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情報の包摂関係の余代数モデル

向井 国昭

慶応義塾大学環境情報学部

オートマトン, 項, 素性構造, 無限木, グラフはそれぞれ適当な関手に対する終余代数をなすことは良く知られている. また, P.Aczel らにより, set-based な関手には——意味領域としての——終余代数が存在することが証明されている (1989 年). 本稿では, これらの結果を使い, まず, (1) 制約, bisimulation, 包摂の概念を関手に相対化し, (2) 制約の解の概念を定義し, (3) 単一化定理を, 「解の存在性が与えられた bisimulation と包摂関係への拡張可能性と同値であること」として定式化する. これらの定式化のもとに, set-based かつ weak pullback を保存する関手について単一化定理が成り立つことを証明する. この定理を応用して, 素性構造上の (外延的) 包摂制約問題が決定可能であることを示す.

本研究は, 情報構造の論理を理解し, それを圏論プログラミングと結びつけるための 基礎的考察の一環である.

A Coalgebra Model for Information Subsumption

Kuniaki Mukai

Faculty of Environmental Information, Keio University
5332 Endo, Fujisawa, Kanagawa 108 JAPAN
E-mail: mukai@sfc.keio.ac.jp

Extensional subsumption relation on feature algebras is generalized for coalgebra for the class functors. For any set-based functor that preserves weak pullbacks, a subsumption constraint has a solution in the final coalgebra for the functor if and only if it has a solution in some coalgebra for the functor. This result is a generalization of the unification lemma formalized and proved by J. Barwise, which assumes the anti-foundation axiom but treats no coalgebra for a functor other than the power class functor.

With some reasonable assumptions, the constraint problem is decidable if the final coalgebra has an additonal binary operation which is associative, commutative, and idempotent with a unit. As a feature algebra is a coalgebra for a special kind of class functors, this result is a generalization of a decidability result on external feature subsumption problems obtained by J. Dörre.

This present work is based on the final coalgebra theorem proved by P.Aczel and N.Mendler.

1 Introduction

Aczel and Mendler [2] proves a final colagebra theorem that for every set-based functor, there is a final coalgebra for the functor. As a simple example of applications of the theorem, it follows that there is a unique solution for the following system of equations on "sets":

$$\begin{aligned} x &= \{y\} \\ y &= \{x\} \end{aligned}$$

Barwise [4] defines a notion of subsumption relation $\sqsubseteq$ on sets and formalizes unification in the universe of hypersets assuming the anti-foundation axiom [1], and proves a theorem called a unification lemma, which will be quoted in the appendix of this paper. As a simple example of applications of the unification lemma, we can verify existence of solutions for the following constraints:

$$\begin{aligned} x &\sqsubseteq \{x, y\} \\ y &\sqsubseteq \{x\} \end{aligned}$$

We notice that the unification lemma is a theorem on the final coalgebra for *pow* (power class functor).

Mukai [9] proposes a constraint logic programming language over hypersets based on Aczel's anti-foundation axiom (AFA). On the other hand, this work is based on the final coalgebra theorem. As the final coalgebra theorem is proved assuming neither AFA nor the foundation axiom, this present work is an extension of the previous work [9], and the unification in final coalgebras proposed in the present paper may be built-in also to the constraint logic programming.

Dörre [5] solved an open problem on feature logic with weak subsumption constraints applying well-known method to transform non-deterministic automata into the deterministic one. We also will solve this problem using our version of unification lemma.

Hagino[7] proposes a categorical programming language. The language treats not only initial algebras but also final coalgebras called DAI-algebras. As a future work, it will be interesting to implement the final coalgebras for set-based functors in the language.

The main source of technical ideas in this paper is Aczel and Mendler [2]. For instance, our generalization of subsumption relation is defined using commutative diagrams in a similar way to Aczel and Mendler [2] used for defining bisimulations.

We use terminologies from basic category theory, Barr and Wells [3] and Mac Lane [8], for example. Eilenberg [6] includes category theoretic treatments of automata.

The organization of the paper is as follows. The next section gives preliminaries. In Section 3 the unification in final coalgebras are formalized and the unification lemma for coalgebra version will be formalized and proved.

In Secion 4, the new unification lemma is applied to the decision problem on generalized subsumption constraints.

A full version of the paper will appear including a generalization Colmerauer's infinite tree unification with disequations.

2 Preliminaries

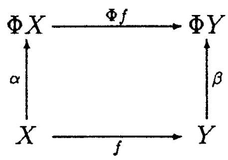
We write fg for the function composition of f and g, i.e., $fgx = f(g(x))$, provided $x \in dom\, g$.

V denotes the class of sets in ZFC^-. Our work space is the category $\mathcal{C}$ of classes of sets: $\mathcal{C} = \{C \mid C \subseteq V\}$.

A functor Φ is *set-based* if for each $a \in \Phi X$ there is some set $x \in V$ and $d \in \Phi x$ such that $x \subseteq X$ and $a = \Phi\iota_{x,X} d$, where $\iota_{x,X}$ is the inclusion map from x into X.

An ordered pair (X, α) of a class X and a function $\alpha: X \to \Phi X$ is called a *coalgebra* for Φ.

Let (X, α) and (Y, β) be coalgebras for Φ. If the following square commutes for a function $f: X \to Y$ then f is called a *homomorphism* from (X, α) into (Y, β):

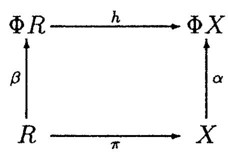


A coalgebra (Y, β) for Φ is called a *final coalgebra* for Φ if for any coalgebra (X, α) for Φ, there is a unique homomorphism from (X, α) into (Y, β).

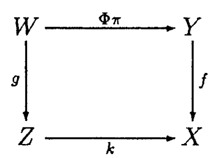
Aczel and Mendler proved the final coalgebra theorem:

Theorem 1 (Final Coalgebra Theorem [1, 2]) *Every set-based functor has a final coalgebra.*

Let π_i be the projection onto the i-th componets. A binary relatin R on X is called a *bisimulation* for Φ if there is a map $\beta: R \to \Phi R$ such that the following square commutes for each $\pi \in \{\pi_1, \pi_2\}$:



A commutative square such as below is called a *weak pullback* if for any $y \in Y$ and $z \in Z$ such that $fy = kz$, there is some $w \in W$ such that $y = hw$ and $z = gw$.



3 Unification in Final Coalgebras

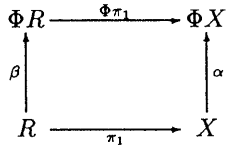
We work informally in the universe V of ZFC$^-$ (without the foundation axiom) with sufficiently many urelements.

Let $x, y \in \Phi X$. x is called a *subobject* of y if for any inclusion map $\iota: Y \to X$ and any $a \in \Phi Y$ such that $y = \Phi\iota a$, there is some $b \in \Phi Y$ such that $x = \Phi\iota b$. We write $x \trianglelefteq y$ when x is a subobject of y.

For example, let $\Phi = pow$. It is easy to see that $x \trianglelefteq y$ if and olnly if $x \subseteq y$.

A binary relation R on X induces a binary relation R' on ΦX so that $aR'b$ if and only if $\Phi\pi_1 x = a$ and $\Phi\pi_2 x \trianglelefteq b$ for some $x \in \Phi X$.

A binary relation R on X is called a *subsumption* if there is a map $\beta: R \to \Phi R$ such that the square



commutes and

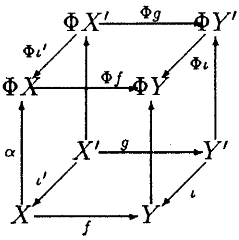
$$\Phi\pi_2\beta d \trianglelefteq \alpha\pi_2 d$$

for any $d \in R$.

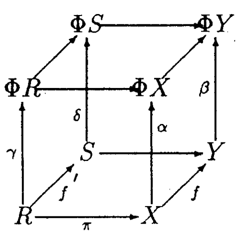
We show that if the functor Φ preserves weak pullbacks then the subobject relation is preserved under any morphism. Assume that Φ preserves weak pullbacks. Let f be a homomorphism between coalgebras (X, α) and (Y, β). Let $x, y \in \Phi X$ and $a, b \in \Phi Y$ so that $a = \Phi fx$, $b = \Phi fy$, and $x \trianglelefteq y$.

Suppose $\iota: Y' \to Y$ is an inclusion map. Let $X' = f^{-1}Y'$ and g be the restriction of f to X'. Then the lower square is a weak pullback.

As we have assumed that Φ preserves weak pullbacks, also the upper square is a weak pullback. Let $u \in \Phi Y'$ such that $b = \Phi\iota u$. So there is some $d \in \Phi X'$ such that $\Phi\iota' d = y$ and $\Phi g d = u$. It follows from $x \trianglelefteq y$ that there is some $e \in \Phi X'$ such that $\Phi\iota' e = x$. Let $v = \Phi g e$. Then $\Phi\iota v = \Phi\iota\Phi e = a$. Hence by definition, a is a subobject of b. Therefore every homomorphism between coalgebras preserves the subobject relation $\trianglelefteq$.



We see that the image of a bisimulation mapped by a homorphism is also a bisimulation. Let π_1 and π_2 be projections to the first and second component, respectively. Let $f: X \to Y$ and let $S = \{(fx, fy) \mid xRy\}$ and define $\delta: S \to \Phi S$ so that $\delta f' = \Phi f'\gamma$. As f' is surjective, δ must exist. For each $\pi \in \{\pi_1, \pi_2\}$, by simple diagram chase reasoning, we have $\beta\pi f' = \Phi\pi\delta f'$. As f' is surjecteve, we have $\beta\pi = \Phi\pi\delta$.



Assuming that that Φ preserves weak pullbacks, we show that the image of a subsumption is also a subsumption. Let δ, S, f' be as the above. As f' is surjective, δ must exist. For $\pi = \pi_1$, the square in the back of the cube commutes as shown above. For $\pi = \pi_2$, by diagram chasing, we have $\beta\pi_2 f' = \Phi f \Phi\pi\gamma$ and $\beta\pi f' = \Phi f \alpha\pi$. Let $x = \Phi\pi\delta s$ and $y = \beta\pi s$. As f' is surjective, there is some $r \in R$ such that $s = f'r$. So it follows from the above functional equations that $x = \Phi f u$ and $y = \Phi f v$ where $u = \Phi\pi\gamma r$ and $v = \alpha\pi r$. As R is a subsumption on X, u is a subobject of v by definition. As every homomorphism preserves the subobject relation, x is a subobject of y.

Let A be a final coalgebra for set-based Φ. Aczel and Mendler [2] proves that there is the maximum congruence relation on A. Also they proves that every congruence relation is a bisimulation, provided Φ preserves weak pullbacks.

There is also the maximum subsumption in the final coalgebra A. To see this let S be the union of small subsumptions. For any $d \in S$, there is a small subsumption R and a coalgebra (R, β). Choose any such coalgebra. Then we have a coalgebra (S, γ) so that $\gamma d = \Phi\iota_{R,S}\beta d$. It is easy to see that (S, γ) satisfies the condition of subsumption. So the existence of the maximum subsumption in A is concluded.

Definition 1

- $R^\alpha = \{(x', y') \mid xRy\}$, where u' means αu if $u \in X$ or u otherwise.
- $R^= = \{(\Phi\pi_1 a, \Phi\pi_2 a) \mid a \in \Phi R\}$.
- $R^\sqsubseteq = \{(\Phi\pi_1 a, y) \mid a \in \Phi R, y \in \Phi X, \Phi\pi_2 a \trianglelefteq y\}$
- $fR = \{(x', y') \mid xRy\}$, where u' means $\Phi f u$ if $u \in \Phi X$ or fu if $u \in X$.

□

A pair (R, S) of a bisimulation R and a subsumption S on a coalgebra is called a *simulation pair* on the coalgebra if $R \subseteq S$. It is not difficult to see that this definition is equivalent to that given by Barwise [4] for the coalgberas for the power class functor *pow*: If a " subsumes" b then for every $x \in a$ there is $y \in b$ such that x "subsumes" y, where a, b, x, y are sets.

Definition 2 Let R,S be relations on $X \cup \Phi X$, (A, α) a coalgebra for Φ, and $f: X \to A$. f is called a *solution in* (A, α) *with a simulation pair* (P, Q) if $(fR)^\alpha \subseteq P^=$ and $(fS)^\alpha \subseteq Q^\sqsubseteq$. □

Theorem 2 *If Φ is set-based then the following are equivalent for any constraint (R, S) for Φ.*

1. *(R, S) has a solution in the final coalgebra for Φ with $(=, \sqsubseteq)$.*
2. *(R, S) has a solution in a coalgebra for Φ.*

Proof Suppose (R, S) is a relation on $X \cup \Phi X$.
(1) $\Longrightarrow$ (2): Obvious.
(2) $\Longrightarrow$ (1): Let $f: X \to A$ be a solution in a coalgera (A, α) with a simulation pair (P, Q). There is a unique homomorphism g from (A, α) into the final coalgebra for Φ. As every simulation pair is preserved by any homorphism, and $(=, \sqsubseteq)$ is the maximum simulation pair of the final coalgebra for Φ with respect to the class inclusion, it follows that gf is a solution in the final coalgebra with $(=, \sqsubseteq)$. □

Let (A, α), P and Q be the final coalgebra for Φ, the identity relation $=$ on A, and the maximum simulation $\sqsubseteq$, respectively. Then $f: X \rightarrow A$ is a solution in the final coalgebra with $(=, \sqsubseteq)$ if and only if the following hold:

1. If $afRb$ then $a = b$.

2. If $afSb$ then $a \sqsubseteq b$.

As we have assumed that there are sufficiently many indeterminates, we can assume without loss of generality that all parametric object are an element of $V_A[\mathcal{X}] \cup \mathcal{X}$.

4 Decidability of Subsumption Constraints

A binary operation on a coalgebra (A, α) for Φ is called a *merge* denoted by $\oplus$ on a coalgebra if the diagram commute for some coalgebra $(A \times A, \beta)$:

$$\begin{array}{ccc} \Phi A \times A & \xrightarrow{\Phi\oplus} & \Phi A \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \alpha} \\ A \times A & \xrightarrow[\oplus]{} & A \end{array}$$

We add several assumptions on the set-based functor. Firstly, we assume that Φf is an injection for any inclusion map f. Then it is easy to see that the inverse of a homophism preserves the subobject relation $\trianglelefteq$. Hence by a similar but dual way of diagram chasing in the above, we can prove that the subsumption relations are preserved under the inverse image of a homomphism. Also by simple diagram chasing, we can show that every bisimulation is preserved under the inverse of homomorphism. Secondly we assume Φf preserves weak pullbacks. Then as shown before, every bisimulation and subsumption are preserved by any homomophism. For the third, we assume that ΦX is finite whenever X is a finite set. Finally , we assume that the final coalgebra (A, α) for Φ has a merge operation and that every closed subclass B of A with respect to the merge $\oplus$ forms a subcoalgebra of (A, α). It is not difficult to show from these assumptions that the given constraint (R, S) on $X \cup \Phi X$ has a solution in the final coalgebra if and only if it has a solution in some subcoalgebra of $(X^{\oplus}, \beta)$, where $X^{\oplus}$ is the free merge algebra. In fact, clearly, $(X^{\oplus}, \oplus)$ is isomorphic to the power set algebra $(pow\, X, \cup)$. There are only a finite number of the subcoalgebras of $(X^{\oplus}, \oplus)$, and for each subcoalgebra it is decidable whether the given constraint has a solution in the subcoalgebra. So we can decide whether the constraint has a solution in the final coalgebra (A, α) or not.

Example 1 We define a merge operation on feature structures. Let Φ be the functor which assigns to the class X the class of all partial functions from B into X, where B is a class of *attributes* and Φf is defined so that $\Phi frb = x$ if and only if $frb = x$.

Define $\beta(r, s) = t$ so that $tb = (x, y)$ if and only if $b \in dom\, r \cup dom\, s$, $x = rb$ or $x = \emptyset$ if $x \notin dom\, r$, and $y = sb$ or $x = \emptyset$ if $x \notin dom\, s$. Then $(A \times A, \beta)$ is a colagebra for Φ. $\oplus$ is the unique homomorphism from the coalgebra $A \times A$ into the final coalgebra for Φ. As A is a final coalgebra for Φ, the four equations follow straightforwardly.

$$\begin{aligned}(x \oplus y) \oplus z &= x \oplus (y \oplus z) \\ x \oplus y &= y \oplus x \\ x \oplus x &= x \\ \emptyset \oplus x &= x \oplus \emptyset\end{aligned}$$

Thus we have defined an algebra on feature structures with a unit which is associative, commutative, and idempotent.

□

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Appendix

Here is the original form of the unification lemma formalized and proved by Barwise [4]. Let $\mathcal{A}$ be a collection of primitive atoms. Let $\mathcal{X}$ be an additional collections of atoms thought as indeterminates. $V_{\mathcal{A}}$ is the universe of hypersets on $\mathcal{A}$. $V_{\mathcal{A}}[\mathcal{X}]$ is the universe of hypersets on $\mathcal{A} \cup \mathcal{X}$.

Definition 3 By a *bisimulation* we mean an equivalence relation $\sim$ on some subclass of $V_{\mathcal{A}}[\mathcal{X}] \cup \mathcal{A} \cup \mathcal{X}$ satisfying the following condition. If $u \sim v$, then:

1. If $u \in \mathcal{A}$ then $u = v$.

2. If u and v are both sets, then for every $u' \in u$ there is a $v' \in v$ such that $u' \sim v'$.

3. If x is an indeterminate in the field of $\sim$, then there is some *set* u such that $x \sim u$.

□

Given any object $a \in V_{\mathcal{A}}[\mathcal{X}]$ the set $para(a) = \{x \in \mathcal{X} \mid x \in TC(a)\}$ is called the set of parameters of a, where $TC(a)$ means the transitive closure of a with respect to the membership relation. By an *anchor* we mean any function f with $dom(f) \subseteq \mathcal{X}$ and $ran(f) \subseteq V_{\mathcal{A}} \setminus \mathcal{A}$.

Definition 4 By a *simulation pair* we mean a pair of relations $\preceq$, $\sim$ with the same field, satisfying the following conditions:

1. $\sim$ is a bisimulation relation.

2. $\preceq$ is a simulation relation. That is:

   - If $u \preceq v$, and either u or v is an element of $\mathcal{A}$, then $u = v$
   - If $u \preceq v$, and u, v are both sets, then for all $u' \in u$ there is a $v' \in v$ such that $u' \preceq v'$.

3. $\sim$ is a congruence relation with respect to $\preceq$. That is,

   - $u \sim v$ implies $u \preceq v$;
   - $u \preceq v$, $u \sim u'$, and $v \sim v'$ implies $u' \preceq v'$.

□

Theorem 3 (Unification Lemma) *Suppose we are given indexed families $\{a_i(x, y, \ldots) \in V_{\mathcal{A}}[\mathcal{X}] \mid i \in I \cup J\}$ and $\{b_i(x, y, \ldots) \in V_{\mathcal{A}}[\mathcal{X}] \mid i \in I \cup J\}$. Then the following are equivalent:*

1. *There is an anchor f such that for each $i \in I$, $a_i(f) = b_i(f)$, and for each $j \in J$, $a_j(f) \sqsubseteq b_j(f)$.*

2. *There is a simulation pair $\preceq$, $\sim$ satisfying $a_i \sim b_i$ for each $i \in I$ and $a_j \preceq b_j$ for $j \in J$.*