アルゴリズム 29－3
（1992. 8. 25）

動的最小木、最大木問題

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本論文は空間上を各点が直線に沿って定速度で移動するとき、点集合に対する最小木・最大木の変化回数について調べる。d 次元空間内の点集合 $S=\{p_1,p_2,\ldots,p_n\}$ が与えられ、各点は各々異なる直線上を定速度で（点によってその速度は異なる）移動するものとする。すなわち点 p_i の位置はパラメータ t の一次関数である。本論文は d は定数と仮定して，t が$-\infty$ から $+\infty$ まで変化するとき、最大木・最小木の変化回数を調べる。$O(n^2)$ 個の点対があり、２つの点対間距離の大小が変化するときのみ、最小木・最大木は変化し得るので、明らかに $O(n^4)$ という変化回数の上限を得る。本論文は L_1 および L_∞ 距離に対してこれらの上限を改善する。$c_p(n,\min)$ および $c_p(n,\max)$ を L_p 距離における最小木・最大木の変化回数とすると、次の結果を得る。
$c_1(n,\min)=O(n^{5/2}\alpha(n))$, $c_\infty(n,\min)=O(n^{5/2}\alpha(n))$, $c_1(n,\max)=\Theta(n^2)$ and $c_\infty(n,\max)=\Theta(n^2)$
ここで$\alpha(n)$ は Ackermann 関数の逆関数である。また特別な場合として、n 点の移動速度ベクトルのうち異なるものは c 個しかない場合や、n 点のうち k 点しか動かない場合に対しても自明でない結果を示す。

On Minimum and Maximum Spanning Trees of Linearly Moving Points

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In this paper, we shall investigate the upper bounds on the numbers of transitions of minimum and maximum spanning trees (MinST and MaxST for short) for linearly moving points. Suppose that we are given a set of n points in general d-dimensional space, $S=\{p_1,p_2,\ldots,p_n\}$, and that all points move along different straight lines at different but fixed speeds, i.e., the position of p_i is a linear function of a real parameter t. We shall investigate the numbers of transitions of MinST and MaxST when t increases from $-\infty$ to $+\infty$. We assume that the dimension d is a fixed constant. Since there are $O(n^2)$ distances among n points, there are naively $O(n^4)$ transitions of MinST and MaxST. We shall improve these trivial upper bounds for L_1 and L_∞ distance metrics.

Let $c_p(n,\min)$ (resp. $c_p(n,\max)$) be the number of maximum possible transitions of MinST (resp. MaxST) in L_p metric for n linearly moving points. We shall give the following results in this paper: $c_1(n,\min)=O(n^{5/2}\alpha(n))$, $c_\infty(n,\min)=O(n^{5/2}\alpha(n))$, $c_1(n,\max)=\Theta(n^2)$ and $c_\infty(n,\max)=\Theta(n^2)$ where $\alpha(n)$ is the inverse Ackermann's function. We shall also investigate two restricted cases, i.e., the c-oriented case in which there are only c distinct velocity vectors for moving n points, and the case in which only k points move.

1 Introduction

Computational geometry problems for moving objects are theoretically interesting and have important applications in motion planning of robotics. The pioneering work in this field was done by Atallah [2], who gave nontrivial upper bounds on combinatorial transitions of several fundamental geometric structures such as convex hulls for moving points. Voronoi diagrams and Delaunay triangulations for moving points have recently been investigated by Imai and Imai [6] and Guibas et al. [3].

Although the two-dimensional minimum spanning tree (MinST) is a sub-graph of the Delaunay triangulation, it is not even clear that the number of transitions of MinST is smaller than that of Delaunay triangulation. Recently, Monma and Suri [10] have investigated the case where only one point is allowed to move in an arbitrary manner, and gave an $O(n^{2d})$ bound (especially a $\Theta(n^2)$ tight bound in Euclidean two-dimensional space) for transitions of MinST. However, to the authors' knowledge, no one has ever succeeded in improving naive bounds on the numbers of combinatorial transitions of MinST and the maximum spanning tree (MaxST) when all points move linearly.

In this paper, we shall investigate the upper bounds on the numbers of transitions of MinST and MaxST for linearly moving points. Our paper is the first to give nontrivial upper bounds for these numbers.

Let us formulate the problem: Suppose that we are given a set of n points in general d-dimensional space, $S = \{p_1, p_2, \ldots, p_n\}$, and that all points move along different straight lines at different but fixed speeds, i.e., the position of p_i is a linear function of a real parameter t. We shall investigate the numbers of transitions of MinST and MaxST when t increases from $-\infty$ to $+\infty$. We assume that the dimension d is a fixed constant.

When t is fixed, MinST and MaxST are determined only by the relative order of edge lengths. This implies that MinST (resp. MaxST) changes only if the relative order of the lengths for some pair of edges changes.

Since there are $O(n^2)$ distances among n points, there are naively $O(n^4)$ transitions of MinST and MaxST. On the other hand, it is easy to construct an example that requires $\Omega(n^2)$ transitions for each of MinST and MaxST. Therefore, there is a rather big gap between lower and upper bounds of such transitions. Note that known bounds for the number of transitions of a planar Delaunay triangulation are $O(n^3)$ and $\Omega(n^2)$ [6,3].

Let $c_p(n, \min)$ (resp. $c_p(n, \max)$) be the number of maximum possible transitions of MinST (resp. MaxST) in L_p metric for n linearly moving points. In this paper, we shall restrict ourselves to the cases of $p = 1$ and ∞ (except in Section 3), and give improved bounds for them as follows:

$$c_1(n, \min) = O(n^{5/2}\alpha(n)), \quad c_\infty(n, \min) = O(n^{5/2}\alpha(n)),$$

$$c_1(n, \max) = \Theta(n^2), \quad c_\infty(n, \max) = \Theta(n^2),$$

where $\alpha(n)$ is the inverse Ackermann's function and is very slowly growing [5]. In particular, a $\Theta(n^2)$ tight bound for MaxST is attained.

We shall then consider two restricted cases. The first is the c-oriented case in which there are only c distinct velocity vectors for moving n points. The second is the case in which only k points move, while the other points remain in their original positions. We shall improve the above upper bounds for these cases.

L_1 and L_∞ metrics are referred to as linear metrics in the subsequent discussion. The common technique we use to derive our upper bounds is the generalization of the combinatorial results obtained by Gusfield [4], and Katoh and Ibaraki [7] for the number of transitions of the minimum (or maximum) weight base in a matroid in which the weights of all elements are linear functions of a single parameter t. Note that the minimum (or maximum) weight base in a matroid is an abstract notion of MinST and MaxST for general graphs.

Since the distance between two points is piecewise linear convex in t for linear metrics, we must generalize the result of [4,7] to the piecewise linear convex case. For this purpose, we introduce a minimum (resp. maximum) weight base problem for matroids appropriately defined on certain multigraphs such that the weights of all elements are linear in t, and the transition of the minimum (resp. maximum) weight base occurs if the transition of MinST (resp. MaxST) for the original graphs occurs.

From this, we obtain $O(m^{3/2})$ and $O(m\sqrt{n})$ nontrivial upper bounds on the numbers of transitions of MinST and MaxST, respectively, for general graphs with n vertices and m edges in which the edge lengths are piecewise linear convex in a single parameter t. As a direct consequence of these results, we have $c_1(n, \min) = O(n^3)$ and $c_1(n, \max) = O(n^{5/2})$. These bounds are further improved to those stated above through geometric insights into the structures of the problems. In particular, we use Yao's lemma [13], with which [13] developed efficient algorithms for the Euclidean MinST.

Finally we study the problem of finding the value of t at which the total length of MaxST is minimized. For linear and L_2 metrics, we give nearly-linear time algorithms for moving points in a plane, based on the parametric search technique developed by Megiddo [8].

2 Linear metric spanning trees of moving points

We shall derive the upper bounds on the number of transitions of MinST and MaxST in L_1 and L_∞ metrics. Since the results we obtain and the techniques we use are the same for both metrics and for any d-dimensional space, we shall concentrate only on L_1 metric case and on $d = 2$. Let $p_i(t) = (x_i(t), y_i(t))$ denote the position of point p_i at t, where $x_i(t)$ and $y_i(t)$ are linear functions of t. The L_1 distance between two points in the plane is a piecewise linear convex function in t with at most two breakpoints. Here t' is said to be a *breakpoint* of a piecewise linear function if the slope of the function changes at t'. The L_p distance between points p_i and p_j is denoted by $d_p(p_i, p_j)$. Since $d_p(p_i, p_j)$ is a function of t, it should be written as $d_p(p_i(t), p_j(t))$, but for convenience we shall omit the argument t unless there is a possibility of confusion.

2.1 Number of distinct MinST's and MaxST's with piecewise linear convex weight functions

First, we introduce a theorem on the minimum weight base of a linearly weighted matroid, previously presented by Gusfield [4], and Katoh and Ibaraki [7]. Let E be a finite set and $\mathcal{B}$ a family of subsets of E. The pair $(E, \mathcal{B})$ is called a *matroid* $M(E, \mathcal{B})$, and the elements of $\mathcal{B}$ are the *bases* of $M(E, \mathcal{B})$, if the following two axioms hold [11]:

(A1) For any $B, C \subset E$ with $B \neq C$, if $B \in \mathcal{B}$ and $C \subset B$, $C \notin \mathcal{B}$.

(A2) For any $B, B' \subset \mathcal{B}$ with $B \neq B'$ and for any $e \in B - B'$, there exists $e' \in B' - B$ such that $(B - \{e\}) \cup \{e'\} \in \mathcal{B}$.

For instance, let $\mathcal{T}$ be a set of spanning trees in an undirected connected graph $G = (V, E)$; then $(E, \mathcal{T})$ forms a matroid and $\mathcal{T}$ is a set of bases [11].

The number $|B|$ of elements of a base $B \in \mathcal{B}$ is independent of the choice of B [11], and is denoted by p. Let $m = |E|$, and assume the elements of E to be indexed from 1 through m. We assume that each element i has a real-valued weight $w_i(t) = a_i t + b_i$ that is linear in the parameter t. The minimum (resp. maximum) weight base is the one in which the sum of weights of elements is minimum (resp. maximum). It is known [11] that the minimum (resp. maximum) weight base changes only if the relative order of weights of some two elements i and j changes.

Since the weight functions of two elements have at most one intersection, we have an $O(m^2)$ trivial upper bound on the number of transitions of the minimum (resp. maximum) weight base of $M(E, \mathcal{B})$. This was improved by [4,7], as will be shown in the following theorem.

Theorem 2.1 ([4,7]) *When all $w_i(t)$ are linear in t, the number of transitions is*

$$O(m \min\{\sqrt{p}, \sqrt{m - p}\}). \tag{1}$$

Next, let us apply the above theorem to analyze the number of transitions of MinST (and MaxST) of a graph with piecewise-linear convex weights.

The weight $w_i(t)$ of an edge i of a graph $G = (V, E)$ is a piecewise linear convex function of a parameter t. Let l_i denote the number of breakpoints of $w_i(t)$, and let

$$M = \sum_{i \in E} (l_i + 1). \tag{2}$$

When t increases from $-\infty$ to $+\infty$, we want to estimate the numbers N_{min} and N_{max} of transitions of MinST and MaxST of G. Notice that MinST (resp. MaxST) changes only if the relative order between the weights for a pair of edges changes. For each pair of edges i and j, the functions $w_i(t)$ and $w_j(t)$ have at most $l_i + l_j + 1$ intersections. Therefore, the trivial upper bound for both of N_{max} and N_{min} is $O(\sum_{i \neq j} l_i + l_j + 1) = O(Mm)$

In order to improve this bound, we construct a multigraph $G' = (V, E')$ from the original graph $G = (V, E)$ in such a way that the vertex set of G' is V, the weight of each edge of G' is linear in t, and the minimum (or maximum) weight base of an appropriate matroid defined on G' changes if (not necessarily only if) the topology of MinST (or MaxST) changes. Thus, apparently the number of transition of the matroid is at least the number of transition of MinST (or MaxST).

The convex function $w_i(t)$ can be thought of as the upper envelope of $l_i + 1$ linear functions. Let such $l_i + 1$ linear functions be

$$z_i^k(t) = a_i^k t + b_i^k, \quad k = 1, \ldots, l_i + 1. \tag{3}$$

The edge set E' consists of l_i+1 multiple edges $e_i^1, e_i^2, \ldots, e_i^{l_i+1}$ connecting two endpoints for each edge i of G. The edge e_j^k has the linear weight $z_i^k(t)$ defined by (3). $|E'| = M$ holds from the definition of M. Fig. 1 illustrates an example of graph G and its corresponding multigraph G'. As illustrated in the left side of the figure, G has three vertices and three edges indexed from 1 through 3, and the weight of each edge is piecewise linear convex with one breakpoint. The corresponding multigraph G' has six edges with linear weights as illustrated in the figure.

Lemma 2.2 *(i) Let C be a subset of E' such that at most one edge among $\{e_i^1, e_i^2, \ldots, e_i^{l_i+1}\}$ does not belong to C for each i, and the set*

$$\{i \in E \mid \text{all edges } e_i^1, e_i^2, \ldots, e_i^{l_i+1} \text{ belong to } C\} \tag{4}$$

is a spanning tree in G. Let $\mathcal{C}$ be the set of all such C's. Then $(E', \mathcal{C})$ is a matroid.

(ii) Let $\mathcal{T}$ be the set of spanning trees of the multigraph G'. Then $(E', \mathcal{T})$ is a matroid.

Proof. Since (ii) is obvious, we prove only (i). Since any $C \in \mathcal{C}$ has $M - (m-n+1)$ edges, the axiom (A1) holds. For the axiom (A2), let us consider $C, C' \in \mathcal{C}$ with $C \neq C'$. Choose an arbitrary $e_i^k \in C - C'$. Let T and T' be two spanning trees defined by (4) for C and C' respectively. The following two cases are possible.

Case 1: $i \notin T$. There exists some $e_i^{k'} \in C' - C$, because there are exactly l_i edges with subscript i in C from $i \notin T$, and at least l_i edges with subscript i in C' from the definition of $\mathcal{C}$. Thus $(C \cup e_i^{k'}) - e_i^k$ again belongs to $\mathcal{C}$.

Case 2: $i \in T$. Since $e_i^k \notin C'$, $i \notin T'$ follows. Thus, there exists a unique path on T' connecting both endpoints of i. Choose an edge j on the path such that $j \notin T$ (such an edge always exists). Then $(T \cup \{j\}) - \{i\}$ is again a spanning tree. Since there exists a unique edge $e_j^{k'}$ such that $e_j^{k'} \in C' - C$ from $j \in T' - T$, $(C \cup e_j^{k'}) - e_i^k$ again belongs to $\mathcal{C}$. □

Theorem 2.3 *For an undirected graph $G = (V, E)$ in which the edge weights are piecewise linear convex functions of a single parameter t, (i) there exist $O(M\sqrt{m})$ transitions of MinST, and (ii) there exist $O(M\sqrt{n})$ transitions of MaxST.*

Proof. (i) Consider the matroid in Lemma 2.2(i). Given an MinST, T, at a certain value t, the set C defined by

$$C = E' - \{e_i^{k'} | i \notin T, z_i^{k'}(t) = \max_{1 \leq k \leq l_i+1} z_i^k(t)\}$$

is a minimum weight base in the matroid as can be easily shown. Conversely, for a minimum weight base in the matroid at a certain t, the corresponding spanning tree defined by (4) is MinST for the same t. Therefore, if MinST changes, the corresponding minimum weight base in the matroid always changes. Thus, since every base in the matroid has $M - (m-n+1)$ elements, the theorem follows from Theorem 2.1.

(ii) Consider the matroid defined in Lemma 2.2(ii). It is clear that if MaxST changes, the corresponding maximum weight base in the matroid always changes. Thus, since every base in the matroid has $n-1$ elements, the theorem follows from Theorem 2.1. □

2.2 Number of transitions of MinST

Based on Theorem 2.3(i), the following theorem is immediate.

Theorem 2.4 $c_1(n, \min) = O(n^3)$.

Proof. Consider a complete graph $G = (S, S \times S)$, where S is the set of n points in the plane, and the length of an edge between two points is measured in L_1 metric. Since the L_1 distance between two points is a piecewise linear convex function in t with at most two breakpoints, we have $M = 3n(n-1)/2$ from (2). Thus, the theorem follows from Theorem 2.3(i). □

This bound is further improved by using the technique developed by Yao [13]. We shall first define an L_1-version of Yao's graph introduced by Yao [13]. For a given t and a given point p_i, we divide the plane into eight regions relative to p_i. The regions are formed by four lines passing through p_i and forming angles of 0°, 45°, 90°, 135°, respectively, with the x-axis. We number the regions counterclockwise, and use $R_l(p_i)$ to denote the set of points in the l-th region (including the boundary), for $1 \leq l \leq 8$. We then have the following lemma:

Lemma 2.5 ([13]) *If p_j and p_k are points in $R_l(p_i)$ for some l, then $d_1(p_j, p_k) \leq \max\{d_1(p_i, p_j), d_1(p_i, p_k)\}$.* □

For each $R_l(p_i)$, let p_k be the one such that $d_1(p_i, p_k) = \min\{d_1(p_i, p_j) \mid j \neq i, p_j \in R_l(p_i)\}$. The point p_k is called the *nearest neighbor to* p_i *in* $R_l(p_i)$. An L_1-version of Yao's graph, $G = (S, E)$, is the one such that S is the set of n points in the plane, and $(p_i, p_j) \in E$ if and only if p_j is the nearest neighbor to p_i in $R_l(p_i)$ for some l with $1 \leq l \leq 8$. $G = (S, E)$ contains at most $8n$ edges.

Lemma 2.6 ([13]) *The edge set E of $G = (S, E)$ contains an MinST in L_1-metric.* □

Since $G = (S, E)$ depends on the parameter t, we shall write it as $G(t) = (V, E(t))$. How many times does $E(t)$ change as t inceases from $-\infty$ to $+\infty$? The lemma below follows from the theory of the upper envelope of line segments [12].

Lemma 2.7 *For each p_i and each l with $1 \leq l \leq 8$, the nearest neighbor to p_i in $R_l(p_i)$ changes $O(n\alpha(n))$ times when t moves from $-\infty$ to $+\infty$, where the function $\alpha(n)$ is the inverse Ackermann's function [5].*

Thus, we have the following lemma:

Lemma 2.8 *The edge set $E(t)$ of $G(t) = (V, E(t))$ changes $O(n^2\alpha(n))$ times.* □

Letting $t_1, t_2, \ldots, t_r$ with $t_1 < t_2 < \cdots < t_r$ be the sequence of t's at which $E(t)$ changes, $[-\infty, +\infty]$ is divided into $O(n\alpha(n))$ disjoint intervals $I_1, I_2, \ldots, I_{n\alpha(n)}$ so that each interval contains $O(n)$ t_k's. Now let us consider the interval I_k and define

$$E_k = \{(p_i, p_j) \mid (p_i, p_j) \in E(t) \text{ for some } t \in I_k\}. \tag{5}$$

Then $|E_k| = O(n)$ follows. Consider the graph $G_k = (S, E_k)$ in which the weight of each edge (p_i, p_j) is equal to $d_1(p_i(t), p_j(t))$. Note that MinST of $G_k = (S, E_k)$ changes at some $t \in I_k$ if and only if MinST for the same point set in the plane changes. From $|E_k| = O(n)$ and Theorem 2.3(i), the number of transitions of MinST of G_k over the interval I_k is $O(n^{3/2})$. Therefore, we have the following theorem:

Theorem 2.9 $c_1(n, \min) = O(n^{5/2}\alpha(n))$. □

The Euclidean (i.e., L_2) MinST is also contained in the edge set of Yao's graph (of L_2 norm). We can easily show that the number of transitions of Yao's graph is $\lambda_4(n) = \Theta(n^2 2^{\alpha(n)})$ for L_2 norm by using the result of [5], where $\lambda_4(n)$ is the maximum length of a Davenport-Schinzel sequence of order 4. Thus, by applying the argument similar to the one given after Lemma 2.8, we immediately obtain the following:

Proposition 2.10 $c_2(n, \min) = O(n^3 2^{\alpha(n)})$.

2.3 Number of transitions of MaxST

As a direct consequence of Theorem 2.3(ii), we get $c_1(n, \max) = O(n^{5/2})$ by the same argument as in the proof of Theorem 2.4. This upper bound is further improved to $O(n^2)$. We shall also prove that $c_1(n, \max) = \Omega(n^2)$. Thus we establish the tight bound $c_1(n, \max) = \Theta(n^2)$.

Theorem 2.11 $c_1(n, \max) = \Theta(n^2)$.

Proof. The $\Omega(n^2)$ lower bound is easily given, even in the one-dimensional case (the details are omitted in this version). We show the $O(n^2)$ upper bound for the planar case. It is not difficult to generalize it for any fixed-dimensional case. As shown by [9], MaxST contains the furthest neighbour graph (FNG). The L_1 hull of S is the set of points which maximizes one of the linear form $x + y$, $x - y$, $-x + y$, and $-x - y$. From the definition, the L_1 furthest neighbour of a point of S is located on the L_1 hull. It is easy to see that the number of transitions of the L_1 hull is $O(n)$, and that the number of transitions of the FNG is $O(n^2)$.

The FNG contains at most two connected components. Let l be the longest distance between the connected components. Then, l is the distance between a point in the L_1 hull of one component and a point in the L_1 hull of the other. MaxST of S is the union of the furthest neighbour graph and l. The number of transitions of the L_1 hull of each connected component is $O(n^2)$. Since at most four points are located on the L_1 hull if the points are in general position, the edge l is changed $O(1)$ times for a fixed topology of the L_1 hulls of components. Thus, we obtain the $O(n^2)$ upper bound. □

2.4 Restricted cases

We shall consider in this section two restricted cases: the c-oriented case and the case where only k points move. We are interested in the case where c and k are small compared with n. In order to deal with the c-oriented case, we shall first give the following lemma which is a counterpart of Theorem 2.1 for the c-oriented case.

Lemma 2.12 *Let $M(E,\mathcal{B})$ be a matroid with m elements in which the weight of each element is linear in t, and suppose that there are only c' distinct slopes among all weight functions. Then the number of changes of the minimum (resp. maximum) weight base is $O((c'mp)^{1/2}+c'p)$, where p denotes the number of elements in a base.*

Now the number of transitions of MinST in the c-oriented case can be analyzed in the same fashion as in Section 2.2. It is easy to show that the number of transitions of Yao's graph is $O(cn^2)$ for the c-oriented case. Thus we establish the following theorem from Lemma 2.12:

Theorem 2.13 $c_1(n,\min)=O(c^3n^2)$ *holds in the c-oriented case.* □

The above bound is tight for fixed c, since it is easy to show the $\Omega(n^2)$ lower bound for the 2-oriented case.

Now let us analyze the number of transitions of MaxST in the c-oriented case. Consider the complete graph $G=(S,S\times S)$ defined in the proof of Theorem 2.4, and the corresponding multigraph G' introduced in Section 2.1. It is easy to see that there are $O(c^2)$ distinct slopes among $O(n^2)$ edge weights. Thus, from Lemma 2.12, we have the following theorem:

Theorem 2.14 $c_1(n,\max)=O(cn^{3/2})$ *holds in the c-oriented case.* □

We shall now consider the case where there are only k moving points. Other points are called *fixed*. Let S' and S'' be the sets of k moving points and $n-k$ fixed points. Let MaxST(S'') (resp. MinST(S'')) be the MaxST (resp. MinST) for the point set S''. This does not change even if t changes, since the points in S'' are fixed. MaxST (resp. MinST) for any t is contained in the set of the union of MaxST(S'') (resp. MinST(S'')) and edges connecting the points in S' and $S(=S'\cup S'')$. There are $O(kn)$ edges in this set. Furthermore, since the situation can be regarded as the $(k+1)$-oriented case, we have the following theorems from Lemma 2.12:

Theorem 2.15 $c_1(n,\max)=O(k^2n)$ *holds when only k points linearly move.* □

Theorem 2.16 $c_1(n,\min)=O(k^3n)$ *holds when only k points linearly move.* □

3 Finding the smallest MaxST

It is an interesting problem to find the value of t when the MaxST of linearly moving points satisfies some minimality condition. In this section, we give efficient algorithms for finding the value of t when the total edge length of MaxST becomes minimum.

Theorem 3.1 *We can find the value of t when the total edge length of MaxST becomes minimum in $O(n\log^2 n)$ time and $O(n\log^4 n)$ time for L_1 and L_2 metrics, respectively.*

Proof. First, note that the total length of a given spanning tree is a convex function in t. Thus, the total length of MaxST is also convex in t, since it is an upper envelope of convex functions each of which corresponds to the total length of a spanning tree. Therefore, the optimal value t^* can be found as the supremum of t such that the slope of the function representing the total length of MaxST at t is negative. Thus, for a given t, we can tell whether $t^*<t$, $t^*>t$, or $t^*=t$ in $O(n\log n)$ time by computing the MaxST at t by using the algorithm given by [9] (for L_2 metric). The time complexity can be reduced to $O(n)$ for the L_1 case.

Next, by directly parallelizing the algorithm of [9], we obtain an $O(\log n)$ time and an $O(n)$ processor algorithm to compute the L_1 MaxST, and an $O(\log^2 n)$ time and an $O(n)$ processor algorithm to compute the L_2 MaxST. We use $O(\log^2 n)$ time and $O(n/\log n)$ processor algorithm to compute the furthest Voronoi diagram of Aggarwal [1].

From the above observations, it is now an easy exercise to apply Megiddo's parametric search [8] in order to obtain the results. □

Notice that these results are valid only for $d=2$, since no parallel algorithm with the above running time is known for the general d-dimensional case.

4 Concluding remarks

We have investigated the upper bounds on the number of transitions of dynamic MinST and MaxST of points moving linearly in a fixed-dimensional space. For linear metrics, we have obtained a tight bound $\Theta(n^2)$ for the MaxST case. On the other hand, for the MinST case, there is still a gap of $\sqrt{n}\alpha(n)$ factor between the lower and upper bounds. We conjecture that the bound for the MinST is also $\Theta(n^2)$.

It is important to investigate the problem for the Euclidean L_2 metric. So far, we have only been able to show an $O(n^3 2^{\alpha(n)})$ bound for MinST, and a trivial $O(n^4)$ bound for MaxST. However, we believe that these bounds will significantly be improved in future.

We also investigated the problem of finding the minimum length of MaxST for moving points, and proposed an efficient algorithm with $O(n \log^4 n)$ running time for L_2 metric. The MinST version of this problem is much important in practical applications. However, the total length of MinST is neither concave nor convex in t, and it is left for future research to design subquadratic algorithms for finding the value of t minimizing the total edge length of the dynamic MinST.

Acknowledgements

The authors would like to thank Prof. Hiroshi Imai for his helpful comments.

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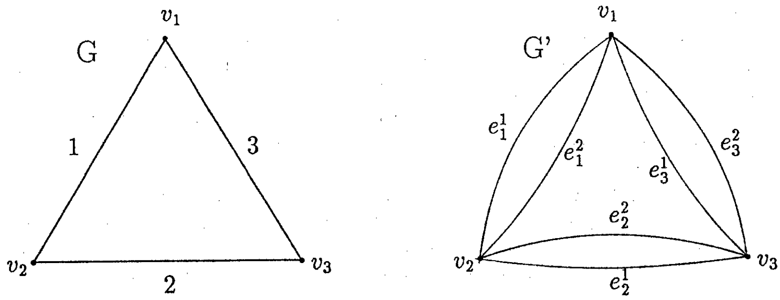

$$w_1(t) = |t - 1| \qquad z_1^1(t) = -t + 1, \quad z_1^2(t) = t - 1$$

$$w_2(t) = |2t - 4| \qquad z_2^1(t) = -2t + 4, \quad z_2^2(t) = 2t - 4$$

$$w_3(t) = |3t - 5| \qquad z_3^1(t) = -3t + 5, \quad z_3^2(t) = 3t - 5$$

Fig. 1 An example of graphs G and G'.