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アレンジメントと有向マトロイドにおける面列挙アルゴリズム

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$f_k(\mathcal{F})$ を d 次元アレンジメント $\mathcal{F}$ または d 次元有向マトロイド $\mathcal{F}$ の k 次元面の数とする。まず面数に関する不等式 $f_k(\mathcal{F}) \leq \binom{d}{k} f_d(\mathcal{F})$ を示し、これを用いて有向マトロイドの極大面の集合から全ての面を列挙する多項式時間アルゴリズムを得る。このアルゴリズムは、d 次元射影空間や d 次元ユークリッド空間内の超平面のアレンジメントにも応用できる。このアルゴリズムと Cordovil & Fukuda からアレンジメントの双対グラフから全ての面の位置ベクトルを多項式時間内で再構成することができる。ここで、双対グラフの頂点はこのアレンジメントの d 次元面に対応し、二つの d 次元面が共通の $(d-1)$ 次元面を持つ時対応する二つの頂点が枝で結ばれている。さらに、このアルゴリズムを用いて与えられた $(+,0,-)$- ベクトル集合がある有向マトロイドの極大面集合になるかの判定が多項式時間内にできる。

Combinatorial Face Enumeration in Arrangements and Oriented Matroids

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Let $f_k(\mathcal{F})$ denote the number of k-dimensional faces of a d-arrangement $\mathcal{F}$ or a d-dimensional oriented matroid $\mathcal{F}$. It is shown that $f_k(\mathcal{F}) \leq \binom{d}{k} f_d(\mathcal{F})$ for $0 \leq k \leq d$. Using the result, we obtain a polynomial algorithm to enumerate all faces from the maximal faces of an oriented matroid. This algorithm can be applied to any arrangement of hyperplanes in P^d or in Euclidean space E^d. Combining this with a result of Cordovil and Fukuda, it is also shown that, given the dual graph of an arrangement (where the vertices are the d-faces and two vertices are adjacent if they intersect in a $(d-1)$-face), one can reconstruct the location vectors of all faces of the arrangement up to isomorphism in a polynomial time. This in turn enables one to test in a polynomial time whether a given set of $(+,0,-)$-vectors is the set of maximal vectors of an oriented matroid.

1 Introduction

A d-arrangement of spheres is a collection $\mathcal{A} = \{S_1, S_2, \cdots, S_n\}$ of n unit $(d-1)$-spheres in unit d-sphere S^d with $\bigcap_{i=1}^{n} S_i = \emptyset$. Each d-arrangement $\mathcal{A}$ is naturally associated with a cell complex of relatively open regions, each of which is called a *face* or k-face if its dimension is k. The f-vector of a d-arrangement is the vector $f(\mathcal{A}) = (f_0(\mathcal{A}), f_1(\mathcal{A}), \cdots, f_d(\mathcal{A}))$ where $f_k(\mathcal{A})$ is the number of k-faces. It is well-known that the f-vector of a d-arrangement is polytopal, i.e., there is a d-polytope whose f-vector is exactly $f(\mathcal{A})$. This implies that the f-vector of an arrangement is a quite restricted vector. However, as it was shown in [5], the class of f-vectors is much more restricted than that of f-vectors of d-polytopes; for any d-arrangement $\mathcal{A}$, and for each $k = 0, 1, \cdots, d$,

$$f_k(\mathcal{A}) \leq \binom{d}{k} f_d(\mathcal{A}). \tag{1}$$

In particular, this implies the number f_k of k-faces is bounded by a polynomial function of f_d since $\binom{d}{k} \leq f_d(\mathcal{A})$.

We first generalize this result to oriented matroids or equivalently to sphere systems [7] (arrangement of topological spheres):

Theorem 1.1. *Let $\mathcal{F}$ be the set of faces of a d-dimensional (corank $d+1$) oriented matroid, and let $f_k(\mathcal{F})$ be the number of k-faces of $\mathcal{F}$. Then for each $k = 0, 1, \cdots, d$,*

$$f_k(\mathcal{F}) \leq \binom{d}{k} f_d(\mathcal{F}). \tag{2}$$

Then we apply this result to obtain a polynomial algorithm for enumerating all faces of an oriented matroid from the given set of d-faces. This implies that one can test in a polynomial time whether a given set of $(+, 0, -)$-vectors is the set of maximal faces of an oriented matroid. Combining this with a recent result of Cordovil and Fukuda [2], we show that one can test efficiently whether a given graph is the adjacency graph of maximal faces of an oriented matroid, i.e., "almost" the graph of a zonotope.

2 Oriented Matroids

Let E be a finite set. A *signed vector* X on E is a vector $(X_e : e \in E)$ with $X_e \in \{+, 0, -\}$. The *negative* $-X$ of X is defined in the obvious way. The *composition* $X \circ Y$ of two signed vectors X and Y is defined by $(X \circ Y)_e = X_e$ if $X_e \neq 0$, and

$(X \circ Y)_e = Y_e$ if $X_e = 0$. An element e of E *separates* X and Y if $X_e = -Y_e \neq 0$. Let $D(X,Y)$ denote the set of elements separating X and Y. An *oriented matroid* [1, 3, 4, 6, 7] on E is defined as a set $\mathcal{F}$ of signed vectors on E satisfying the following axioms (*face axioms*):

(F1) $\tilde{0} \in \mathcal{F}$;

(F2) $X \in \mathcal{F} \Rightarrow -X \in \mathcal{F}$;

(F3) $X, Y \in \mathcal{F} \Rightarrow X \circ Y \in \mathcal{F}$;

(F4) $X, Y \in \mathcal{F}$ and $j \in D(X,Y) \Rightarrow \left(\begin{array}{ll} ^{\exists}Z \in \mathcal{F} & \text{such that} \\ Z_j = 0 & \text{and} \\ Z_e = (X \circ Y)_e & \text{for all } e \notin D(X,Y) \end{array} \right)$.

Here $\tilde{0}$ denotes the signed vector of all 0's. We call a member of $\mathcal{F}$ a *face*. Note that the set $\mathcal{F}$ is also called as the signed span of cocircuits of an oriented matroid [1].

Let $\mathcal{A} = \{S_1, S_2, \cdots, S_n\}$ be a d-arrangement, and let $S_i^+ \cup S_i \cup S_i^-$ be a partition of S^d such that S_i^+ and S_i^- are the two open hemispheres of S^d bounded by S_i for each $i = 1, 2, \cdots, n$. For each vector $\boldsymbol{x} \in S^d$, $\sigma(\boldsymbol{x})$ denotes the location vector of $\boldsymbol{x}$, that is,

$$\sigma(\boldsymbol{x})_i = \begin{cases} + & \text{if } \boldsymbol{x} \in S_i^+ \\ 0 & \text{if } \boldsymbol{x} \in S_i \\ - & \text{if } \boldsymbol{x} \in S_i^- \end{cases}.$$

Then it is easy to verify that the set $\{\sigma(\boldsymbol{x}) \mid \boldsymbol{x} \in S^d\} \cup \{\tilde{0}\}$ satisfies the face axioms.

For signed vectors X and Y, we say that X *conforms to* Y (denoted by $X \preceq Y$) if $X_e = Y_e$ or 0 for each $e \in E$. Note that $\prec$ means the strict conformal relation, i.e., $X \prec Y$ if $X \preceq Y$ and if there exists $e \in E$ such that $0 = X_e \neq Y_e$. The set $\mathcal{F}$ of faces of an oriented matroid satisfies the Jordan-Dedekind property with respect to the partial order $\preceq$. For each $X \in \mathcal{F}$, let $\rho(X)$ be the length of maximal chain between $\tilde{0}$ and X. We call $\dim(X) = \rho(X) - 1$ a *dimension* of X in $\mathcal{F}$. The *dimension* $\dim(\mathcal{F})$ of the oriented matroid is defined as $\max\{\rho(X) - 1 \mid X \in \mathcal{F}\}$.

3 Face Enumerating Algorithm

In this section, we propose an algorithm enumerating all faces of an oriented matroid from the set of maximal faces.

Let X and Y be k-faces of an oriented matroid with $\underline{X} = \underline{Y}$. If $D(X,Y)$ is a parallel class of the set $\mathcal{G}$ of faces whose supports are equal to $\underline{X}$ then $Z = X + D(X,Y)^0$

is a $(k-1)$-face. Here $Z = X + D(X,Y)^0$ is the signed vector obtained from X by replacing $X_{D(X,Y)}$ by $\tilde{0}$. Conversely, any $(k-1)$-face can be constructed by such a way. This follows from the fact that $\mathcal{G}$ is also the set of maximal faces of some minor of the oriented matroid. A function enumerating all $(k-1)$-faces from all k-faces is described as follows.

function lower_face($\mathcal{F}^k$)

Input $\mathcal{F}^k \subseteq \{+,0,-\}^n$;
Output the set $\mathcal{F}^{k-1}$ of $(k-1)$-faces if $\mathcal{F}^k$ is the set of k-faces of an oriented matroid;

```
begin
  Partition F^k into {F^k_1, F^k_2, ..., F^k_m} such that          O(n|F^k|)
   F^k_i is the set of elements of F^k having the same support ;
  F^{k-1} := ∅ ;                                                     O(1)
  for i := 1 to m do begin
    Compute the collection of parallel classes of F^k_i ;           O(n|F^k_i|)
    for each X, Y ∈ F^k_i do begin
      if D(X,Y) is a parallel class of F^k_i then                   O(n)
        F^{k-1} := F^{k-1} ∪ {X + D(X,Y)^0} ;                       O(n log |F^{k-1}|)
    end ;
  end ;
  return( F^{k-1} ) ;
end.
```

One can enumerate all faces from all maximal faces of an oriented matroid by using the above function lower_face() at most n times.

Face Enumerating Algorithm

Input an integer n and $\mathcal{T} \subseteq \{+,-\}^n$;
Output the set $\mathcal{F}$ of faces if $\mathcal{T}$ is the set of maximal faces of an oriented matroid;

```
begin
  F := T;  W := T;
  for i := 1 to n do begin
    if W ≠ ∅ then begin
      W := lower_face( W );
      F := F ∪ W;
    end ;
  end ;
end.
```

To evaluate complexity of the procedure, first assume $\mathcal{T}$ is the set of maximal faces of some d-dimensional oriented matroid on an n-set. The time complexity of each statement of the function lower_face( ) can be easily evaluated as shown. The function lower_face( )

requires at most $O(n|\mathcal{F}^k|^2 \log|\mathcal{F}^{k-1}|)$ time to enumerate all $(k-1)$-faces from all k-faces. Theorem 1.1 says $|\mathcal{F}^k| \leq \binom{d}{k}|\mathcal{T}| \leq |\mathcal{T}|^2$ for any k. Hence the face enumerating algorithm has the total complexity of $O(2^{2d}n|\mathcal{T}|^2 \log|\mathcal{T}|)$ under the assumption.

If $\mathcal{T}$ is not the set of maximal faces of any oriented matroid, complexity of lower_face( ) cannot be bounded by the above formula because d cannot be defined for such a set and because the size $|\mathcal{F}|$ may not be polynomially bounded by $|\mathcal{T}|$. However, Theorem 1.1 guarantees that if $|\mathcal{F}| \geq |\mathcal{T}|^2$ then $\mathcal{T}$ is not the set of maximal faces of any oriented matroid. We consider a modified face enumerating algorithm which stops as soon as $|\mathcal{F}| \geq |\mathcal{T}|^2$ is detected. Complexity of the modified procedure is $O(n|\mathcal{T}|^4 \log|\mathcal{T}|)$.

The represented algorithm requires $O(|\mathcal{F}_i^k|^2)$ comparisons of signed vectors to enumerate $(k-1)$-faces obtained from $\mathcal{F}_i^k$. We can get a better complexity by efficiently implementing this part. Suppose that $\mathcal{F}_i^k$ is sorted according to a natural lexicographic order and that S is a parallel class of $\mathcal{F}_i^k$. Without loss of generality, we assume that $X_S = (+,\cdots,+)$ or $(-,\cdots,-)$ for all $X \in \mathcal{F}_i^k$. Let $\mathcal{F}_i^k(S+) = \{X \backslash S \mid X \in \mathcal{F}_i^k \text{ and } X_S = (+,\cdots,+)\}$ and $\mathcal{F}_i^k(S-) = \{X \backslash S \mid X \in \mathcal{F}_i^k \text{ and } X_S = (-,\cdots,-)\}$. Remark that these are sorted according to the ordering because the deletion preserves it. Generally, in order to find all numbers contained in two given strictly monotone sequences, the number of comparisons is at most the amount of lengths of these sequences. Thus the number of comparisons of signed vectors to enumerate all $(k-1)$-faces obtained from $\mathcal{F}_i^k$ by replacing restrictions on S by $\tilde{0}$ is bounded by $|\mathcal{F}_i^k| = |\mathcal{F}_i^k(S+)| + |\mathcal{F}_i^k(S-)|$. Since the number of parallel classes of $\mathcal{F}_i^k$ is at most n, the new algorithm requires $O(n|\mathcal{F}_i^k|)$ comparisons of signed vectors. Furthermore, the set of $(k-1)$-faces obtained from $\mathcal{F}_i^k(S+)$ and $\mathcal{F}_i^k(S-)$ is sorted according to the lexicographic order. Hence we need to sort $\mathcal{T}$ only once.

Since $(k-1)$-faces are enumerated from some distinct pairs of k-faces several times in lower_face( ), $O(n\log|\mathcal{F}^{k-1}|)$ time is required to update $\mathcal{F}^{k-1}$. However, we can avoid enumerating each $(k-1)$-face twice. Since it is a little bit complicated, we omit an explanation. By this, the number of arithmetic operations to update $\mathcal{F}^{k-1}$ becomes $O(1)$.

By the above consideration, we obtain an efficient face enumerating algorithm whose complexity is $O(n^3|\mathcal{T}|^2)$.

4 Concluding Remarks

Let $\mathcal{O}$ be a set of signed vectors on an n-set. By circuit axioms [1], one can test in $O(n^2|\mathcal{O}|^3)$ time whether $\mathcal{O}$ is the set of cocircuit of some oriented matroid. From Theorem 1.1, the number of cocircuits of an oriented matroid is less than or equal to the number of maximal faces. Then, by using the efficient algorithm in the previous section, we can test in $O(\max\{n^2|\mathcal{T}|^3, n^3|\mathcal{T}|^2\})$ time whether a given set $\mathcal{T}$ of signed vectors on an n-set is the set of maximal faces of an oriented matroid.

There is a graph naturally associated with an arrangement, called the *dual graph* of the arrangement. The vertices of the graph are the d-faces (maximal faces) of the arrangement, and two vertices are adjacent if they intersect in a $(d-1)$-face. A recent result of Cordovil and Fukuda [2] says: given the dual graph of an arrangement, one can reconstruct the location vectors of all d-faces of the arrangement up to isomorphism in $O(ve\log v)$ time. Here v and e denote the number of vertices and the number of edges of the graph, respectively. The enumerated location vectors have exactly n components where n is the diameter of the graph. From Theorem 1.1, $e \le dv \le nv$ holds. Combining this with the efficient face enumerating algorithm, we can reconstruct the location vectors of all faces from the graph in $O(\max\{nv^2\log v, n^3v^2\})$ time.

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