 4, July, pp. 163-169, 1988.

6: Françon J. , Discrete Combinatorial Surface, Graphical Models and Image Processing, Vol. 57, No. 1, January, pp. 20-26, 1995

7: Aleksandrov P. S. , *Combinatorial Topology*, Vol. 1, Graylock Press, Rochester, N.Y., 1956.

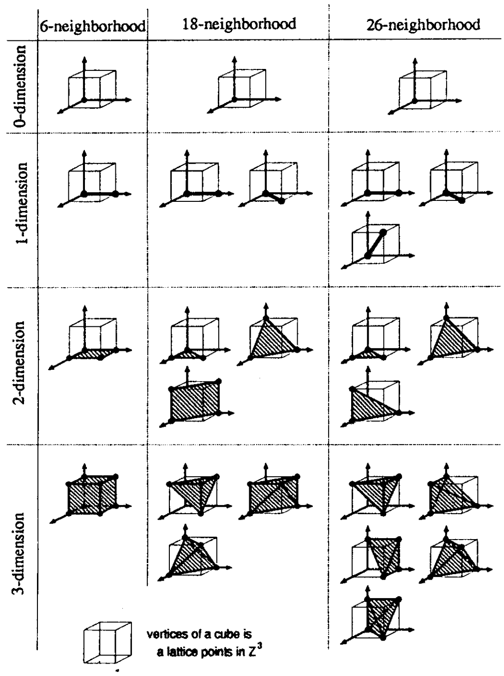

Fig.1: d-simplexes in Z^3.

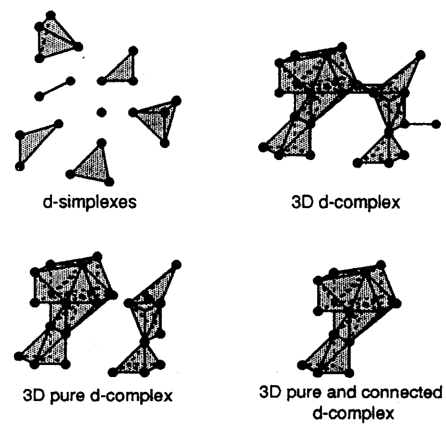

Fig.2: d-simplexes, a d-complex, a pure d-complex and a connected and pure d-complex. The neiborhood in this figure is 26-neighborhood. A 3-dimensional d-complex is a combination of d-simplexes whose dimensions are 3 or lower. In a pure 3-dimensional d-complex, d-simplexes of dimension less than 3 do not exist unless they belong to the faces of some 3-dimensional d-simplexes. In a connected and pure d-complex, d-simplexes are connected with each other.

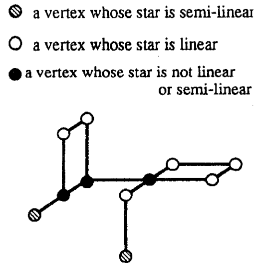

Fig.3: Vertices of which stars are semi-linear and linear. The neighborhood in this figure is 6-neighborhood.

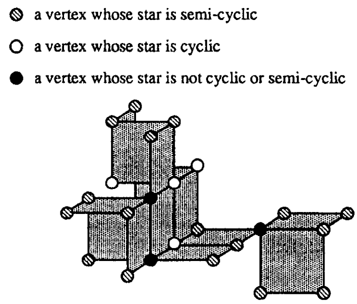

Fig.4: Vertices of which stars are semi-cyclic and cyclic. The neighborhood in this figure is 6-neighborhood.

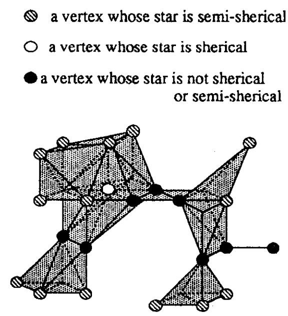

Fig.5: Vertices of which stars are semi-spherical and spherical. The neighborhood in this figure is 26-neighborhood.

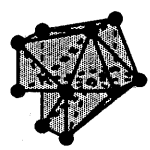

Fig.6: A discrete object. The neighborhood in this figure is 26-neighborhood.