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離散物体の単体ラベリングとその応用

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３次元物体を計算機内で表現する場合，システムの解像度が有限であることから，物体が孤立点や鬚のような３次元ではない特異な部分を持つことがある．本論文では，まず，組合せ位相幾何学の複体を取り入れてこのような特異な部分の性質を明らかにする．そして，この特異な部分を計測データから一意に抽出する算法を提案する．このような特異な部分は解像度を上げると特異でなくなることがある．従って，可変解像度解析では，各解像度において特異な部分を抽出しておく必要がある．

Discrete Complex and Its Application

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In computer vision, one of the ultimate purposes is the acquisition of geometric concepts of a 3-dimensional world from measured data. As an expression intermediate between measured data and geometric concepts, we need a representation of objects in computers because in a computer we can only manipulate finite-precision numbers. Objects represented in computers often have isolated points, needles and thin walls because of the finite resolution. Such isolated points, needles and thin walls are treated as irregular parts in the classical boundary detection. However, these irregular parts possess local geometric properties of objects. Thus, we extract irregular parts of objects by using properties of a complex which we defined in the previous paper.

1 Introduction

In computer vision, one of the ultimate purposes is the acquisition of geometric concepts of a 3-dimensional world from measured data. As an expression intermediate between measured raw data and geometric concepts, we need a representation of objects in computers. Computers express and calculate data using finite-precision arithmetic. Thus, we need a method to express geometric objects in a computer memory using only finite-precision data. Data expressed using finite-precision arithmetic are equivalent to data expressed using only integers. This leads to the conclusion that geometric data should be expressed mathematically as lattice points in a 3-dimensional space. We call such geometric data obtained from an object in a 3-dimensional world a discrete object because a space of lattice points is a discrete space.

This paper is the third of a series on discrete geometry. In the first paper[1], we presented the representation of a discrete object and its boundary. In the second paper[2], we proposed an algorithm for the unique extraction of boundaries of discrete objects from a given subset of a discrete space. This algorithm automatically excludes the irregular parts, such as isolated points, needles and thin walls. In contrast, in this paper, we extract such irregular parts by using properties of a complex because irregular parts of a discrete object determine local geometric and topological properties.

There are many methods of representing discrete objects [3, 4, 5, 6, 7, 8]. They focus only on the elimination of irregular points. However, in this paper, we develop a method to extract these irregular points. Irregular points will classify the local geometric and topological properties of objects because the irregularity of points depends on the resolution of a system used to obtain discrete data. Thus, if we change the resolution of the system, some irregular points will change their properties. The extraction of n-dimensional irregular parts of an object is a direct application of discrete combinatorial geometry which we proposed in the previous papers.

In this paper, we use topological properties of a complex. We define a discrete complex in section 2. In section 3, we show that we can always construct a discrete complex from a given subset of a discrete space. We also propose an algorithm for the construction of a discrete complex from a set of lattice points. In section 4, we give proofs of theorems and lemmas whose results we show in section 3. In section 5, we extract n-dimensional parts from a discrete complex.

2 Discrete Complex

We define a discrete space as $\mathbf{Z}^3$ where $\mathbf{Z}$ is the set of all integers. $\mathbf{Z}^3$ is a subset of 3-dimensional Euclidean space $\mathbf{R}^3$. In $\mathbf{Z}^3$, we define neighborhoods as follows.

Definition 1 (neighborhood) *Let $\boldsymbol{x} = (i,j,k)$ be a point in $\mathbf{Z}^3$. Two kinds of neighborhoods of $\boldsymbol{x}$ are defined by*

$$N_6(\boldsymbol{x}) = \{(p,q,r) \mid (i-p)^2 + (j-q)^2 + (k-r)^2 \leq 1, \quad p,q,r \in \mathbf{Z}\} \tag{1}$$

and

$$N_{26}(\boldsymbol{x}) = \{(p,q,r) \mid (i-p)^2 + (j-q)^2 + (k-r)^2 \leq 3, \quad p,q,r \in \mathbf{Z}\}\ . \tag{2}$$

They are called 6-neighborhood and 26-neighborhood, respectively.

In this section, we define a discrete complex in $\mathbf{Z}^3$, which is called a d-complex[1]. According to discrete combinatorial geometry, d-complexes are constructed from d-simplexes[1]. d-simplexes are defined for two kinds of neighborhoods, and their dimensions are from zero to three. If the dimension of a d-simplex is r, we call such a d-simplex an r-d-simplex. We illustrate r-d-simplexes such that $r = 0, \ldots, 3$ for each neighborhood in figure 1. In order to construct a d-complex from d-simplexes, we need to embed d-simplexes in $\mathbf{R}^3$, and we use the notation $|a|$ as an embedded d-simplex corresponding to a d-simplex $[a]$ [1]. Moreover, we also need a function $face([a])$, which defines the set of all d-simplexes included in a d-simplex $[a]$ [1]. The exact definitions of a d-simplex, an embedded d-simplex and a function $face$ are given in reference 1. Next, we define a d-complex as follows.

Definition 2 (d-complex[1]) *A finite set $\mathbf{K}$ whose elements are d-simplexes in $\mathbf{Z}^3$ is called a d-complex, if the following conditions are satisfied.*

1. *If $[a] \in \mathbf{K}$, $face([a]) \in \mathbf{K}$,*
2. *If $[a_1], [a_2] \in \mathbf{K}$ and $|a_1| \cap |a_2| \neq \emptyset$, $[a_1] = [a_2]$,*

where $[a]$ and $|a|$ indicate a d-simplex and its embedded d-simplex, respectively. The dimension of $\mathbf{K}$ is equivalent to the maximum dimension of all d-simplexes which belong to $\mathbf{K}$.

Next, we introduce two properties of d-complexes.

Definition 3 (pure d-complex[1]) *An r-dimensional d-complex* $\mathbf{K}$ *is pure if every s-d-simplex* $[a]$ $(s < r)$ *in* $\mathbf{K}$ *satisfies the formula*

$$[a] \in face([b]), \tag{3}$$

where $[b]$ *is one of the r-d-simplexes in* $\mathbf{K}$.

Definition 4 (connected d-complex[1])
A d-complex $\mathbf{K}$ *is connected if two arbitrary elements* $[a]$ *and* $[b]$ *define a path*

$$[a_1] = [a],\ [a_2],\ \ldots,\ [a_n] = [b], \tag{4}$$

where

$$[a_i] \in \mathbf{K} \tag{5}$$

and

$$[a_i] \cap [a_{i+1}] \neq \emptyset. \tag{6}$$

We define the body of a d-complex as follows.

Definition 5 (body) *If* $\mathbf{K}$ *is a d-complex, the body of* $\mathbf{K}$ *is defined as*

$$\mathbf{B} = \bigcup_{[a]\in\mathbf{K}} [a]. \tag{7}$$

The body of a d-complex is a finite subset of $\mathbf{Z}^3$. While a d-complex is a set of d-simplexes and has topological structures, the body of a d-complex is a set of points which are included in a d-complex and has no structure.

3 Triangulation of a Point Set

We assume that a finite subset of $\mathbf{Z}^3$ is given, and we call such a given subset a point set. In this section, we show that we can always construct a d-complex from any point set. The construction of a d-complex from a point set is equivalent to the determination of the set of all d-simplexes which are embedded in a point set. From the viewpoint of classical combinatorial geometry in $\mathbf{R}^3$, the construction of a complex from a set of points is called triangulation of the set. Therefore, we also call our construction of a d-complex from a point set triangulation of a point set. Thus, in this section, we give the theorem of the existence of triangulation of any point set. We simply show the results of theorems and lemmas in this section and give their proofs in the following section. Furthermore, we propose an algorithm for the construction of a d-complex from a point set.

First, we partition $\mathbf{Z}^3$ as

$$\mathbf{Z}^3 = \bigcup_{\boldsymbol{x}\in\mathbf{Z}^3} D(\boldsymbol{x}), \tag{8}$$

where

$$D(\boldsymbol{x}) = \{(i+\epsilon_1, j+\epsilon_2, k+\epsilon_3) \mid \boldsymbol{x} = (i,j,k), \epsilon_i = 0 \text{ or } 1\}. \tag{9}$$

Each $D(\boldsymbol{x})$ has eight points. Denoting a point set by $\mathbf{V}$ and denoting a function which obtains the number of elements of a set $\mathbf{A}$ by $E(\mathbf{A})$, the relation

$$0 \leq E(\mathbf{V} \cap D(\boldsymbol{x})) \leq 8 \tag{10}$$

holds. For each $D(\boldsymbol{x})$, there are from zero to eight points which belong to $\mathbf{V}$. Eight points in $D(\boldsymbol{x})$ are categorized in two classes. One class consists of points which belong to $\mathbf{V}$, and we call such points black points. The other class consists of points which do not belong to $\mathbf{V}$, and we call such points white points. If a $D(\boldsymbol{x})$ has zero black points and eight white points, the arrangement of points is unique. If a $D(\boldsymbol{x})$ has one black point and seven white points, arrangement patterns are eight because a black point can be located on each point in $D(\boldsymbol{x})$ and $D(\boldsymbol{x})$ has eight points. However, if we consider rotations with respect to a central point of $D(\boldsymbol{x})$, we can regard all patterns of the arrangement as the same. If a $D(\boldsymbol{x})$ has two black points and six white points, there are three possible arrangement patterns of points considering all rotations with respect to a central point of $D(\boldsymbol{x})$. In the same way, we can count all patterns of the arrangement of n black points and $(8-n)$ white points for $0 \leq n \leq 8$. Figure 2 shows all patterns of the arrangement of points in $D(\boldsymbol{x})$. There are twenty-three patterns in total.

Second, we consider triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$. Since the size of every d-simplex which we defined in the previous section is smaller than a region $D(\boldsymbol{x})$, we obtain the next lemma.

Lemma 1 *Triangulation of* $D(\boldsymbol{x}) \cap \mathbf{V}$ *is always possible.*

If we use 6-neighborhood, we can derive the following lemma referring to figure 3.

Lemma 2 *If we use 6-neighborhood, the triangulation pattern of* $D(\boldsymbol{x}) \cap \mathbf{V}$ *is unique.*

In contrast with the case of 6-neighborhood, if we use 26-neighborhood, we can determine a finite number of patterns of the triangulation, not only one but sometimes more than one, corresponding to an arrangement pattern of black and white points. Figure 4 illustrates such a correspondence between triangulation patterns and arrangement patterns of $D(\boldsymbol{x}) \cap \mathbf{V}$.

Let the set of all n-dimensional d-simplexes which are embedded in $D(\boldsymbol{x}) \cap \mathbf{V}$ be $S_n(\boldsymbol{x})$. Then, we obtain the set of all d-simplexes of $D(\boldsymbol{x}) \cap \mathbf{V}$ as

$$C(\boldsymbol{x}) = S_0(\boldsymbol{x}) \cup S_1(\boldsymbol{x}) \cup S_2(\boldsymbol{x}) \cup S_3(\boldsymbol{x}). \tag{11}$$

If we obtain m patterns of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$, we set

$$C^i(\boldsymbol{x}) = S_0^i(\boldsymbol{x}) \cup S_1^i(\boldsymbol{x}) \cup S_2^i(\boldsymbol{x}) \cup S_3^i(\boldsymbol{x}) \qquad (12)$$

where $i = 1, 2, \ldots, m$. Here, since $S_0^i(\boldsymbol{x})$ is equivalent to $D(\boldsymbol{x}) \cap \mathbf{V}$, the relation

$$S_0^i(\boldsymbol{x}) = S_0^j(\boldsymbol{x}) \qquad (13)$$

is established where $i \neq j$.

Third, we consider the combination of triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ with respect to all $\boldsymbol{x}$ in $\mathbf{Z}^3$. In other words, if $D(\boldsymbol{y})$ is a region adjacent to $D(\boldsymbol{x})$, and a pattern of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ is given, we must find a pattern of the triangulation of $D(\boldsymbol{y}) \cap \mathbf{V}$ so that there is no contradiction between this pattern of the triangulation of $D(\boldsymbol{y}) \cap \mathbf{V}$ and the pattern of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$.

If $\boldsymbol{y}$ is different from $\boldsymbol{x}$, we can categorize the relations of $\boldsymbol{x}$ and $\boldsymbol{y}$ into four types as follows.

$$E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 0, 1, 2 \text{ or } 4, \qquad (14)$$

since $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y}))$ is never 3. If $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 0$, $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ have no common point and they are not adjacent. If $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 1$, $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ have a common point and they are adjacent with respect to this single point. If $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 2$, $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ have two common points and they are adjacent with respect to these two points. These two points are adjacent to each other. If $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 4$, $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ have four common points and they are adjacent with respect to these four points. These four points are adjacent to each other, are located on the same plane and form a square.

If $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ are adjacent, we can consider the triangulation of $D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}$. The triangulation of $D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}$ is denoted by $C(\boldsymbol{x}, \boldsymbol{y})$. Then, if we consider 6-neigborhood, the next lemma is derived.

Lemma 3 *Assume that we use 6-neighborhood. Let a region adjacent to $D(\boldsymbol{x})$ be $D(\boldsymbol{y})$ such that $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) \neq 0$, and the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ and $D(\boldsymbol{y}) \cap \mathbf{V}$ be $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$, respectively. There is no contradiction between $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$.*

If we use 26-neighborhood, the lemma is more complicated.

Lemma 4 *Assume that we use 26-neighborhood. Let a region adjacent to $D(\boldsymbol{x})$ be $D(\boldsymbol{y})$ such that $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) \neq 0$. If the numbers of the triangulation patterns of $D(\boldsymbol{x}) \cap \mathbf{V}$ and $D(\boldsymbol{y}) \cap \mathbf{V}$ are $m(\boldsymbol{x})$ and $m(\boldsymbol{y})$, respectively, we can set the triangulation patterns of $D(\boldsymbol{x}) \cap \mathbf{V}$ and $D(\boldsymbol{y}) \cap \mathbf{V}$ to be $C^i(\boldsymbol{x})$ ($i = 1, 2, \ldots, m(\boldsymbol{x})$) and $C^j(\boldsymbol{y})$ ($j = 1, 2, \ldots, m(\boldsymbol{y})$), respectively. Then, we can choose $C^i(\boldsymbol{x})$ and $C^j(\boldsymbol{y})$ without contradiction between $C^i(\boldsymbol{x})$ and $C^j(\boldsymbol{y})$.*

Then, we can derive the next theorem from previous lemmas.

Theorem 1 *Triangulation of a point set is always possible.*

This theorem implies that the number of the triangulation patterns of a point set is finite because a point set is of finite size, while the number of the triangulation patterns in $\mathbf{R}^3$ is infinite[9]. In particular, if we use 6-neighborhood, the pattern of the triangulation of a point set is unique. Then, an algorithm for the construction of a d-complex from a point set is given as follows.

algorithm 1

1. Partition $\mathbf{Z}^3$ as
$$\mathbf{Z}^3 = \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} D(\boldsymbol{x}).$$
2. Consider $m(\boldsymbol{x})$ patterns of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ and put $C^{i(\boldsymbol{x})}(\boldsymbol{x})$ where $i(\boldsymbol{x}) = 1, 2, \ldots, m(\boldsymbol{x})$.
3. Choose $C^{i(\boldsymbol{x})}(\boldsymbol{x})$ and combine $C^{i(\boldsymbol{x})}(\boldsymbol{x})$ with respect to each $\boldsymbol{x}$ in $\mathbf{Z}^3$. Then, we obtain a d-complex
$$\mathbf{C} = \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} C^{i(\boldsymbol{x})}(\boldsymbol{x}).$$

The principle of algorithm 1 is that we embed as many d-simplexes as possible in a point set. Then, the following theorem is established.

Theorem 2 *The body of a d-complex which is constructed from a point set $\mathbf{V}$, denoted by $\mathbf{B}$, satisfies*

$$\mathbf{B} = \mathbf{V}. \qquad (15)$$

4 Proofs of Theorems

4.1 Proof of Lemma 1

The principle of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ is that we embed as many d-simplexes as possible in $D(\boldsymbol{x}) \cap \mathbf{V}$. Vertices of all d-simplexes must be black points, not white points. If we use 6-neighborhood, we can determine a pattern of the triangulation for each pattern of the arrangement of black and white points. Figure 3 shows the pattern of the triangulation corresponding to each pattern of the arrangement of black and white points.

If we use 26-neighborhood, we can determine a pattern of the triangulation for each pattern of the arrangement of black and white points. However, for some patterns of the arrangement of points, there are two or four patterns of the triangulation. Figure 4 shows patterns of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$. Therefore, with respect to each pattern of the arrangement of points in $D(\boldsymbol{x})$, there is always at least one pattern of the triangulation. Thus, the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ is always possible.

4.2 Proof of Lemma 2

If we use 6-neighborhood, we can determine a unique pattern of the triangulation corresponding to a pattern of the arrangement of points in $D(\boldsymbol{x})$. Figure 3 illustrates the one-to-one correspondence between a pattern of the triangulation and a pattern of the arrangement.

4.3 Proof of Lemma 3

According to lemma 2, $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$ are uniquely determined. We can also uniquely determine $C(\boldsymbol{x}, \boldsymbol{y})$, the triangulation of $D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}$, depending on the combinatorial patterns of black points in $D(\boldsymbol{x}) \cap D(\boldsymbol{y})$. Figure 5 shows the one-to-one correspondence between $C(\boldsymbol{x}, \boldsymbol{y})$ and an arrangement pattern of black points $D(\boldsymbol{x}) \cap D(\boldsymbol{y})$ for $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 1, 2, 4$. Then, $C(\boldsymbol{x}, \boldsymbol{y})$ satisfies the following two formulas.

$$C(\boldsymbol{x}, \boldsymbol{y}) \subset C(\boldsymbol{x}) \tag{16}$$

and

$$C(\boldsymbol{x}, \boldsymbol{y}) \subset C(\boldsymbol{y}). \tag{17}$$

Therefore, there is no contradiction between $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$.

4.4 Proof of Lemma 4

If $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}) \neq 4$, we can uniquely determine $C(\boldsymbol{x}, \boldsymbol{y})$. Figure 5 shows the one-to-one correspondence between $C(\boldsymbol{x}, \boldsymbol{y})$ and a pattern of the arrangement of black points in $D(\boldsymbol{x}) \cap D(\boldsymbol{y})$ for $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) = 1, 2, 4$. Therefore, we can choose any $C^i(\boldsymbol{x})$ and any $C^j(\boldsymbol{y})$ such that there is no contradiction between $C^i(\boldsymbol{x})$ and $C^j(\boldsymbol{y})$.

Now, assume that $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}) = 4$. If $E(D(\boldsymbol{x}) \cap \mathbf{V}) = 8$ and $E(D(\boldsymbol{y}) \cap \mathbf{V}) = 8$, $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$ are uniquely determined from figure 4, and there is no contradiction between $C(\boldsymbol{x})$ and $C(\boldsymbol{y})$. If $E(D(\boldsymbol{x}) \cap \mathbf{V}) = 7$ and $C(\boldsymbol{y})$ is given, we can choose $C^i(\boldsymbol{x})$ without contradiction between $C^i(\boldsymbol{x})$ and $C(\boldsymbol{y})$. Because figure 4 shows all patterns of triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$, we can choose a suitable one among all four patterns. Similar reasoning holds in the case of $E(D(\boldsymbol{x}) \cap \mathbf{V}) = 6, 5$ and 4. Therefore, we can choose $C^i(\boldsymbol{x})$ and $C^j(\boldsymbol{y})$ without cotradiction between $C^i(\boldsymbol{x})$ and $C^j(\boldsymbol{y})$.

4.5 Proof of Theorem 1

If we use 6-neighborhood, we obtain $C(\boldsymbol{x})$ with respect to each $\boldsymbol{x}$ according to lemma 2. Furthermore, according to lemma 3 we can combine $C(\boldsymbol{x})$ and obtain a d-complex

$$\mathbf{C} = \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} C(\boldsymbol{x}). \tag{18}$$

If we use 26-neighborhood, we obtain $C^i(\boldsymbol{x})$ ($i = 1, 2, \ldots, m(\boldsymbol{x})$) with respect to each $\boldsymbol{x}$ according to lemma 1. Furthermore, according to lemma 4 we can choose $C^{i(\boldsymbol{x})}(\boldsymbol{x})$ and obtain a d-complex

$$\mathbf{C}^j = \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} C^{i(\boldsymbol{x})}(\boldsymbol{x}) \tag{19}$$

where $j = 1, 2, \ldots, l$.

4.6 Proof of Theorem 2

If we use 6-neighborhood, we can uniquely obtain $C(\boldsymbol{x})$ with respect to each $\boldsymbol{x}$ in $\mathbf{Z}^3$. If $\mathbf{B}(\boldsymbol{x})$ is the body of $C(\boldsymbol{x})$, it is clear that

$$\mathbf{B}(\boldsymbol{x}) = D(\boldsymbol{x}) \cap \mathbf{V}. \tag{20}$$

Therefore,

$$\begin{aligned} \mathbf{B} &= \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} \mathbf{B}(\boldsymbol{x}) \\ &= \bigcup_{\boldsymbol{x} \in \mathbf{Z}^3} D(\boldsymbol{x}) \cap \mathbf{V} \\ &= \mathbf{V}. \end{aligned} \tag{21}$$

If we use 26-neighborhood, we can obtain $C^{i(\boldsymbol{x})}(\boldsymbol{x})$ ($i(\boldsymbol{x}) = 1, 2, \ldots, m(\boldsymbol{x})$) with respect to each $\boldsymbol{x}$ in $\mathbf{Z}^3$. If $\mathbf{B}^{i(\boldsymbol{x})}(\boldsymbol{x})$ is the body of $C^{i(\boldsymbol{x})}(\boldsymbol{x})$, it is clear that

$$\mathbf{B}^{i(\boldsymbol{x})}(\boldsymbol{x}) = D(\boldsymbol{x}) \cap \mathbf{V}. \tag{22}$$

Then, we can regard $\mathbf{B}^{i(\boldsymbol{x})}(\boldsymbol{x})$ as $\mathbf{B}(\boldsymbol{x})$. It is clear that

$$\mathbf{B}(\boldsymbol{x}) = D(\boldsymbol{x}) \cap \mathbf{V}. \tag{23}$$

Therefore

$$\mathbf{B} = \mathbf{V}. \tag{24}$$

5 Extraction of n-Dimensional Parts

By algorithm 1, we can obtain a d-complex $\mathbf{C}$ from a point set $\mathbf{V}$. Assume that the dimension of $\mathbf{C}$ is three.

Since $\mathbf{C}$ is a set of d-simplexes, we divide $\mathbf{C}$ into sets of n-d-simplexes as

$$\mathbf{C} = S_0 \cup S_1 \cup S_2 \cup S_3 \tag{25}$$

where S_n is the set of all n-d-simplexes in $\mathbf{C}$. If $\mathbf{C}$ is not pure, $\mathbf{C}$ has some parts with dimension less than three. If P_n denotes n-dimensional parts of $\mathbf{C}$, we divide $\mathbf{C}$ in another way as

$$\mathbf{C} = P_0 \cup P_1 \cup P_2 \cup P_3. \tag{26}$$

P_0 is a set of isolated points. P_1 is a set of 1-d-simplexes and their faces which form needlelike regions. P_2 is a set of 2-d-simplexes and their faces which form thin wall-like regions. P_3 is a set of 3-d-simplexes and their faces which form objectlike regions. The exact formulas generating P_n are as follows:

$$P_3 = S_3 \cup (\bigcup_{[a]\in S_3} face([a])), \tag{27}$$

$$P_2 = S_2' \cup (\bigcup_{[a]\in S_2'} face([a])) \tag{28}$$

where

$$S_2' = S_2 \setminus \bigcup_{[a]\in S_3} face([a]), \tag{29}$$

$$P_1 = S_1' \cup (\bigcup_{[a]\in S_1'} face([a])) \tag{30}$$

where

$$S_1' = S_1 \setminus \bigcup_{[a]\in S_2} face([a]) \tag{31}$$

and

$$P_0 = S_0 \setminus \bigcup_{[a]\in S_1} face([a]). \tag{32}$$

The following theorem is derived.

Theorem 3 *P_n, the set of n-dimensional parts of a d-complex, is pure.*

(proof) P_n is clearly an n-dimensional d-complex because the maximum dimension of all d-simplexes in P_n is n. Furthermore, every d-simplex whose dimension is less than n in P_n is included in at least one n-d-simplex in P_n. Therefore, P_n is pure. *(Q.E.D)*

Thus, the extraction of n-dimensional parts from a d-complex is equivalent to the extraction of an n-dimensional pure d-complex from a d-complex.

If we consider 6-neighborhood, we obtain a unique d-complex $\mathbf{C}$. Thus, P_n is also unique, corresponding to a point set. If we consider 26-neighborhood, we choose a d-complex $\mathbf{C}$ among a finite number of $\mathbf{C}^j$. Therefore, if we choose $\mathbf{C}^j$ as a d-complex, an n-dimensional pure d-complex in $\mathbf{C}^j$ is denoted by P_n^j. P_n^j is sometimes different from P_n^k where $j \neq k$. However, we obtain the next theorem.

Theorem 4 *Assume that we use 26-neighborhood and we obtain two different d-complexes from a point set. If the two d-complexes are $\mathbf{C}^j$ and $\mathbf{C}^k$, we obtain P_n^j and P_n^k, respectively. B_n^j and B_n^k, the bodies of P_n^j and P_n^k, respectively, satisfy the formula*

$$B_n^j = B_n^k. \tag{33}$$

(proof) We can obtain S_n^j through the formula

$$S_n^j = \bigcup_{\boldsymbol{x}\in \mathbf{Z}^3} S_n^{j(\boldsymbol{x})}(\boldsymbol{x}) \tag{34}$$

where $S_n^{j(\boldsymbol{x})}(\boldsymbol{x})$ is the set of all n-d-simplexes in $C^{j(\boldsymbol{x})}(\boldsymbol{x})$. If $a \neq b$, $S_n^a(\boldsymbol{x})$ is not equal to $S_n^b(\boldsymbol{x})$. However, figure 4 shows that

$$B_n^a(\boldsymbol{x}) = B_n^b(\boldsymbol{x}) \tag{35}$$

where $B_n^a(\boldsymbol{x})$ and $B_n^b(\boldsymbol{x})$ are the bodies of $S_n^a(\boldsymbol{x})$ and $S_n^b(\boldsymbol{x})$, respectively. Let the body of $S_n^{j(\boldsymbol{x})}(\boldsymbol{x})$ be $B_n^{j(\boldsymbol{x})}(\boldsymbol{x})$, then, since

$$B_n^j = \bigcup_{\boldsymbol{x}\in \mathbf{Z}^3} B_n^{j(\boldsymbol{x})}(\boldsymbol{x}) \tag{36}$$

and

$$B_n^k = \bigcup_{\boldsymbol{x}\in \mathbf{Z}^3} B_n^{k(\boldsymbol{x})}(\boldsymbol{x}), \tag{37}$$

$$B_n^j = B_n^k. \tag{38}$$

(Q.E.D)

Furthermore, if we use connectivity of a d-complex, we can separate P_n into n-dimensional connected pure d-complexes.

6 Conclusions

In this paper, we showed that we can always construct a discrete complex from a given subset of a discrete space. From the viewpoint of classical combinatorial topology, the construction of a discrete complex means the existence of triangulation of a given subset. The number of the triangulation patterns of a subset is finite in a discrete space. If we use 6-neighborhood, the pattern of the triangulation is unique and we uniquely construct a discrete complex. Furthermore, we also proposed an algorithm for the construction of a discrete complex. Finally, using these discrete complexes, we extracted n-dimensional parts where $n = 0, 1, 2, 3$, respectively. We also showed that a set of points included in n-dimensional parts is uniquely determined from a given subset. n-dimensional parts of an object are generally regarded as irregular parts. These irregular parts are caused by the finite resolution of measured data. Therefore, if there are parts whose dimension is less than three, we set higher resolution locally in these parts. Then, we can change all n-dimensional irregular parts to 3-dimensional parts by using multiresolution. Moreover, we can regard n-dimensional parts of an object as the local geometric and topological properties of an object. Therefore, we can investigate the geometric and topological structure of measured data using discrete complexes. For example, if we attempt to find some parts whose shapes are needlelike, we should extract 1-dimensional parts from an object.

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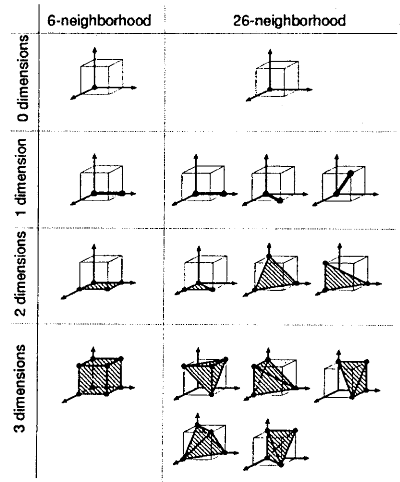

Fig.1: d-simplexes in $\mathbf{Z}^3$. Vertices of a cube are lattice points in $\mathbf{Z}^3$.

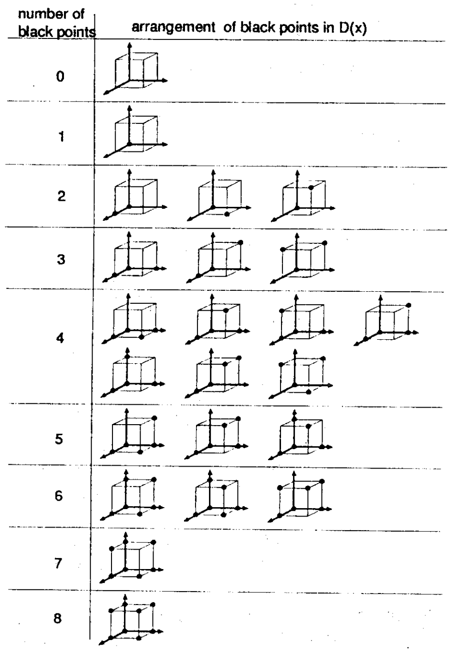

Fig.2: All patterns of arrangement of black points in a region $D(\boldsymbol{x})$. Black points indicate points which belong to a point set. A cube is a region $D(\boldsymbol{x})$ and each vertex of a cube corresponds to a lattice point. We take into account congruence transformations.

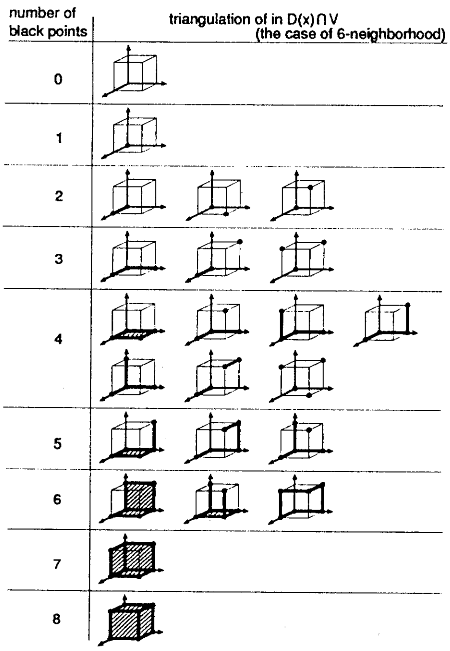

Fig.3: All patterns of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ corresponding to a pattern of the arrangement of black points in $D(\boldsymbol{x})$. In this figure, it is assumed that we use 6-neighborhood. The correspondence between patterns of triangulation and patterns of arrangement of black points is one-to-one.

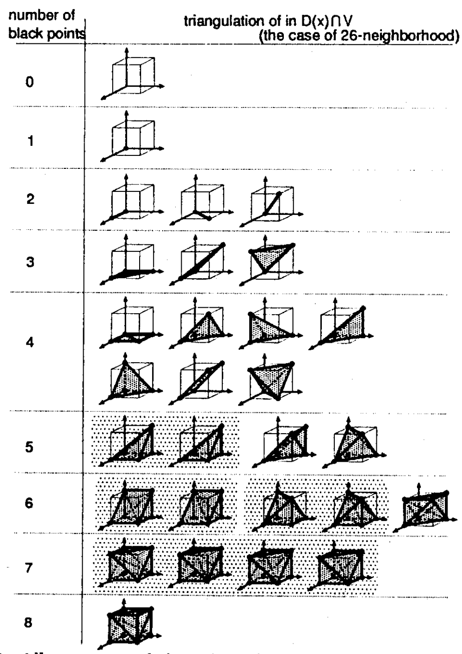

Fig.4: All patterns of the triangulation of $D(\boldsymbol{x}) \cap \mathbf{V}$ corresponding to a pattern of the arrangement of black points in $D(\boldsymbol{x})$. In this figure, it is assumed that we use 26-neighborhood. The correspondence between patterns of triangulation and patterns of arrangement of black points is not one-to-one. The patterns of the triangulation in a hatched area correspond to a pattern of the arrangement.

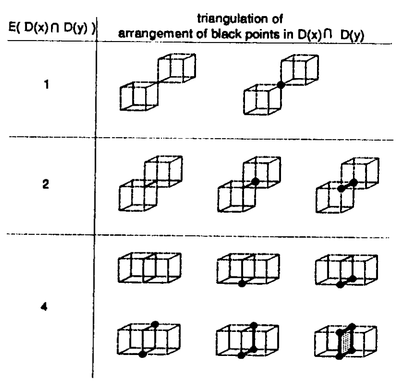

Fig.5: All patterns of the triangulation of $D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}$ where $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) \neq 0$. In this figure, it is assumed that we use 6-neighborhood. The correspondence between patterns of triangulation and patterns of arrangement of black points is one-to-one.

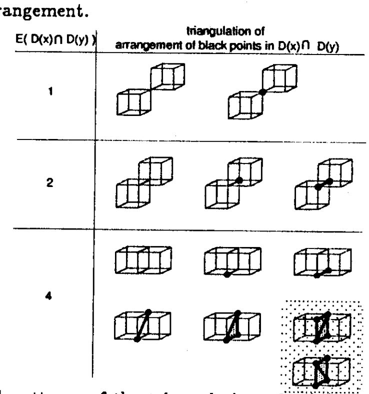

Fig.5: All patterns of the triangulation of $D(\boldsymbol{x}) \cap D(\boldsymbol{y}) \cap \mathbf{V}$ where $E(D(\boldsymbol{x}) \cap D(\boldsymbol{y})) \neq 0$. In this figure, it is assumed that we use 26-neighborhood. The correspondence between patterns of triangulation and patterns of arrangement of black points is not one-to-one. The patterns of the triangulation in a hatched area correspond to a pattern of the arrangement.