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汎関数の多項式時間階層について

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S.R.Buss は、彼の学位論文の中で、計算量理論における Meyer-Stockmeyer の多項式時間階層の汎関数による拡張を行なった。 本論文では、彼の先見的な研究を引き継ぎ、更に汎関数の多項式時間階層の理論を展開する。

ON THE POLYNOMIAL-TIME HIERARCHY OF FUNCTIONALS

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In his dissertation, S.R.Buss initiated the study of the functional extension of the Meyer-Stockmeyer polynomial-time hierarchy in Complexity Theory. This paper follows his forsighted investigations and intends to develop the theory of the polynomial-time hierarchy of functionals.

1. FOUNDATIONS

In the present paper, the reader's familiarity of Buss(*1986*), Hinman (*1978*) and Stockmeyer(*1976*) is assumed. Here we explain some others.

Instead of the boldface letters, let us now use a vector form $\vec{x}$ to denote a finite sequence. Especially $\vec{\emptyset}$ expresses the empty sequence. The notation $len(\vec{x})$ denotes the length of a finite sequence $\vec{x}$. We write the i-th component of $\vec{x}$ by x_i whenever $i<len(\vec{x})$. For a nonempty finite sequence $\vec{m}$ from ω, $|\vec{m}|$ means the sum $\sum_{i=0}^{k-1}|m_i|$, where $k=len(\vec{m})$. We set $|\vec{\emptyset}|=0$ as a special case.

Here we make use of an arbitrary polynomial-time computable coding function $\langle\ \rangle$ from $\cup\{{}^k\omega|k\in\omega\}$ to ω for the coding of finite sequences from ω. Relating to this coding, moreover we use $Seq(v)$, $len(v)$ and $(v)_j$ in order to express "v is a valid code of a finite sequence", "the length of a finite sequence v" and "the j-th element in a finite sequence v", respectively.

For a relation R of rank (k,l), let us denote by $\neg R$ the complement of R with respect to ${}^{k,l}\omega$. We write χ_R for the characteristic functional of rank (k,l) for R.

Now suppose $s,v,k,l\in\omega$, $\alpha\in{}^{\omega}\omega$, and $\vec{m},\vec{n}$ is a finite sequences from ω. The notation $[\vec{m},\vec{n}]^s$ stands for the set of all function $\alpha\in{}^{\omega}\omega$ such that, for any $i<len(\vec{m})$,

$$(n_i\geq s\wedge\alpha(m_i)\geq s)\vee(n_i<s\wedge\alpha(m_i)=n_i)$$

when $len(\vec{m})=len(\vec{n})$. We denote by $\alpha\restriction\vec{m}$ the finite sequence

$$(\alpha(m_0),\alpha(m_1),\cdots,\alpha(m_{k-1})),$$

where $k=len(\vec{m})$. Then, in paticular, $\alpha\restriction\vec{\emptyset}$ expresses the empty sequence. Moreover we define $\alpha[v]^s$ as

$$\alpha[v]^s=\begin{cases}\langle min\{s,\alpha((v)_0)\},\cdots,min\{s,\alpha((v)_{k-1})\}\rangle & \text{if } Seq(v)\wedge len(v)>0,\\ 0 & \text{otherwise,}\end{cases}$$

where $k=len(v)$.

For any ω-sequence $\mathcal{S}=\langle S_i\subseteq{}^{k,l}\omega|\,i\in\omega\rangle$ of relations, we write $\neg\mathcal{S}$ to stand for the ω-sequence $\langle\neg S_i\,|\,i\in\omega\rangle$. We call a relation R the p-union of $\mathcal{S}$ by a polynomial q if the equivalence

$$(\vec{m},\vec{\alpha})\in R\iff\exists i<2^{q(|\vec{m}|)}((\vec{m},\vec{\alpha})\in S_i)$$

holds for all $(\vec{m},\vec{\alpha})\in{}^{k,l}\omega$. If $\neg R$ is the p-union of $\neg\mathcal{S}$ by q, then R is said the p-intersection of $\mathcal{S}$ by q.

2. THE POLYNOMIAL-TIME HIERARCHY OF FUNCTIONALS

A functional version of polynomial-time hierarchy has already introduced by Buss(*1986*) as a natural extension of the Meyer-Stockmeyer polynomial-time hierarchy in Complexity Theory.

To the simplicity, this section gives a slightly different definition of the polynomial-time hierarchy of functionals from Buss' original one. We begin by seeing the concept of the polynomial-time computable functionals.

DEFINITION *2.1*. For any $k,l\in\omega$, $a<b<c<k$, $d<e<k$, $i<k$, $j<l$ and any $(\vec{m},\vec{\alpha})\in{}^{k,l}\omega$, let

(*1*) *the initial functionals*:

(i) $Const^{k,l}(\vec{m},\vec{\alpha})=0$;

(ii) $Rshift_i^{k,l}(\vec{m},\vec{\alpha})=\lfloor m_i/2\rfloor$;

(iii) $Lshift0_i^{k,l}(\vec{m},\vec{\alpha})=2m_i$;

(iv) $Lshift1_i^{k,l}(\vec{m},\vec{\alpha})=2m_i+1$;

(v) $Parity_i^{k,l}(\vec{m},\vec{\alpha})=\begin{cases}0 & \text{if } m_i \text{ is even,}\\ 1 & \text{otherwise,}\end{cases}$

(vi) $Choice_{a,b,c}^{k,l}(\vec{m},\vec{\alpha})=\begin{cases}m_b & \text{if } m_a>0,\\ m_c & \text{otherwise,}\end{cases}$

(vii) $Ap_{d,e,j}^{k,l}(\vec{m},\vec{\alpha})=min\{m_d,\alpha_j(m_e)\}$.

(*2*) *functional composition*:

for any $n\in\omega$ and any functionals $G,H_0,\cdots,H_{n-1}$, $FComp_n^{k,l}(G,H_0,\cdots,H_{n-1})$ is the functional F of rank (k,l) such that

(i) if G is of rank (n,l) and $H_0,\cdots,H_{n-1}$ are all of rank (k,l), then

$$F(\vec{m},\vec{\alpha})=G(H_0(\vec{m},\vec{\alpha}),\cdots,H_{n-1}(\vec{m},\vec{\alpha}),\vec{\alpha}),$$

(ii) otherwise, $F(\vec{m},\vec{\alpha})=0$.

(*3*) *polynomial-time recursion*:

for any functionals G,H,K, $PRec^{k+1,l}(G,H,K)$ is the functional F of rank $(k+1,l)$ such that

(i) if G is of rank (k,l), H is of rank $(k+2,l)$, K is of rank $(k+1,l)$ and $|H(s,t,\vec{n},\vec{\beta})|\leq|s|+|K(t,\vec{n},\vec{\beta})|$ holds for all $(s,t,\vec{n},\vec{\beta})\in{}^{k+2,l}\omega$, then

$$F(0,\vec{m},\vec{\alpha})=G(\vec{m},\vec{\alpha}),$$

$$F(n,\vec{m},\vec{\alpha})=H(F(\lfloor n/2\rfloor,\vec{m},\vec{\alpha}),n,\vec{m},\vec{\alpha}) \quad \text{if } n>0,$$

(ii) otherwise, $F(n,\vec{m},\vec{\alpha})=0$.

DEFINITION *2.2*. Let C be any class of relations. Pcf(C) denotes the smallest class which contains the initial functionals and all χ_R for any $R \in C$, and is closed under functional composition and polynomial-time recursion. The class Pcf of *polynomial-time computable functionals* is Pcf($\emptyset$).

Notice that the ordinary polynomial-time computable functions are special cases of these polynomial-time computable functionals.

Now let us define the polynomial-time hierarchy of functionals as follows.

DEFINITION *2.3*. For any relation R, R is said to be *polynomial-time computable* if and only if χ_R is polynomial-time computable.

DEFINITION *2.4*. (The Polynomial-time Hierarchy of Functionals) For all $k \in \omega$, define

(*1*) $\Delta_0^{0,\mathrm{P}} = \Sigma_0^{0,\mathrm{P}} = \Pi_0^{0,\mathrm{P}} =$ the class of all polynomial-time computable relations;

(*2*) $\Box_0^{0,\mathrm{P}} = \mathrm{Pcf}$;

(*3*) $\Sigma_{k+1}^{0,\mathrm{P}} = \{ (\exists x \leq F(\vec{m},\vec{\alpha})) R(x,\vec{m},\vec{\alpha}) \mid R \in \Pi_k^{0,\mathrm{P}}, F \in \mathrm{Pcf} \}$;

(*4*) $\Pi_{k+1}^{0,\mathrm{P}} = \{ (\forall x \leq F(\vec{m},\vec{\alpha})) R(x,\vec{m},\vec{\alpha}) \mid R \in \Sigma_k^{0,\mathrm{P}}, F \in \mathrm{Pcf} \}$;

(*5*) $\Box_{k+1}^{0,\mathrm{P}} = \mathrm{Pcf}(\Sigma_k^{0,\mathrm{P}})$;

(*6*) $\Delta_{k+1}^{0,\mathrm{P}} = \{ R \mid \chi_R \in \Box_{k+1}^{0,\mathrm{P}} \}$;

(*7*) $\Box_{<\omega}^{0,\mathrm{P}} = \cup \{ \Box_{k+1}^{0,\mathrm{P}} \mid k \in \omega \}$;

(*8*) $\Delta_{<\omega}^{0,\mathrm{P}} = \cup \{ \Sigma_k^{0,\mathrm{P}} \cup \Pi_k^{0,\mathrm{P}} \mid k \in \omega \}$.

It follows from the definition that

$$\Box_1^{0,\mathrm{P}} = \Box_0^{0,\mathrm{P}}, \quad \Delta_1^{0,\mathrm{P}} = \Delta_0^{0,\mathrm{P}} \text{ and } \Sigma_n^{0,\mathrm{P}} \cup \Pi_n^{0,\mathrm{P}} \subseteq \Delta_{n+1}^{0,\mathrm{P}} \subseteq \Sigma_{n+1}^{0,\mathrm{P}} \cap \Pi_{n+1}^{0,\mathrm{P}}$$

for any $n \in \omega$. For the relationships to the Meyer-Stockmeyer polynomial-time hierarchy, we only note the fact that $\Delta_n^{\mathrm{P}} = \Sigma_n^{\mathrm{P}}$ is equivalent to $\mathrm{Pcf}(\Sigma_n^{\mathrm{P}}) \subseteq \Box_n^{0,\mathrm{P}}$, where $n > 0$.

After Yao's(*1985*) announcement, Håstad(*1987*) first demonstrated the existence of an oracle which separates all complex classes in the Meyer-Stockmeyer polynomial-time hierarchy. Founded on his proof method, we can easily obtain the following theorem, so-called the hierarchy theorem.

THEOREM *2.5.* (Hierarchy Theorem) *For all* $n>0$, $\Delta_n^{0,\mathrm{p}} \neq \Sigma_n^{0,\mathrm{p}} \neq \Pi_n^{0,\mathrm{p}}$ holds.

3. THE BOLDFACE POLYNOMIAL-TIME HIERARCHY OF FUNCTIONALS

Here we turn our attentions to the relativization of the polynomial-time hierarchy of functionals. This section gives a natural relativization of this hierarchy and introduces the boldface hierarchy, say the boldface polynomial-time hierarchy of functionals, as an analogue of the boldface arithmetical hierarchy.

DEFINITION *3.1.* Let F be any functional, $\boldsymbol{R}$ a relation and $\vec{\beta}$ a finite sequence from ${}^{\omega}\omega$.

(*1*) We say F is *polynomial-time computable in* $\vec{\beta}$ if and only if there exists a $G\in\mathrm{Pcf}$ so that $F(\vec{m},\vec{\alpha})=G(\vec{m},\vec{\alpha},\vec{\beta})$ for all $(\vec{m},\vec{\alpha})\in{}^{k,l}\omega$. $\mathrm{Pcf}[\vec{\beta}]$ denotes the class of all functionals which are polynomial-time computable in $\vec{\beta}$.

(*2*) $\boldsymbol{R}$ is said to be *polynomial-time computable in* $\vec{\beta}$ if and only if $\chi_R\in\mathrm{Pcf}[\vec{\beta}]$.

DEFINITION *3.2.* (The Relativized Polynomial-time Hierarchy of Functionals) For any finite sequence $\vec{\beta}$ from ${}^{\omega}\omega$ and all $n\in\omega$, let

(*1*) $\Delta_0^{0,\mathrm{p}}[\vec{\beta}]=\Sigma_0^{0,\mathrm{p}}[\vec{\beta}]=\Pi_0^{0,\mathrm{p}}[\vec{\beta}]=$ the class of all relations which are polynomial-time computable in $\vec{\beta}$;

(*2*) $\square_0^{0,\mathrm{p}}=\mathrm{Pcf}[\vec{\beta}]$;

(*3*) $\Sigma_{n+1}^{0,\mathrm{p}}[\vec{\beta}]=\{(\exists x\leq F(\vec{m},\vec{\alpha}))R(x,\vec{m},\vec{\alpha})\mid R\in\Pi_n^{0,\mathrm{p}}[\vec{\beta}],F\in\mathrm{Pcf}\}$;

(*4*) $\Pi_{n+1}^{0,\mathrm{p}}[\vec{\beta}]=\{(\forall x\leq F(\vec{m},\vec{\alpha}))R(x,\vec{m},\vec{\alpha})\mid R\in\Sigma_n^{0,\mathrm{p}}[\vec{\beta}],F\in\mathrm{Pcf}\}$;

(*5*) $\square_{n+1}^{0,\mathrm{p}}[\vec{\beta}]=\mathrm{Pcf}(\Sigma_{n+1}^{0,\mathrm{p}}[\vec{\beta}])$;

(*6*) $\Delta_{n+1}^{0,\mathrm{p}}[\vec{\beta}]=\{R\mid\chi_R\in\square_{n+1}^{0,\mathrm{p}}[\vec{\beta}]\}$.

It is remarked that, for instance, $\Sigma_n^{0,\mathrm{p}}[\beta]$ coincides with $\Sigma_n^{0,\mathrm{p}}$ for any $n\in\omega$ if β is polynomial-time computable.

The boldface polynomial-time hierarchy of functionals is defined by the collection of all classes belonging to each relativized hierarchy.

DEFINITION *3.3.* (The Boldface Polynomial-time Hierarchy of Functionals)

For all $n\in\omega$, we define

(1) $\boldsymbol{\Sigma}_n^{0,\mathrm{p}} = \cup\{\Sigma_n^{0,\mathrm{p}}[\beta] \mid \beta\in{}^{\omega}\omega\}$;

(2) $\boldsymbol{\Pi}_n^{0,\mathrm{p}} = \cup\{\Pi_n^{0,\mathrm{p}}[\beta] \mid \beta\in{}^{\omega}\omega\}$;

(3) $\boldsymbol{\Delta}_n^{0,\mathrm{p}} = \cup\{\Delta_n^{0,\mathrm{p}}[\beta] \mid \beta\in{}^{\omega}\omega\}$;

(4) $\boldsymbol{\square}_n^{0,\mathrm{p}} = \cup\{\square_n^{0,\mathrm{p}}[\beta] \mid \beta\in{}^{\omega}\omega\}$.

Clearly it holds that $\boldsymbol{\Sigma}_n^{0,\mathrm{p}}\cup\boldsymbol{\Pi}_n^{0,\mathrm{p}}\subseteq\boldsymbol{\Delta}_{n+1}^{0,\mathrm{p}}\subseteq\boldsymbol{\Sigma}_{n+1}^{0,\mathrm{p}}\cap\boldsymbol{\Pi}_{n+1}^{0,\mathrm{p}}$ for all $n\in\omega$. Similar to Theorem *2.5*, the following hierarchy theorem also holds for the boldface polynomial-time hierarchy of functionals.

THEOREM *3.4*. (Hierarchy Theorem) *For all* $n>0$,

(1) $\boldsymbol{\Delta}_n^{0,\mathrm{p}}\neq\boldsymbol{\Sigma}_n^{0,\mathrm{p}}\neq\boldsymbol{\Pi}_n^{0,\mathrm{p}}$,

(2) $\boldsymbol{\Sigma}_{n+1}^{0,\mathrm{p}}\nsubseteq\boldsymbol{\Sigma}_n^{0,\mathrm{p}}$ *and* $\boldsymbol{\Pi}_{n+1}^{0,\mathrm{p}}\nsubseteq\boldsymbol{\Pi}_n^{0,\mathrm{p}}$.

4. TOPOLOGICAL CHARACTERIZATIONS

In the previous section, we defined the boldface polynomial-time hierarchy of functionals. Then we here intend to investigate several topological characterizations of the classes in this boldface hierarchy.

We first introduce the basic concept of the p-openness which is a polynomial version of the open relations in Descriptive Set Theory.

DEFINITION *4.1*. Let $k,l\in\omega$, $\vec{m},\vec{n}$ be any finite sequences from ω, q be a polynomial and R be any relation of rank (k,l).

(1) We say R is *p-open with* q if and only if, for any $(\vec{m},\vec{\alpha})\in{}^{k,l}\omega$, there is a finite sequence $\vec{s}$ from ω such that

(i) $|\vec{s}|\le q(|\vec{m}|)$, and

(ii) $(\vec{m},\vec{\alpha})\in R$ implies $\{m_0\}\times\cdots\times\{m_{k-1}\}\times[\vec{s},\alpha_0\upharpoonright\vec{s}]^t\times\cdots\times[\vec{s},\alpha_{l-1}\upharpoonright\vec{s}]^t\subseteq R$,

where $t=2^{q(|\vec{m}|)}$. R is said to be *p-closed* with q if $\neg R$ is p-open with q. R is *p-closed-open* (abbr. *p-clopen*) *with* q if both R and $\neg R$ are p-open with q.

(2) We simply say R is *p-open* whenever R is p-open with some suitable polynomial, R is *p-closed* whenever $\neg R$ is p-open. If R and $\neg R$ are p-open, then R is said to be *p-clopen*.

DEFINITION *4.2*. Let F be a partial functional.

(1) F is *polynomially bounded* if and only if there exists a polynomial q satisfying that $|F(\vec{m},\vec{\alpha})|\leq q(|\vec{m}|)$ for all $(\vec{m},\vec{\alpha})\in\mathrm{Dm}(F)$.

(2) F is *partially p-continuous* if F is polynomally bounded and there is a polynomial q such that for any $n\in\omega$, $F^{-1}(\{n\})$ is p-open with $\lambda x.q(x+|n|)$. We say F is *p-continuous* if F is partially p-continuous and total.

Using this p-continuity, we then establish the following characterizations of the classes in the first level of the boldface polynomial-time hierarchy of functionals.

THEOREM 4.3. *Suppose R is a relation and F is a functional.*

(1) *$R\in\Sigma_1^{0,\mathrm{p}}\cap\Pi_1^{0,\mathrm{p}}$ if and only if χ_R is p-continuous.*

(2) *$R\in\Sigma_1^{0,\mathrm{p}}$ if and only if R is the domain of some partial p-continuous functional.*

(3) *$F\in\Sigma_1^{0,\mathrm{p}}$ if and only if F is p-continuous.*

Next we consider the concepts of polynomial σ-size and π-size founded on the p-openness.

DEFINITION 4.4. Let $k,l,r\in\omega$, R be any relation of rank (k,l), q,q_0,q_1 be suitable polynomials and $\mathcal{S}$ be any ω-sequence $\langle S_i\subseteq^{k,l}\omega\,|\,i\in\omega\rangle$ of relations.

(1) A *σ-size* (*π-size*) of R is a pair of r and q which is defined as follows.

(i) R must have a δ-size $(0,q)$ and also a π-size $(0,q)$ if and only if R is p-clopen with q.

(ii) If $\mathcal{S}$ is π-homogeneous by (r,q_0) and R is the p-union of $\mathcal{S}$ by q_1 then R has a σ-size $(r,\lambda x.q_0(x+q_1(x)))$.

(iii) If $\mathcal{S}$ is σ-homogeneous by (r,q_0) and R is the p-intersection of $\mathcal{S}$ by q_1 then R has a π-size $(r,\lambda x.q_0(x+q_1(x)))$.

(2) We say $\mathcal{S}$ is *σ-homogeneous* (resp. *π-homogeneous*) by (r,q) if and only if for any $i\in\omega$, there is a $n_i<r$ such that each σ-size (resp. π-size) of S_i is $(n_i,\lambda x.q(x+|i|))$.

(3) R is simply said to have a *polynomial σ-size* (resp. *polynomial π-size*) n if and only if R has a σ-size (resp. π-size) (n,q) for some suitable polynomial q.

Finally we show the main consequence of a topological characterization of all classes in the boldface polynomial-time hierarchy of functionals.

THEOREM *4.5. Let n>0 and R be any relation.*

(1) $R \in \Sigma_n^{0,\mathrm{p}}$ *if and only if R has a polynomial σ-size n.*

(2) $R \in \Pi_n^{0,\mathrm{p}}$ *if and only if R has a polynomial π-size n.*

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