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グラフの各辺両端点の小さい方の次数和に関する上限の初等的証明と幾何学問題への応用

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本稿ではまず、点集合 $V(|V|>14)$, 辺集合 E なる平面グラフ $G=(V,E)$ について、

$$\sum_{e=(u,v)\in E} \min\{\deg(u),\ \deg(v)\} \leq 6|E|-36,$$

であること、および、この結果の最良性を示す。ここで $\deg(v)$ は、点 $v\in V$ の次数を意味する。さらに、長さ3のサイクルを持たない平面グラフにおいて、この和は高々 $4|E|-16$ となり、外平面グラフに対しては高々 $4|E|-12$ となることを証明する。次数和に関するこの性質は, ボロノイ図を扱う計算上ロバストなアルゴリズムの解析や、線分, 曲線の交差を求める最適ランダム化アルゴリズムを得る上で有用である。

A Simple Proof of a Tight Bound of the Sum of Smaller Endpoint Degree over Edges of Graphs and Its Applications to Geometric Problems

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In this paper we first show that, for a planar graph $G=(V,E)$ with vertex set V ($|V|>14$) and edge set E,

$$\sum_{e=(u,v)\in E} \min\{\deg(u),\ \deg(v)\} \leq 6|E|-36,$$

where $\deg(v)$ is the degree of a vertex $v\in V$, and that this is tight. Also, for a planar graph having no cycle of length 3, the summation is shown to be at most $4|E|-16$, and for an outerplanar graph, at most $4|E|-12$. This degree property can be used in the analysis of a computationally robust algorithm for Voronoi diagrams and also to obtain another optimal randomized algorithm for finding the intersections among line segments and curves.

1. Introduction

In computational geometry, many geometric structures are represented as graphs, especially planar graphs, and there arise new graph problems in analyzing geometric algorithms. In this paper, we are interested in obtaining bounds of the sum of smaller endpoint degrees over edges of graphs, and utilize them to devise efficient algorithms for geometric problems.

For an undirected graph $G = (V, E)$ with vertex set V and edge set E, define $D(G)$ by

$$D(G) = \sum_{e=(u,v)\in E} \min\{\deg(u),\ \deg(v)\},$$

where $\deg(v)$ is the degree of a vertex v. Chiba and Nishizeki [2] show that $D(G)$ is at most $2a(G)|E|$ where $a(G)$ is the arboricity of G (the minimum number of trees covering G). Since the arboricity of a planar graph is at most 3, this upper bound for planar graphs becomes $6|E|$. This paper investigates $D(G)$ for planar graph G in more detail, and proves the following tight bounds.

Theorem. For a planar undirected graph $G = (V, E)$, which is simple and connected, with more than 14 vertices, $D(G) \le 6|E| - 36$. For a planar graph $G' = (V', E')$ with more than 10 vertices and without any cycle of length 3, $D(G') \le 4|E'| - 16$. For an outerplanar graph $G'' = (V'', E'')$ with more than 7 vertices, $D(G'') \le 4|E''| - 12$. Also, there exist graphs G, G' and G'' satisfying $D(G) = 6|E| - 36$, $D(G') = 4|E'| - 16$ and $D(G'') = 4|E''| - 12$. □

This theorem can be used in the analysis of a computationally robust divide-and-conquer algorithm for constructing the Delaunay triangulation (Oishi and Sugihara [7]). Furthermore, this can be directly used to develop an optimal randomized algorithm for constructing the intersections (or arrangements) of line segments and curves. In this paper, we provide only an outline of this algorithm. Although such optimal (randomized) algorithms are already known to exist (Chazelle, Edelsbrunner [1], Mulmuley [6]), this indicates the usefulness of the theorem in developing new geometric algorithms.

2. Upper Bounds

Let $G = (V, E)$ be a simple planar graph. In considering the upper bound for $D(G)$, we can assume G is edge maximal, because adding edges to G does not reduce the value of $D(G)$. We denote the maximum degree of G by Δ. For each $i = 1, \ldots, \Delta$, define V_i to be the set of vertices whose degree is i, E_i to be the set of edges whose smaller endpoint degree is i, and let $\tilde{V}_i = \bigcup_{j=i}^{\Delta} V_j$, and let $\tilde{E}_i = \bigcup_{j=i}^{\Delta} E_j$. We also define the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 3,\ c(2) = 5,\ c(k) = 6 \ \text{for}\ k \ge 3.$$

Since each subgraph H of G is also planar, we can get $|E(H)| \le 3|V(H)| - c(|V(H)|)$ for all subgraph H of G by using Euler's relation. Then we have

$$\begin{aligned} D(G) &= \sum_{i=1}^{\Delta} i|E_i| = \sum_{i=1}^{\Delta}\sum_{j=i}^{\Delta} |E_i| = \sum_{i=1}^{\Delta} |\tilde{E}_i| \\ &\le \sum_{i=1}^{\Delta} 3|\tilde{V}_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) = 3\sum_{i=1}^{\Delta}\sum_{j=i}^{\Delta} |V_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \\ &= 3\sum_{i=1}^{\Delta} i|V_i| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) = 6|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|). \end{aligned}$$

Now we denote the second maximum degree and the third maximum degree by Δ_2 and Δ_3

,respectively. Then we get

$$\sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \geq \Delta c(1) + \Delta_2\{c(2) - c(1)\} + \Delta_3\{c(3) - c(2)\}$$
$$= 3\Delta + 2\Delta_2 + \Delta_3$$

Using that G is edge maximal planar graph with more than 3 vertices, we can show that $|E(G)| = 3|V(G)| - 6$ and that $\Delta \geq \Delta_2 \geq \Delta_3 \geq 3$. If $\Delta > 8$ or $\Delta_2 > 6$ or $\Delta_3 > 5$, then $3\Delta+2\Delta_2+\Delta_3 \geq 36$. Hence we can assume $\Delta \leq 8$ and $\Delta_2 \leq 6$ and $\Delta_3 \leq 5$. If $\Delta_3 < 5$, then $6|V(G)| - 12 = 2|E(G)| \leq \Delta_3(|V(G)| - 2) + \Delta_2 + \Delta \leq 4(|V(G)| - 2) + 14$, therefore $|V(G)| \leq 9$. As $|V(G)| > 14$, we can assume $\Delta_3 = 5$. If $\Delta > 6$, then $3\Delta + 2\Delta_2 + \Delta_3 \geq 36$. Thus we can assume that $\Delta = 6$. And we get $6|V(G)| - 12 \leq \Delta_3(|V(G)| - 2) + \Delta_2 + \Delta = 5(|V(G)| - 2) + \Delta_2 + 6$, that is, $|V(G)| \leq 8 + \Delta_2 \leq 14$. This implies that $3\Delta + 2\Delta_2 + \Delta_3 \geq 36$ for $|V(G)| > 14$. We thus have the upper bound for $D(G)$ in Theorem. Note that this proof also indicates that the exceptional case for $|V(G)| = 14$ must consist of 2 adjacent vertices with degree 6 and other 12 vertices with degree 5. This, however, can not be planar. This means that we can get the same bound for $|V(G)| > 13$.

Now, suppose that G does not have any cycle of length 3. We redefine the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 2,\ c(2) = 3,\ c(k) = 4 \ \text{for}\ k \geq 3.$$

Then, Euler's relation states that, for any subgraph H of G,

$$|E(H)| \leq 2|V(H)| - c(|V(H)|).$$

Applying the above arguments to this case, we can obtain the upper bound. We will give an outline of the proof. First we get

$$D(G) \leq 4|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \leq 4|E(G)| - 2\Delta - \Delta_2 - \Delta_3.$$

To show $2\Delta + \Delta_2 + \Delta_3 \geq 16$, we can assume $\Delta \leq 5$ and $\Delta_2 \leq 4$ and $\Delta_3 \leq 3$. Using the same argument, it is shown that $\Delta_3 = 3$ and $\Delta = 4$, and we get $|V(G)| \leq 6 + \Delta_2 \leq 10$. We thus have the upper bound for $D(G')$ in Theorem.

Finally, suppose that G is outerplanar. We redefine the integer function $c(k)$ for every positive integer k as follows:

$$c(1) = 2,\ c(k) = 3 \ \text{for}\ k \geq 2.$$

Then, for any subgraph H of G,

$$|E(H)| \leq 2|V(H)| - c(|V(H)|).$$

Applying the same arguments to this case, we can obtain the upper bound. We will give an outline of the proof. Here we get

$$D(G) \leq 4|E(G)| - \sum_{i=1}^{\Delta} c(|\tilde{V}_i|) \leq 4|E(G)| - 2\Delta - \Delta_2.$$

To show $2\Delta + \Delta_2 \geq 12$, we can assume $\Delta \leq 4$ and $\Delta_2 \leq 3$. The same argument shows that $\Delta_2 = 3$ and $\Delta = 4$, and we get $|V(G)| \leq 7$. We thus have the upper bound for $D(G'')$ in Theorem.

3. Lower Bounds

We next consider lower bounds of the summation in the theorem for planar graphs.

Consider a regular tetrahedron T_0 whose edges are of unit length. Each face is a regular triangle, and there are 4 faces, 6 edges and 4 vertices of degree 3. By connecting the midpoints of the edges, each triangle may be partitioned into four regular subtriangles; by repeating this process k times (denote the resultant polyhedron by T_k), each original

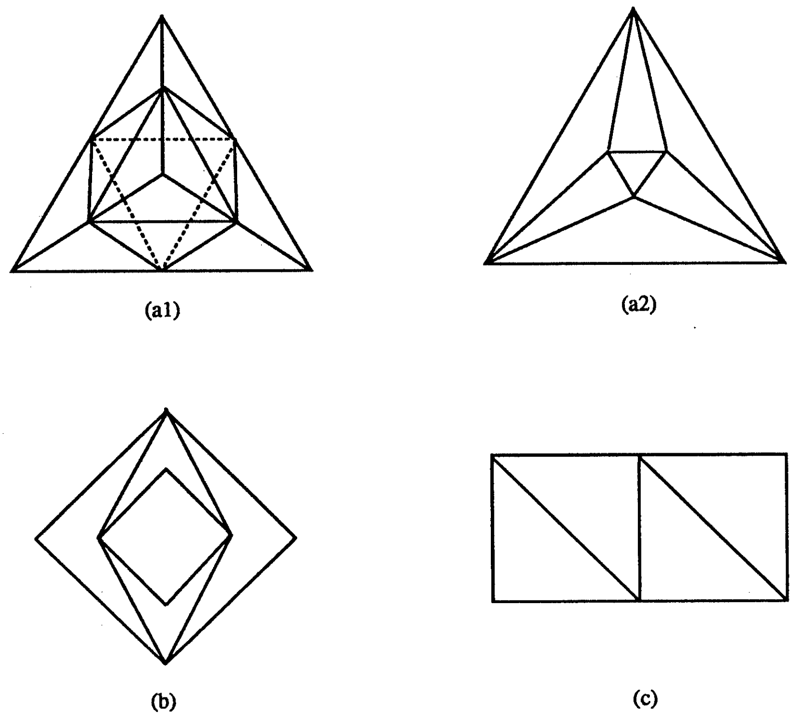

Figure 3.1. (a1) T_1, (a2) P_2, (b) S_2 and (c) R_2

triangle is divided into 4^k regular subtriangles with edges of length 2^{-k}. Figure 3.1(a1) depicts T_1. Then, in the interior of each original face, there are $1+2^{2k-1}-3\cdot 2^{k-1}$ vertices of degree 6. Hence, in total, among $n = 4(1+2^{2k-1}-3\cdot 2^{k-1})+6(2^k-1)+4 = 2+2^{2k+1}$ vertices, only 4 vertices have degee 3 and the others degree 6, and no edge connects vertices of degree 3. Therefore, for such $n = 2+2\cdot 4^k$ and $m = 6\cdot 4^k$ with $k = 2, 3, \ldots$, there exists a planar graph with n vertices and m edges for which the summation in the theorem is $6m - 36$. We can construct another series of graphs which attain the lower bound. P_1 is a triangle. P_{i+1} is constructed from P_i by adding new larger triangle surrounding P_i and connect each new vertices in the larger triangle to the corresponding two vertices on the outer triangle of P_i. Figure 3.1(a2) illustrates P_2. P_i has $n = 3i$ vertices and $m = 9i - 6$ edges. $D(P_i) = 6m - 36$ for $i \geq 3$.

To obtain a lower bound for planar graphs without cycle of length 3, construct the following series of graphs. S_0 is a square. We regard a pair of diagonal vertices in S_0 as new vertices. S_{i+1} is constructed from S_i by adding two new vertices and connect each of them with the new vertices in S_i. Figure 3.1(b) illustrates S_2. S_i has $n = 4 + 2i$ vertices and $m = 4 + 4i$ edges. $D(S_i) = 4m - 16$ for $i \geq 1$.

To obtain a lower bound for outerplanar graphs, construct the following series of graphs. R_1 is a square with a diagonal. R_{i+1} is constructed from R_i by adding two new vertices to make a copy of R_1 outside on the right edge of R_i. Figure 3.1(c) illustrates R_2. R_i has $n = 2 + 2i$ vertices and $m = 1 + 4i$ edges. $D(S_i) = 4m - 12$ for $i \geq 2$.

4. An Optimal Randomized Algorithm for Arrangements of Curves

In this paper, we will describe an $O(N^2)$-time randomized algorithm for constructing an arrangement of N lines, without using the zone theorem for lines. This illustrates, for the problem of constructing an arrangement of N curves such that any two curves intersect at a constant number of points, how a simple incremental algorithm using a careful search technique with $O(N^2)$ randomized time complexity may be devised based on the inequality in Theorem.

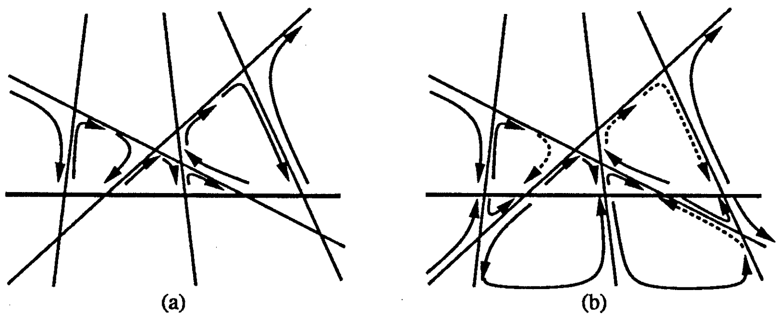

Figure 4.1. (a) A simple incremental algorithm and (b) an incremental algorithm which chooses shorter paths

An incremental algorithm for constructing the arrangement of N lines $l_1, l_2, \ldots, l_N$ works roughly as follows: at the first stage, construct a trivial arrangement of one line l_1; at the ith stage ($i = 2, 3, \ldots, N$), add line l_i to the arrangement of lines $l_1, \ldots, l_{i-1}$, which has been computed already, to obtain the arrangement of lines $l_1, \ldots, l_{i-1}, l_i$. Here, the arrangement is represented by a standard data structure for planar subdivisions.

The main step here is to add l_i to the arrangement A_{i-1} of $l_1, \ldots, l_{i-1}$. To do this, we find an edge e of the arrangement A_{i-1} that is just above l_i at $x = -\infty$, and a cell c intersecting l_i at $x = -\infty$. This can be done in linear time by finding, from among the lines $l_1, \ldots, l_{i-1}$, the line of largest slope less than that of l_i. We then traverse A_{i-1} along l_i by following edges of the cell c in clockwise order, starting with e, to find a new intersection point of l_i with an edge e' of the cell. We iterate for $e := e'$ and $c :=$cell adjacent to c at e' until a cell is found intersecting l_i at $x = +\infty$. See Figure 4.1(a).

The time complexity of adding l_i to A_{i-1} is proportional to the number of edges of cells in A_{i-1} intersecting l_i (these cells form a zone of l_i, and this number is the complexity of the zone). The well-known zone theorem for lines (e.g., [3]) states that the complexity of this zone is $O(i)$. Hence, it takes $O(i)$ time to insert l_i to A_{i-1} to construct A_i, and in total the arrangement of N lines can be constructed in $O(N^2)$ time. Note that this time complexity is worst-case optimal, since the size of a simple arrangement is $\Theta(N^2)$.

In the above algorithm, of the cells intersecting l_i, only the portion above l_i is traversed. Instead of this, we may traverse edges of the upper and lower parts of a cell intersecting l_i one by one simultaneously so that a new intersection point of l_i with the cell may be found in time proportional to the length of the shorter of the two paths (upper and lower) from the old intersection point to the new one. See Figure 4.1(b). This way of traversing cells is sometimes used in other geometric algorithms.

Now, let $e(l_j)$ be the number of edges traversed in adding l_j to the arrangement of lines $\{l_1, \ldots, l_i\} - \{l_j\}$ with choosing shorter paths as above ($j = 1, \ldots, i$). Consider the dual graph of the arrangement of $l_1, \ldots, l_i$ as a planar graph. This dual graph has at most i^2 edges, and does not have any cycle of length 3. Hence, applying the Theorem, it is seen that

$$\sum_{j=1}^{i} e(l_j) \leq 2 \cdot 4i^2.$$

By randomizing the order of insertion of lines in this modified incremental algorithm, the number of edges traversed in adding the ith line is at most $8i$ on the average. This implies that in total this algorithm constructs the arrangement of N lines in $O(N^2)$ average time.

This idea can be carried over for the case of the arrangement of curves, for which we need some of the techniques developed in [4,6], say the vertical decomposition of the arrangement. Besides these applications and that of [7], Theorem could be useful in the analysis of other geometric and graph problems.

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