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イメージ切り出しに関するアルゴリズム

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要旨. 濃淡画像からイメージを切り出すことはパターン認識や画像処理で大切な問題である（図1）。本論文では連結したイメージを画像から取り出す問題で、基礎的な場合について多項式時間で解くアルゴリズムを与える。取り扱うのは、イメージの境界が一本ないしは二本の横方向に単調な曲線である場合である（二本の場合、可容イメージ切り出しと言う）。一本の場合は $O(n^2)$ 二本の場合は $O(n^{2.5})$ の時間で解くアルゴリズム、及び、二本の場合に、ϵ 誤差で近似する高速 ($O(\epsilon^{-1} n \log\log n \log(nH)$ 時間) アルゴリズムを与える。

Polynomial-time Solutions to Image Segmentation

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Abstract.

Detecting an object in an image is a central problem in pattern recognition and computer vision. This paper presents a polynomial-time algorithm for computing an optimal separation of a connected region from the background in the criterion due to discriminant analysis in two basic cases. In one case an image is partitioned into two parts by $x-$monotone curve so that the interclass variance is maximized. In the other it is separated by two $x-$monotone curves. The latter problem is named admissible partition, and equivalent to separating a connected region which is vertical-convex (that is, intersection with each column is connected). Proposed algorithms run in $O(n^2)$-time and $O(n^{2.5})$-time, respectively, for an image consisting of n pixels. We also give an efficient approximation scheme to obtain an ϵ-approximation solution in $O(\epsilon^{-1} n \log\log n \log(nH))$ time, where H is the total sum of brightness levels of an image.

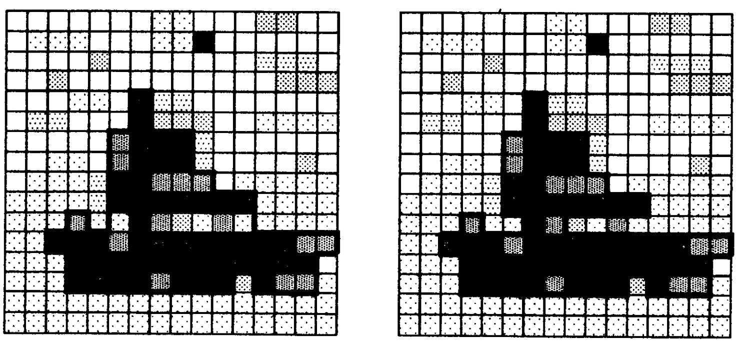

図 1: イメージ切り出し問題（左が可容イメージ切り出し）

1 Introduction

One of the most important operations that should be performed by a computer vision system is the separation of objects from the background. This operation is commonly called "segmentation". For the segmentation of intensity images, there are five main approaches [3]: threshold techniques, edge-based methods, region-based methods, hybrid techniques, and connectivity-preserving relaxation methods.

(A) The threshold techniques [16] are effective only if all pixels in objects have brightness levels within a certain range which can be distinguished from those of the background. Since all of the spatial information is neglected, they do not cope with blurring at region boundaries. (B) The edge-based methods [5] highly depend on edge-detection. A difficulty is in how to connect disconnected edges to form a closed curve especially in blurred portion. (C) A typical region-based method [17, 8] consists of the following steps: Partition an entire image into connected regions by grouping neighboring pixels having similar brightness levels, and merge two adjacent regions under some criterion such as homogeneity [15, 4] or weakness of region boundaries. A strict criterion generally leads to creation of many small regions, while a loose one may easily merge two regions which should be separated but adjacent by blurred boundaries. (D) There are also several methods [13, 3] in which the above two criteria are combined. (E) "Snakes: Active Contour Models" [10] starts with some initial shape of boundary represented by Spline curves and changes their shape according to some energy function. One of the disadvantages of the method is that it may fall into a local optimum.

Different from those previous ones, our basic standpoint is to formulate the segmentation problem as an optimization problem under certain geometric constraints. One important constraint on the segment is the connectivity. Unofortunately, the following problem is NP-hard.

Connected-image segmentation problem: Given a positive integer $K \leq n$, find a connected region of G containing K pixels which maximizes total sum of brightness levels.

The NP-hardness can be proven by reducing to the connected covering problem of a planar graph with maximum degree 4 [6]. Accordingly, it looks difficult to solve optimization problems on the segmentation under the connectivity constraint.

Hence, we strengthen the constraint to be vertical-convex as well as connected, and solve the associated optimization problems in polynomial time. Here, a region S is called *vertical-convex* if the intersection of S with any vertical line is connected. Equivalently, the boundary of S consists of two $x-$monotone chains. In short, we call a region *admissible* if it is vertical-convex and connected. We mainly consider the following problem:

Admissible-image segmentation problem: Partition an image into an admissible region S_0 and its complement S_1 so as to maximize *interclass variance* $V(S_0, S_1)$ (see Section 2 for its definition) used in discriminant analysis [7].

We first consider an easier case where the boundary of S_0 consists of a single x-monotone chain (together with boundary of the grid). We present $O(n^2)$-time and $O(n)$-space algorithm for finding an optimal partition for an image consisting of n pixels. Then, we extend the algorithm so that an image is partitioned by two x-monotone chains into an admissible region and its complement. The proposed algorithm runs in $O(n^{2.5})$ time. We also give an efficient algorithm to compute an ϵ-approximation solution in $O(\epsilon^{-1} n \log\log n \log(nH))$ time, where H is the total sum of the brightness levels.

2 Segmentation Problem

Let G be an $N \times N$ grid plane, i.e., $G = \{(i,j) \mid i = 1,2,\ldots,N,\ j = 1,2,\ldots,N\}$ and g_{ij} be the brightness level of a lattice point (i,j) of a given image on G.

Let n denote the number of pixels of G. Throughout the paper, we assume that we are interested in all pixels in the grid, thus $n = N \times N$. The image is usually required to be connected.

We want to find a partition (S_0, S_1) of G satisfying certain connectivity constraints so that a given objective function is maximized. Let $\mu = (\sum_{(i,j)\in G} g_{ij})/n$ be the average brightness level over the whole image. Let n_i and μ_i $(i = 0,1)$ be the cardinality of a set S_i and the average brightness level in the set S_i, respectively. Formally, they are defined by $n_0 = |S_0|$, $n_1 = |S_1|$, $\mu_0 = \frac{1}{n_0}\sum_{(i,j)\in S_0} g_{ij}$, $\mu_1 = \frac{1}{n_1}\sum_{(i,j)\in S_1} g_{ij}$.

A typical objective function is

$$V(S_0, S_1) = n_0(\mu - \mu_0)^2 + n_1(\mu - \mu_1)^2,$$

which is called the ***interclass variance*** in ***discriminant analysis*** [7]. The interclass variance is proportional to the sum of squares of ***standardized means*** [12].

The maximization of $V(S_0, S_1)$ is known to be equivalent [7] to the minimization of the ***intraclass variance*** $U(S_0, S_1) = \sum_{(i,j)\in S_0}(g_{ij} - \mu_0)^2 + \sum_{(i,j)\in S_1}(g_{ij} - \mu_1)^2$, which is another typical objective function used in clustering (e.g.[9]).

3 Optimal separation by a monotone chain

3.1 Rewriting the objective function

The objective function $V(S_0, S_1)$ can be rewritten as $n_0 n_1(\mu_0 - \mu_1)^2/n$ (we omit the details). Thus, our problem can be written as

$$\text{maximize } D(S_0, S_1) \equiv \sqrt{n_0 n_1}|\mu_0 - \mu_1| = \sqrt{n_0/n_1}|(1 + n_1/n_0)\textstyle\sum_{(i,j)\in S_0} g_{ij} - H|,$$

where $H = \sum_{(i,j)\in G} g_{i,j}$.

Let $P(k)$ denote the subproblem that is to maximize the above objective function under the constraint that n_0 is fixed to k. The algorithm solves subproblems $P(k)$ for all $k = 1,2,\ldots,n$. Among those solutions, an optimal solution to the original problem is included. We ignore the connectivity of S_0 and S_1 for a while, and show an algorithm for solving $P(k)$.

For each $j = 1,2,\ldots,N$, define $f_j(x_j) = \sum_{i=1}^{x_j} g_{ij}$, $x_j = 0,1,\ldots,N$. The sum is understood to be 0 if $x_j = 0$. Letting $t = n_1/n_0$, $P(k)$ can be formulated as follows:

$$P(k) : \text{maximize } \sqrt{1/t}\ \ |(1+t)\sum_{j=1}^{N} f_j(x_j) - H| \quad \text{subject to } \sum_{j=1}^{N} x_j = k.$$

In this formulation, x_j is a decision variable associated with the jth column implying that in the jth column in the grid plane G, lattice points $(1,j),(2,j),\ldots,(x_j,j)$ belong to S_0, while $(x_j+1,j),(x_j+2,j),\ldots,(N,j)$ belong to S_1.

It is clear that $P(k)$ can be solved by computing optimal solutions of the following two problems $P_+(k)$ and $P_-(k)$, and taking the one with the larger objective value.

$P_+(k)$: maximize $\sqrt{1/t}((1+t)\sum_{j=1}^N f_j(x_j) - H)$ subject to $\sum_{j=1}^N x_j = k$.
$P_-(k)$: maximize $\sqrt{1/t}(H - (1+t)\sum_{j=1}^N f_j(x_j))$ subject to $\sum_{j=1}^N x_j = k$.

We shall concentrate on how to solve $P_+(k)$. Since t, k, and H are constants, the objective function of $P_+(k)$ is a function on $k\mu_0 = \sum_{j=1}^N f_j(x_j)$, and monotonically increasing with respect to it. Thus, $P_+(k)$ is equivalent to the following key problem:

Key problem. maximize $\sum_{j=1}^N f_j(x_j)$ subject to $\sum_{j=1}^N x_j = k$.

3.2 Dynamic programming algorithm

To solve the key problem, we define its restricted case where the image S_0 intersects only the first m columns of G.

$$F(k,m) = \max[\ \sum_{j=1}^m f_j(x_j) | \sum_{j=1}^m x_j = k\ ].$$

$F(k,N)$ together with the associated values of x_j for $j = 1,2,..N$ gives the solution of the key problem. It is easy to see the following inductive formula:

$$F(k,m) = \max_{x_m=0,\ldots,N}[f_m(x_m) + F(k-x_m, m-1)].$$

This formula leads to a dynamic-programming-based algorithm for computing $F(k,N)$. Since $m \le N$, $x_m \le N$, and $k \le n$, the complexity of the dynamic programming is $O(N^2 n) = O(n^2)$.

In the algorithm shown above, we ignored connectivity of image segment S_0. However, it is not difficult to fix the algorithm so that it outputs a connected segmentation.

Theorem 1 *Given an image with n pixels, an $x-$monotone path to give an optimal partition can be computed in $O(n^2)$ time and $O(n)$ space. Moreover, it can be computed in $O(\log^2 n)$ time using $O(n^2/\log^2 n)$ processors on an EREW PRAM.*

4 Segmentation by two monotone chains

If two monotone chains cuts a connected region S_0, the partition induced by such monotone chains is called an *admissible partition*. The image segment becomes an admissible image.

We design an algorithm so that it can find an optimal admissible partition with respect to the interclass variance. S_0 is the image-segment, and $S_1 = G - S_0$ is the background.

For each $j = 1,2,\ldots,N$, and for each interval $I \subset [0,N]$, we define $f_j(I) = \sum_{i\in I} g_{ij}$. We allow an interval to be empty. $|I|$ is the number of integers in the interval.

Letting $t = n_1/n_0$, we consider the problem under the cardinality condition, for which we use the same notation $P_+(k)$ as that for the problem in the previous section, since it will not cause any confusion.

$$P_+(k) : \text{maximize } \sqrt{1/t}\{(1+t)\sum_{j=1}^N f_j(I_j) - H\}$$

subject to (i) $\sum_{j=1}^N |I_j| = k$, and (ii) $1 \le \exists a < \exists b \le N$ such that $I_j = \emptyset$ if $j < a$ or $j > b$, and $I_j \cap I_{j+1} \ne \emptyset$ if $a \le j \le b-1$.

In this formulation, if $I_j = [x_j, y_j]$. x_j and y_j are decision variables associated with the jth column implying that lattice points $(x_j, j), (x_j+1, j), \cdots, (y_j, j)$ belong to S_0, while the other points in the column belong to S_1. It is clear from (ii) that a solution of $P_+(k)$ is an admissible partition.

Similarly, we define $P_-(k)$. We shall now concentrate ourselves on how to solve $P_+(k)$.

4.1 Naive implementation

To solve the problem using dynamic programming, we define $F[k, I, m]$ to be the maximum weight of S_0 over all admissible partitions of the columns 1 through m under the constraint that the m-th column of S_0 is an interval I (I can be an empty set). We also define $F(k, t, m)$ to be the maximum weight of S_0 over all admissible partitions of the columns 1 through m under the constraint that the pixel (t, m) must be included in the set S_0. For convenience's sake, we introduce $F(k, 0, m)$ to be the maximum weight of S_0 over all admissible partitions of the column 1 through m containing no pixel in the mth column.

Since $\max_{S_0,|S_0|=k} \sum_{(ij)\in S_0} g_{ij} = \max_{t=0,1,..,N} F(k, t, N)$, it suffices to compute $F(k, t, N)$ for all k and t.

$F(0, t, m) = F[0, I, m] = 0$. For $k > 0$, $F(k, t, m)$ and $F[k, I, m]$ can be computed as follows:

$$F(k, t, m) = \max_{t\in I} F[k, I, m]. \quad (1)$$

Here, we adopt a convention that $0 \in \emptyset$.

$$F[k, I, m] = f_m(I) + \max_{l\in I} F(k - |I|, l, m - 1) \text{ if } I \neq \emptyset \quad (2)$$
$$F[k, \emptyset, m] = \max_{l=0,1,..,N} F(k, l, m - 1).$$

These recursions lead to a dynamic programming algorithm.

There are $O(N^2)$ possible choice of I and $O(N)$ possible choice of l to compute $F(k, t, m)$. The number of combinations of k, t, and m becomes $O(N^4)$. Hence, a naive implementation of such dynamic programming procedure runs in $O(N^7) = O(n^{3.5})$ time.

4.2 Efficient implementation

Let us efficiently compute $F[k, I, m]$ by using formula (2). We define $w(k, I, m) = \max_{l\in I} F(k, l, m-1)$. Then, (2) is written as $F[k, I, m] = f_m(I) + w(k - |I|, I, m)$. Thus, it suffices to compute $w(q, I, m)$ for all I, q and m.

We have the following formula: $w(q, [i, j + 1], m) = \max\{w(q, [i, j], m), F(q, j + 1, m - 1)\}$

We classify the intervals by their starting indices. The $i-$th group is $\{[i, i], [i, i+1], \ldots, [i, N]\}$. Among each group, we compute $w(q, I, m)$ by using the fórmula (3). which can be processed in $O(N)$ time for fixed q and m. Thus, the total time complexity for computing (2) for all k, I and m is $O(N^5) = O(n^{2.5})$.

Next, we consider efficient computation of the formula (1) for fixed k and m. For each interval I, we consider a horizontal segment in the plane whose y-coordinate is $F[k, I, m]$ and projection on the x-axis is I. Then, the computation of formula (1) is equivalent to a visibility problem: "For each of x-coordinate values $\{1, 2, ..N\}$, report the highest segment whose projection contains the value," and can solved in $O(N^2)$ time.

Hence, formula (1) can be computed in $O(N^5) = O(n^{2.5})$ time for all k and m.

Theorem 2 *Given an image in a grid with n pixels. the optimal admissible partition can be computed in $O(n^{2.5})$ time and $O(n^{1.5})$ space.*

5 A Fast Approximation Algorithm

In this section, we give a fast approximation scheme to compute a solution S_0 such that $D(S_0, S_1) \geq (1-\epsilon)D_{opt}$ in $O(\epsilon^{-1} n \log\log n \log(nH))$ time, where $D(S_0, S_1) = \sqrt{V(S_0, S_1)}$ as defined in Section 2. Here, D_{opt} is the optimal value of D, and $H = \sum_{(i,j)\in G} g_{i,j}$.

We formulate the segmentation problem as a parametric optimization problem, and devise the approximation algorithm by using it. Let us consider the cost function $U_\theta(S_0) = n\sum_{(i,j)\in S_0} g_{i,j} - \theta|S_0|$ for a parameter θ.

Problem $Q(\theta)$: Compute the admissible image S_0 which maximizes $U_\theta(S_0)$.

A proof of the following theorem can be found in [2].

Theorem 3 *$Q(\theta)$ can be solved in $O(n \log\log n)$ time.*

We consider relation between U_θ and $D(S_0, S_1)$ as target functions.

Lemma 1 *Let S_0 be the set maximizing $D(S_0, S_1)$. Then, S_0 maximizes U_θ for a θ.*

Proof: Maximizing $D(S_0, S_1)$ is equivalent to maximizing $\sqrt{|S_0||S_1|}(\mu_0 - \mu_1)$. Let us consider the function $F_\theta(k) = \max_{|S_0|=k} U_{H+\theta}(S_0)$ for $k = 0, 1, .., n$. It is easy to veryfy that $U_H(S_0) = \sqrt{|S_0|(|n - |S_0|)}D(S_0, S_1)$.

Suppose the maximum value D_{opt} of $D(S_0, S_1)$ is taken on S_0^* such that $|S_0^*| = \nu$, and consider (x, y)−plane in which all points $(x, F_0(x)), x = 1, 2, \ldots, n-1$ are plotted. Then, the curve $y = D_{opt}(x(n-x))^{1/2}$ touches $(x, F_0(x))$ at $x = \nu$ because it is clear that $F_0(\nu) = \sqrt{|S_0^*||n - S_0^*|}D(S_0^*, G - S_0^*)$, and all the other points $(x, F_0(x))$ lie below (or on) the curve $y = d(x(n-x))^{1/2}$ from the maximality of D_{opt}.

Since the second derivative of $D_{opt}\sqrt{x(n-x)}$ is negative if $0 < x < n$, the curve $y = D_{opt}\sqrt{x(n-x)}$ is concave in x. Hence, all points $(x, F_0(x))$ lie below (or on) the tangent line l of this curve at $(\nu, F_0(\nu))$. Let θ^* be the slope of l. Then, $F_{\theta^*}(k)$ is maximized at $k = \nu$, and the set maximizing $U_{H+\theta^*}$ is S_0^*. □

From Lemma 1, maximizing $D(S_0, S_1)$ reduces to solving the parametric problem $Q(\theta)$ for all possible values of θ. However, there are too many possible values of θ, and hence we shall propose an efficient algorithm to obtain an ϵ-approximation solution. Its running time is $O(\epsilon^{-1} n \log\log n \log(nH))$. Namely, the proposed algorithm solves $Q(\theta)$ for $O(\epsilon^{-1}\log(nH))$ distinct values, and chooses the best one. The idea of the algorithm is similar to the one given in [11]. This algorithm is much faster than previous one when n is large, and is useful for practical purposes. The algorithm is based on the following lemma.

Lemma 2 *Let θ^* denote the optimal parameter value with which an optimal solution S_0 of $Q(H+\theta^*)$ maximizes $D(S_0, S_1)$. If $\theta^* \neq 0$, then an optimal solution of $Q(H + (1+\epsilon)\theta^*)$ produces an ϵ−approximate solution of our segmentation problem.*

Proof. Let $\nu = |S_0^*|$ and D_{opt} be those defined in the proof of the previous lemma. As shown in the proof of the previous lemma, $F_0(\nu) = D_{opt}\sqrt{\nu(n-\nu)}$. The derivative θ^* of the curve

$y = D_{opt}\sqrt{x(n-x)}$ at $x = \nu$ is $\frac{D_{opt}(n-2\nu)}{2\sqrt{\nu(n-\nu)}}$. The tangent line of the curve $y = D_{opt}\sqrt{x(n-x)}$ at $(\nu, F_0(\nu))$ is given by

$$y = \frac{D_{opt}(n-2\nu)}{2\sqrt{\nu(n-\nu)}}(x-\nu) + D_{opt}\sqrt{\nu(n-\nu)}.$$

As shown in the proof of Lemma 5, S_0^* maximizes $U_{H+\theta^*}$. Now let us consider the subset S_0^ϵ that maximizes $U_{H+(1+\epsilon)\theta^*}$. Then we claim that $(S_0^\epsilon, G - S_0^\epsilon)$ is an ϵ−approximate solution of our segmentation problem considered in previous sections. In order to prove this, let us consider a line in (x,y)-plane whose slope is $(1+\epsilon)\theta^*$. Among all points $(x, F_0(x))$ $x = 1, 2, \ldots, n-1$, consider the one that maximizes $-(1+\epsilon)\theta^* x + F_0(x)$ (i.e., the one corresponding to an optimal solution of $Q(H-(1+\epsilon)\theta^*)$). Let it be $(\nu', F_0(\nu'))$. We shall show that $D' = F_0(\nu')/\sqrt{\nu'(n-\nu')}$ is at least $(1-\epsilon)D_{opt}$ (which proves our claim). From the maximality of $(\nu', F_0(\nu'))$, the point $(\nu, F_0(\nu))$ is below the line $y = \frac{(1+\epsilon)D_{opt}(n-2\nu)}{2\sqrt{\nu(n-\nu)}}(x-\nu') + F_0(\nu')$. Hence, the point $(\nu', F_0(\nu'))$ is above (or on) the line

$$l : y = \frac{(1+\epsilon)D_{opt}(n-2\nu)}{2\sqrt{\nu(n-\nu)}}(x-\nu) + D_{opt}\sqrt{\nu(n-\nu)}.$$

Since $(\nu', F_0(\nu'))$ is on the curve $\gamma : y = D'\sqrt{x(n-x)}$, the concave curve γ intersects the line l. Thus, the minimum r_{min} of the values $y/D_{opt}\sqrt{x(n-x)}$ on the line l gives a lower bound on D'/D_{opt}. We shall show that r_{min} is at least $1-\epsilon$.

$$r_{min} = \min\{\frac{1}{\sqrt{x(n-x)}}(\frac{(1+\epsilon)(n-2\nu)}{2\sqrt{\nu(n-\nu)}}(x-\nu) + \sqrt{\nu(n-\nu)}) \mid 0 < x < n\}. \tag{3}$$

The term to be minimized in (3) is rewritten into

$$(a\sqrt{x/(n-x)} + b\sqrt{(n-x)/x})/(2\sqrt{\nu(n-\nu)}). \tag{4}$$

Here, $a = (n-\nu)(n+\epsilon(n-2\nu))/n, b = \nu(n-\epsilon(n-2\nu))/n$.

Letting $t = \sqrt{x/(n-x)}$, the minimum of (4) is attained at $t = \sqrt{b/a}$. Thus, the minimum value r_{min} of (3) is $\sqrt{ab}/\sqrt{\nu(n-\nu)} = \sqrt{n^2 - \epsilon^2(n-2\nu)^2}/n \geq \sqrt{1-\epsilon^2} \geq 1-\epsilon$. This proves the lemma. □

From the assumption of the integrality of $g_{i,j}$, $|\theta^*|$ satisfies $1/2 \leq |\theta^*| \leq nH/2$, unless $\theta^* \neq 0$. Let us define the sequence $\{\theta_l, -\theta_l\}$ of parameters θ by $\theta_l = (1+\epsilon)^{l-1}/2$ for $0 \leq l \leq K = \lfloor \log(nH)/\log(1+\epsilon) \rfloor$.

Then, the approximate algorithm solves parametric problem $P(\theta)$ for all values of the above defined sequence and chooses the one that maximizes $D(S_0, S_1)$. It is clear from Lemma 2 that such a solution is an ϵ−approximation of our problem. The number of parameters generated as above is $O(\log(nH)/\log(1+\epsilon))$. The running is $O(\epsilon^{-1} n \log\log n \log(nH))$, since $\log(1+\epsilon) \approx \epsilon$ for a small ϵ.

6 Discussions

In this paper we have discussed the problem of detecting an object region with x−monotone boundary in an intensity image. We presented polynomial-time algorithms for detecting an optimal region in the sense of the discriminant analysis. To the authors' knowledge, this is the first computational-geometric attempt to the region segmentation problem.

An issue to be pointed out is that the monotonicity of object boundaries is a constraint far from practical applications. However, if we imagine a small floating window over an image in which optimal monotone boundaries are computed, we may obtain a new image-filtering or edge-detection scheme with confidence weighted by inter-cluster distance. We are now implementing experiments in this direction and the results will be presented at the symposium.

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