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ランダム計算による物体の対称軸の抽出

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あらまし 本論文では，ランダム標本化と投票を利用した物体の対称性を抽出する手法を提案する．この手法は完全な物体だけでなく，ノイズや誤差によって一部が欠けた物体の対称性を抽出することができる．まず，点の集合として与えられた入力物体の境界から，点をランダムに選びだし，整合判定を行なう．次に，対称性を表す特徴量を算出し，パラメータ空間に投票する．投票をある回数繰り返した後，パラメータ空間の投票値が周期関数であることを利用して，物体の対称性の抽出ができることを示す．

キーワード 対称性，ランダム標本化，投票，折り畳み数，ケプストラム解析

Symmetry Detection by Random Sampling and Voting Process

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Abstract We propose a randomized method for the detection of symmetry in polyhedra without assuming predetermination of the centroids of the objects. Using a voting procedure, which is the main concept of the Hough transform in image processing, we transform the geometric computation for symmetry detection based on graph theory, to the peak detection problem in an voting space in the context of Hough transform.

Key words: Symmetry, Random sampling, Voting process, Folding number, Cepstrum analysis

1 Introduction

In pattern recognition the symmetry of an object is an important feature because symmetry provides references for recognition and measurement of objects. The symmetry information enables to speed up the recognition process and also to reduce the space for storage of the object models. The symmetry properties of the objects, give valuable information for image understanding. To make use the symmetry property it is necessary to determine the symmetry axes or shape orientations. Moreover, many methods have been proposed to determine the object orientation, such as principal axes [1, 2], mirror-symmetry axes [3, 4], universal principal axes [5]. However, these methods are not suitable when the shape is rotationally symmetry.

In this paper, we propose a symmetry detection method based on random sampling and voting process. The voting process converts direct geometric and analytical computation of features from data to the peak detection problem in a voting space. The method proposed here is an extension of our randomized method for 3D motion detection which we proposed previously [16]. There are ambiguities in the solutions obtained from the 3D motion detection algorithm if the object has symmetry axes and occlusion is not considered. In this paper, we use these ambiguities for the determination of the symmetries of a polyhedron. Although in related works the origin of the coordinate system is assumed to be at the centroid of the object, our method does not require such geometric normalization.

An object is said to be rotationally symmetric, if the object, after being rotated around an axis for an appropriate angle, becomes identical to the original object. Let $\boldsymbol{V}$ be an n-fold rotationally symmetric object with $n \geq 3$, $\boldsymbol{V}$ will be identical to itself after being rotated through any multiple of $\frac{2\pi}{n}$. Lin [6] proposed a method for the determination of shape orientations by the fold-invariant introducing the concepts of fold invariant centroid (FIC) and the fold invariant weighted mean (FIRWM). In this method the rotational symmetry of a shape is defined as the direction of the unique half line starting from the centroid and through FIC and FIRWM.

The number of folds n of a given rotational symmetry shape can be determine by string matching technique [7]. Lin, Tsai and Chen [8] proposed a method to determine the number of folds based on a simple mathematical property. Recently, Lin [9] have proposed also a modification of his previous method in which the matching procedure is discarded. Additionally, we can find others approaches as the proposed for Yip, Lam and Leung [10] who use Hough transform method to determine the rotational symmetry of planar shapes.

Zabrodsky and Weinshall [11] used the symmetry-mirror of 2D projections for the reconstruction of 3D objects, using matching graph. They treated the global symmetry as continous feature, defining a symmetry distance (SD) of a shape as the minimum mean square distance required to move a point from the original position in order to obtain a symmetry one. Minovic, Ishikawa and Kato [12] presented an algorithm for identifying symmetry of a 3D object given by its octree, in which bilateral, axial, rotational and central symmetry can be identified. Jiang and Bunke [13, 14] proposed a method for the determination of rotational polyhedral objects by graph matching. Alt et. al. [15] presented a method for symmetry detection of an object by labeling. Although, there are many algorithms for symmetry detection, these algorithms generally are developed for 2D shape. Moreover, these kind of algorithms must perform matching process and determine the centroid of the shape.

2 Symmetry and Transformation

Symmetry means the congruence of an object under transformations. Here we assume Euclidean transformations. The presence of an axis of symmetry in an object may be considered as the existence of rotational or reflectional symmetry. In this paper, we only consider rotation symmetry and the number of axes of rotation symmetry. The order of rotation symmetry is called the folding number for planar figures. Since we deal with spatial objects, the folding number is the order of a rotation with respect to the invariant direction of a rotation.

The axis of the rotation $\boldsymbol{R}$ is the vector which satisfies the equation $\boldsymbol{Rk} = \boldsymbol{k}$. Therefore, the axis of the rotation $\boldsymbol{k}$ is the invariant direction for rotation matrix $\boldsymbol{R}$. Let $\boldsymbol{U}$ be a rotation matrix such that $\boldsymbol{U}^m = \boldsymbol{I}$, for an appropriate positive integer m such that $m \geq 2$. Setting $\boldsymbol{g}$ to be the centroid of $\mathbf{V}$, we define a set of vectors $\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{g}$, for $\boldsymbol{x} \in \mathbf{V}$. Setting

$$\boldsymbol{U}^k(\boldsymbol{V}) = \{\bar{\boldsymbol{y}} | \bar{\boldsymbol{y}} = \boldsymbol{U}^k \bar{\boldsymbol{x}}, \quad \bar{\boldsymbol{x}} \in \mathbf{V}\}, \tag{1}$$

if $\mathbf{V} = \boldsymbol{U}^k(\mathbf{V})$, then $\boldsymbol{V}$ has a symmetry axis with respect to $\boldsymbol{g}$. Then, $\mathbf{V}$ is n-rotation symmetry, and we call n the folding number of an object $\mathbf{V}$ with respect to the axis of the rotation $\boldsymbol{k}$.

These geometric properties conclude that, if we find the axis of the rotation we can determinr the folding number of an object with respect to this axis.

For a vector $\boldsymbol{a} \in \mathbf{R}^3$ and a 3×3 matrix $\boldsymbol{A}$, setting $\boldsymbol{a}_i$ to be the i-th row vector of $\boldsymbol{A}$, we define the matrix vector product as

$$\boldsymbol{a} \times \boldsymbol{A} = [\boldsymbol{a} \times \boldsymbol{a}_1, \boldsymbol{a} \times \boldsymbol{a}_2, \boldsymbol{a} \times \boldsymbol{a}_3]. \quad (2)$$

For a rotation matrix $\boldsymbol{R}$, setting

$$\begin{aligned} tr\boldsymbol{R} &= 1 + 2\cos\theta \quad (3) \\ \boldsymbol{R} - \boldsymbol{R}^\top &= 2\sin\theta(\boldsymbol{k} \otimes \boldsymbol{I}) \quad (4) \end{aligned}$$

the matrix $\boldsymbol{R}$ defines the rotation angle θ around the vector $\boldsymbol{k}$.

If we set

$$\boldsymbol{R} = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}, \quad (5)$$

we obtain the relations

$$\cos\theta = \frac{(n_x + o_y + a_z) - 1}{2} \quad (6)$$

and

$$\begin{aligned} k_x &= \operatorname{sgn}(o_z - a_y)\sqrt{\frac{n_x - \cos\theta}{1 - \cos\theta}}, \\ k_y &= \operatorname{sgn}(a_x - n_z)\sqrt{\frac{o_y - \cos\theta}{1 - \cos\theta}}, \quad (7) \\ k_z &= \operatorname{sgn}(n_y - o_x)\sqrt{\frac{a_z - \cos\theta}{1 - \cos\theta}}, \end{aligned}$$

for $\boldsymbol{k} = (k_x, k_y, k_z)^\top$. Since we detect the rotation angle θ which is not always zero and π, we do not need to consider the case, $\cos\theta = 1$.

3 Motion Analysis by Sampling and Voting

Setting $\{\boldsymbol{x}_\alpha\}_{\alpha=1}^m$ and $\{\boldsymbol{y}_\beta\}_{\beta=1}^n$ to be points on an object in 3-dimensional Euclidean space $\mathbf{R}^3$, which are observed at time t_1 and t_2, respectively, such that $t_1 < t_2$, we assume that for arbitrary pairs of α and β, $\boldsymbol{x}_\beta$ and $\boldsymbol{y}_\beta$ are connected by a Euclidean motion, that is

$$\boldsymbol{y}_\beta = \boldsymbol{R}\boldsymbol{x}_\alpha + \boldsymbol{t}, \quad (8)$$

where $\boldsymbol{R}$ and $\boldsymbol{t}$ are a rotation matrix and a translation vector, respectively.

In our previous paper [16], we derived an algorithm for 3D motion estimation without predetermination of point correspondences. We assumed that the edges and the vertices of a polyhedron are determined from two image frames. using an appropriate method before we apply the motion analysis algorithm. If we do not know point correspondences between frames, the motion parameters $\boldsymbol{R}$ and $\boldsymbol{t}$ are obtained as the solution which minimizes the criterion

$$E = \min_{\alpha,\beta,\boldsymbol{R},\boldsymbol{t}} |\boldsymbol{y}_\beta - (\boldsymbol{R}\boldsymbol{x}_\alpha + \boldsymbol{t})|, \quad (9)$$

setting

$$\boldsymbol{y}_{\sigma(i)} = \boldsymbol{R}\boldsymbol{x}_i + \boldsymbol{t}, \quad (10)$$

where $\sigma(i)$ is a permutation over $1 \le i \le n$.

The voting procedure is the main idea for the Hough transform which detects lines and conics from noisy digitized samples. The estimation of parameters for lines from sampled data is an inverse problem. Therefore, the Hough transform inferences correct parameters collecting many evidences using the voting procedure. Setting $\boldsymbol{x}$ to be the valuable in n-dimensional Euclidean space, the Hough transform is a method for the estimation of parameters $\{\boldsymbol{a}_i\}_{i=1}^n$ of a collection of equations,

$$f_i(\boldsymbol{a}_i, \boldsymbol{x}) = 0, \ i = 1, 2, \cdots, k \quad (11)$$

from finite many samples $\{\boldsymbol{x}_j\}_{j=1}^m$ such that $m \gg k \ge 1$. using the voting procedure. An equation $f_i(\boldsymbol{x}, \boldsymbol{a}_i) = 0$ is called a model for the parameter estimation. The most typical and traditional models for the Hough transform are planar line and conic if the dimension of space is two. The Euclidean motion equation such that $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$ is expressed in the form,

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} & -\boldsymbol{I} & \mathbf{o} \\ \mathbf{o}^\top & 1 & \mathbf{o}^\top & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \\ \boldsymbol{y} \\ 1 \end{pmatrix} = 0, \quad (12)$$

where, $\boldsymbol{I}$ is the identity matrix and $\mathbf{o}$ is the null vector. This description implies that motion analysis from a collection of noisy corresponding points is a model fitting problem [?] for the model,

$$F(\boldsymbol{A}, \boldsymbol{\nu}) = 0, \quad (13)$$

such that

$$F(\boldsymbol{A}, \boldsymbol{z}) = \boldsymbol{A}\boldsymbol{\nu}, \quad (14)$$

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} & -\boldsymbol{I} & \mathbf{o} \\ \mathbf{o}^\top & 1 & \mathbf{o}^\top & -1 \end{pmatrix}, \\ \boldsymbol{\nu} &= (\boldsymbol{x}^\top, 1, \boldsymbol{y}^\top, 1)^\top \end{aligned}$$

Therefore, it is obvious that the voting procedure achieves the estimation of the motion parameters $\boldsymbol{R}$ and $\boldsymbol{t}$ of a Euclidean motion from finite many samples.

Assuming that $t = 0$, motion analysis algorithms detect rotation of an object. If an object is rotation symmetry, the results of a rotation derives the set of points. Therefore, if we apply motion analysis algorithm to an object which has rotation symmetry, it is possible paremeters which define the symmetry as the motion paremeters of a rotation.

4 Symmetry Detection by Motion Analysis

Rotation symmetry and the folding number of an object define point correspondences with respect to the rotation axes. Therefore, if we detect point corespondences, we can determine symmetry and the folding number with respecto to an axis of rotation of an object. Since the randome sampling and voting method for the motion analysis detect both motion parameters and point correspondences concurently, we apply this method for the detection of symmetry of an object.

Using random sampling, a quadruplet of points is selected from each frame, $\boldsymbol{x}_i$, $\boldsymbol{x}_j$, $\boldsymbol{x}_s$, $\boldsymbol{x}_u$ and $\boldsymbol{y}_i$, $\boldsymbol{y}_j$, $\boldsymbol{y}_s$, $\boldsymbol{y}_u$, respectively. Three of these points are used to construct an orthogonal frame for each image frame. Furthermore, these orthogonal frames are used as references to compute the motion parameters. These orthogonal frames are estimated as follows.

$$\boldsymbol{\xi}_{iu} = \frac{\boldsymbol{x}_{iu}}{|\boldsymbol{x}_{iu}|} \quad \boldsymbol{\xi}_{ju} = \frac{\boldsymbol{x}_{ju} - (\boldsymbol{x}_{ju}^\top \boldsymbol{\xi}_{iu})\boldsymbol{\xi}_{iu}}{|\boldsymbol{x}_{ju} - (\boldsymbol{x}_{ju}^\top \boldsymbol{\xi}_{iu})\boldsymbol{\xi}_{iu}|} \quad \boldsymbol{\xi}_{su} = \frac{\boldsymbol{x}_{iu} \times \boldsymbol{x}_{ju}}{|\boldsymbol{x}_{iu} \times \boldsymbol{x}_{ju}|}, \tag{15}$$

$$\boldsymbol{\eta}_{iu} = \frac{\boldsymbol{y}_{iu}}{|\boldsymbol{y}_{iu}|} \quad \boldsymbol{\eta}_{ju} = \frac{\boldsymbol{y}_{ju} - (\boldsymbol{y}_{ju}^\top \boldsymbol{\eta}_{iu})\boldsymbol{\eta}_{iu}}{|\boldsymbol{y}_{ju} - (\boldsymbol{y}_{ju}^\top \boldsymbol{\eta}_{iu})\boldsymbol{\eta}_{iu}|} \quad \boldsymbol{\eta}_{su} = \frac{\boldsymbol{y}_{iu} \times \boldsymbol{y}_{ju}}{|\boldsymbol{y}_{iu} \times \boldsymbol{y}_{ju}|}. \tag{16}$$

Thus, setting

$$\begin{aligned} \boldsymbol{x}_{iu} &= \alpha_{iu}^1 \boldsymbol{\xi}_{iu}, \\ \boldsymbol{x}_{ju} &= \alpha_{iu}^2 \boldsymbol{\xi}_{iu} + \alpha_{ju}^2 \boldsymbol{\xi}_{ju}, \\ \boldsymbol{x}_{su} &= \alpha_{iu}^3 \boldsymbol{\xi}_{iu} + \alpha_{ju}^3 \boldsymbol{\xi}_{ju} + \alpha_{su}^3 \boldsymbol{\xi}_{su}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \boldsymbol{y}_{iu} &= \beta_{iu}^1 \boldsymbol{\eta}_{iu}, \\ \boldsymbol{y}_{ju} &= \beta_{iu}^2 \boldsymbol{\eta}_{iu} + \beta_{ju}^2 \boldsymbol{\eta}_{ju}, \\ \boldsymbol{y}_{su} &= \beta_{iu}^3 \boldsymbol{\eta}_{iu} + \beta_{ju}^3 \boldsymbol{\eta}_{ju} + \beta_{ju}^3 \boldsymbol{\xi}_{ju}, \end{aligned} \tag{18}$$

for a pair of real-valued hexaplet

$$\boldsymbol{\alpha} = (\alpha_{iu}^1, \alpha_{iu}^2, \alpha_{ju}^2, \alpha_{iu}^3, \alpha_{ju}^3, \alpha_{su}^3), \tag{19}$$

and

$$\boldsymbol{\beta} = (\beta_{iu}^1, \beta_{iu}^2, \beta_{ju}^2, \beta_{iu}^3, \beta_{ju}^3, \beta_{su}^3) \tag{20}$$

which are determined from the quadruplets of vectors, the following theorem holds.

Theorem 1 *If $\boldsymbol{y}_h = \boldsymbol{R}\boldsymbol{x}_h$, $\boldsymbol{\alpha} = \boldsymbol{\beta}$.*

If a pair of hexplets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfies the relation $\boldsymbol{\alpha} = \boldsymbol{\beta}$, a pair of quadrplets $\{\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_s, \boldsymbol{x}_u\}$ and $\{\boldsymbol{y}_i, \boldsymbol{y}_j, \boldsymbol{y}_s, \boldsymbol{y}_u\}$ forms cogruent tetrahedrons each other. If $\mathbf{V}$ and $\mathbf{V}'$ are conguruent, that is, there exist rotation matrices which hold the relations $\boldsymbol{R}(\mathbf{V}) = \mathbf{V}'$ for $k = 1, 2, cdots, n$, it is possible to find a pari of quadrplets which forms a conguruent tetrahedrons such that $\{\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_s, \boldsymbol{x}_u\} \in \mathbf{V}$ and $\{\boldsymbol{y}_i, \boldsymbol{y}_j, \boldsymbol{y}_s, \boldsymbol{y}_u\} \in \mathbf{V}'$. However, even if $\alpha_{\gamma u} = \beta_{\gamma u}$, we cannot conclude that $\boldsymbol{y}_h = \boldsymbol{R}\boldsymbol{x}_h$, since $\boldsymbol{\alpha} = \boldsymbol{\beta}$ is not a sufficient condition for Euclidean transformation [16]. Therefore, using the idea of voting, we can define an inverse transformation from $\alpha_{\gamma u} = \beta_{\gamma u}$ to $\boldsymbol{y}_h = \boldsymbol{R}\boldsymbol{x}_h$.

Algorithm for the computation of the rotation

1 Randomly select quadruplets of points $\{\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_s, \boldsymbol{x}_u\} \in \mathbf{V}$ and $\{\boldsymbol{y}_i, \boldsymbol{y}_j, \boldsymbol{y}_s, \boldsymbol{y}_u\} \in \mathbf{V}'$, where $\mathbf{V}$ and $\mathbf{V}'$ are the sets of points on edges and vertices.

2 Compute the vectors $\{\boldsymbol{x}_{\gamma u}, \boldsymbol{y}_{\gamma u}\}$ where $\gamma = i, j, s$.

3 Compute the bases and scalars, $\{\boldsymbol{\xi}_{\gamma u}, \alpha_{\gamma u}^p\}$ and $\{\boldsymbol{\eta}_{\gamma u}, \beta_{\gamma u}^p\}$.

4 If $|\boldsymbol{\alpha} - \boldsymbol{\beta}| < \varepsilon$ then set $\boldsymbol{R} = [\boldsymbol{\eta}_{iu}\, \boldsymbol{\eta}_{ju}\, \boldsymbol{\eta}_{su}]\, [\boldsymbol{\xi}_{iu}\, \boldsymbol{\xi}_{ju}\, \boldsymbol{\xi}_{su}]^\top$ where $\boldsymbol{\xi}_{tu}$ and $\boldsymbol{\eta}_{tu}$ are obtained using an orthogonalization process.

5 If $\det|\boldsymbol{R}| \approx 1$, then increment the accumulation space of $\boldsymbol{R}$ by one.

6 If a threshold in the accumulation space $(\boldsymbol{R})$ is reached, then stop, otherwise go to step 1.

Assuming that the translation vector of the motion is zero vector, that is $\boldsymbol{t} = \mathbf{o}$, our algorithm detects all $\boldsymbol{U}^k\boldsymbol{R}$ for $k = 1, 2, \cdots, n$, where $\boldsymbol{R}$ is the true rotation matrix. However, all estimated matrices have the same translation vector. Using this ambiguity we detect symmetry axes and the folding numbers of spatial objects. After a sufficient number of iterations this algorithm detects all rotation matrices $\{\boldsymbol{U}^k\}_{k=1}^m$ such that $\boldsymbol{U}^n(\mathbf{V}) = \mathbf{V}$.

5 Detection of Symmetry

5.1 Detection of Symmetry Axis

Since a pair of tetrahedrons determines a rotation matrix $\boldsymbol{R}$, using eqs. (3) and (4), we compute the invariant axis $\boldsymbol{k}$ of a rotation and the rotation angles θ with respect to this axis. We set these vector and angles as a pair $\boldsymbol{a} = \langle \hat{\boldsymbol{k}}, \theta \rangle$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. This pair implies that our accumulator space is mathematically

$A = S^2 \times S$, where S^2 and S are the unit spheres in $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. Furthermore, we express $\boldsymbol{a}$ in the manner of the binary tree on A, using the dictionary order of quadruplets. For $\boldsymbol{a} = \langle(\alpha, \beta, \gamma)^\top, \theta\rangle$ and $\boldsymbol{a}' = \langle(\alpha', \beta', \gamma')^\top, \theta'\rangle$ the dictionary order is defined as

if $\alpha > \alpha'$ **then** $\boldsymbol{a} \succ \boldsymbol{a}'$
else if $\beta > \beta'$ **then** $\boldsymbol{a} \succ \boldsymbol{a}'$
else if $\gamma > \gamma'$ **then** $\boldsymbol{a} \succ \boldsymbol{a}'$
else if $\theta > \theta'$ **then** $\boldsymbol{a} \succ \boldsymbol{a}'$.

After an appropriate number of iterations, we define the list of the angles for each $\hat{\boldsymbol{k}}$ such as

$$\hat{\boldsymbol{k}}_i = \langle\theta_{i1}, \theta_{i2}, \cdots, \theta_{ik}\rangle, \ \ \theta_{in} < \theta_{i\,n+1} \tag{21}$$

Setting $scor(\theta, \hat{\boldsymbol{k}}_i)$ to be the number of votes to $\langle \boldsymbol{k}_i, \theta$ on the accumulator space, for each i

$$k_i(\theta) = scor(\theta, \hat{\boldsymbol{k}}_i) \tag{22}$$

defines a discrete function which is periodic with respect to 2π and $2\pi/n$ if the folding number with respect to the rotation axis $\hat{\boldsymbol{k}}_i$is n. These relations derive the following algorithm for the detection of the folding number and the rotation axes.

Algorithm for the computation of the symmetry

1 For each $\hat{\boldsymbol{k}}_i$ on the accumulator space such that $scor(\boldsymbol{a})$ is larger than 1, construct the function $k_i(\theta)$.

2 Detect the pitch of $k_i(\theta)$ as $2\pi/n$.

3 Return $\hat{\boldsymbol{k}}_i$ and n as the axis of the rotation and the holding number with respect to this axis, respectively.

5.2 Detection of the Folding Number

If the periodic function $f(x)$ such that $f(x) = f(x+2\pi)$ satisfies the property that $f(x) = f(x + 2\pi/k)$ for an integer k such that $k \geq 2$, we derive a method to estimate k from samples of $f(x)$. Since the function $f(x)$ has two periods 2π and $2\pi/k$, we obtain the following two Fourier expansions,

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} a_m e^{im\frac{2\pi}{a}x} \\ &= \sum_{m=-\infty}^{\infty} a_m e^{ikmx} \end{aligned} \tag{23}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \tag{24}$$

Since

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx, \tag{25}$$

setting

$$\delta(n, km) = \frac{\sin \pi(n - km)}{\pi(n - km)}, \tag{26}$$

we obtain the ralations

$$\alpha_n = \delta(n, km) a_m. \tag{27}$$

Furthermore, considering (26), we have the relation,

$$\alpha_n = \begin{cases} a_m, & n = km \\ 0, & n \neq km. \end{cases} \tag{28}$$

If we assume that we measure $g(x)$ which is the results of the covolution between a degrading system $h(x)$ and the true signal $f(x)$,

$$g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - y) h(y) dy, \tag{29}$$

setting

$$g(x) = \sum_{n=-\infty}^{\infty} b_m e^{inx} \tag{30}$$

$$h(x) = \sum_{n=-\infty}^{\infty} c_m e^{inx}, \tag{31}$$

we have the relation,

$$b_{km} = a_{km} c_{km} \tag{32}$$

for the Fourier coefficents. For the power spectrum of three functions, we have the relation.

$$\ln | b_{km} |^2 = \ln | a_{km} |^2 + \ln | c_{km} |^2. \tag{33}$$

Therefore, if $| c_{km} | < 1$, then we have the inequality $\ln | c_{km} |^2 < 0$. These relations imply the relation

$$\ln | b_{km} |^2 = \begin{cases} > 0, & n = km \\ \leq 0, & n \neq km. \end{cases} \tag{34}$$

These properties of the periodic function derive the following algorithms for the estimation of the folding number with respect to the rotation axis since it is possible to compute the Fourier coefficient using the discrete Fourier transform.

Algorithm for the detection of the folding number

1 Compute the DFT(the Discrete Fourier Transform) of $k_i(\theta)$ using the FFT(the Fast Fourier Transform).

2 Compute power spectrum of $k_i(\theta)$ from the result of Step 1, and set it $K_i(n)$

3 Compute the logarithm of the power spectrum of $k_i(\theta)$ from the result of Step 2.

4 Detect the positive peak of $K_i(n)$ for the smallest n and set it n_i^*

5 Adopt n_i^* as the folding number with respect to the rotation axis $\hat{\boldsymbol{k}}_i$.

6 Numerical Examples

Assuming that there exist k axes of rotation, we set

$$\boldsymbol{R}_i \boldsymbol{k}_i = \boldsymbol{k}_i,\ i = 1, 2, \cdots, k. \tag{35}$$

Furthermore, if the folding number of the object with respect to the axis $\boldsymbol{k}_i$ is $l(i)$, then each matrix $\boldsymbol{R}_i$ satisfies the equality

$$\boldsymbol{R}_i^{l(i)} = \boldsymbol{I}. \tag{36}$$

Setting the total number of sample-points of object to be m, we assume that m satisfies the equality $m = k \times l(i) timesn(i)$. This relation means that for each axis of rotation, there exist $l(i) \times n(i)$ sample points. Furthermore, we assume that $l(i) \times n(i)$ sample points consist from $l(i)$ collections of $n(i)$ points and assume that each collection of sample points has no common point each other. Moreover, we assume that the average of $\{l(i)\}_i^k$ is l and $m = k \times l \times n$.

Using these notations, the total number of point selection for the pair of quadruplets is $M =_m C_4 \times_m C_4$. Furthermore, the number of the selections of an axis is $_kC_1$. Moreover, for each axis, the number of selection of points which define a pair of congruent triangle with respect to an axis is

$$K =_l C_{3n}C_1 +_l C_2({}_nC_2 +_n C_1) +_l C_{1n}C_3. \tag{37}$$

These numbers conclude that the possibility which determines a true rotation matrix is

$$P = e\frac{K}{M}, \tag{38}$$

where e is the probability that a quadruplet forms a tetrahedron. Therefore, setting N and p to be the total number of iterations and the height of peaks in the accumulator space, we have the equality

$$PN \geq p. \tag{39}$$

This equation concludes that $N \cong O(m^8)$ if $p \cong m$. For the numerical examples, we set $m = a \times 10$, for $2 \leq a < 10$. This condition leads that $N \cong 10^8$.

Figures show results of symmetry analysis for a tetrahedron, a cube and a quint-pipedon. Our algorithm detects all axes of rotation for these objects. In figures, small squares express sample points on edges and vertices. and the axes of rotation are expressed as lines. These lines show the axes of rotation for the folding numbers which are larger than 2.

7 Conclusions

In this work, we developed a randomized algorithm for detection of rotational symmetry in polyhedra without assuming the predetermination of the centroids of the objects. Our algorithm is simple because we converted the matching problem for detection of symmetry to the peak detection in a voting space. This result showed that the voting process is a suitable approach to simplify matching problems. In our numerical examples, we adopted sampl points on the edges and the vertices as feature points. Our algorithm detects all symmetry axes of a cube.

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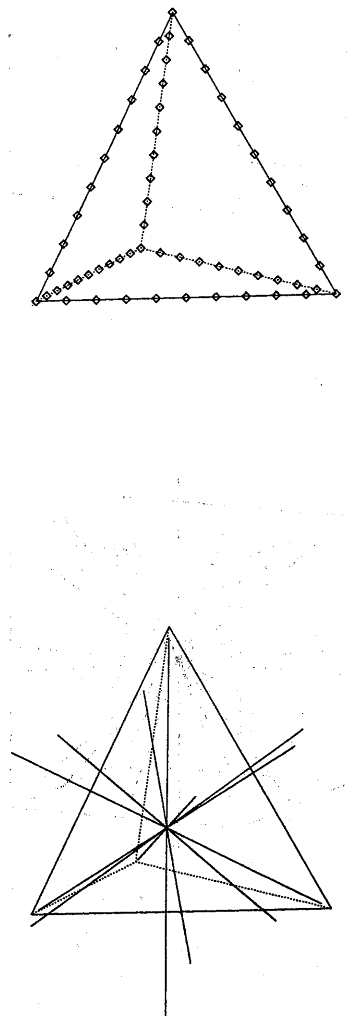

Figure 1: Symmetry axes of a tetrahedron.

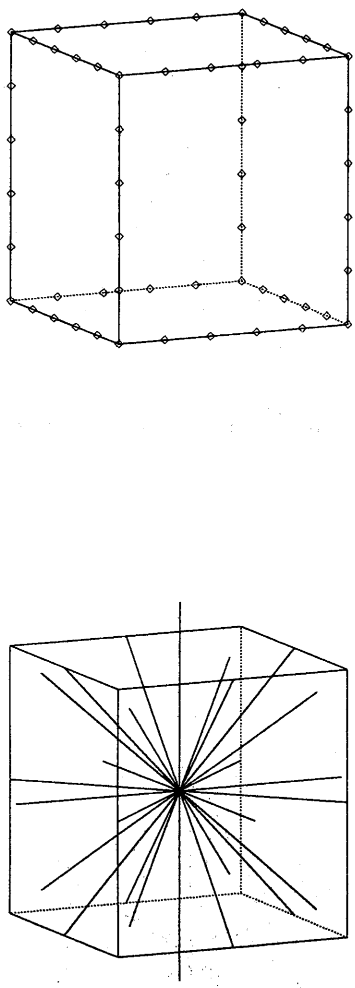

Figure 2: Symmetry axes of a cube.

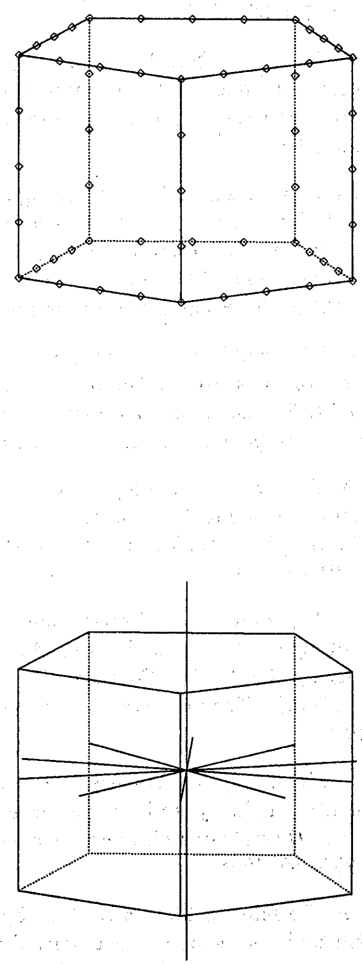

Figure 3: Symmetry axes of a quintpipdon.