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# 積グラフの独立全域木について

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あらまし

グラフ $G$ が頂点 $r$ を根とする $n$ 本の独立全域木が存在するならば, $G$ を $r$ での $n$ チャネルグラフという. 全ての頂点 $u$ に対して, $G$ が $u$ での $n$ チャネルグラフならば, $G$ を単に $n$ チャネルグラフという. 独立全域木は, 耐故障ブロードキャスティングにおいて重要である。$G$ が独立全域木についてある条件を満たすならば, well-formed という. 本稿では, $G_1$ が well-formed $n_1$ チャネルグラフで $G_2$ が well-formed $n_2$ チャネルグラフならば, $G_1 \times G_2$ は, well-formed $(n_1+n_2)$ チャネルグラフであることを示す. $G_1$ の $n_1$ 本の独立全域木 と $G_2$ の $n_2$ 本の独立全域木から $G_1 \times G_2$ の $n_1+n_2$ 本の独立全域木を構成することにより証明する.

キーワード

ブロードキャスト，チャネルグラフ，耐故障，独立全域木，積グラフ

# Independent Spanning Trees of Product Graphs

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## Abstract

A graph $G$ is called an $n$-channel graph at vertex $r$ if there are $n$ independent spanning trees rooted at $r$. A graph $G$ is called an $n$-channel graph if for every vertex $u$, $G$ is an $n$-channel graph at $u$. Independent spanning trees of a graph play an important role in fault-tolerant broadcasting in the graph. A graph $G$ is said to be well-formed if $G$ satisfy a certain condition about its independent spanning trees. In this paper we show that if $G_1$ is a well-formed $n_1$-channel graph and $G_2$ is a well-formed $n_2$-channel graph, then $G_1 \times G_2$ is a well-formed $(n_1+n_2)$-channel graph. We prove this fact by constructing $n_1+n_2$ independent spanning trees of $G_1 \times G_2$ satisfying the condition from $n_1$ independent spanning trees of $G_1$ and $n_2$ independent spanning trees of $G_2$.

**key words** broadcasts , channel graphs , fault-tolerance , independent spanning trees , product graphs

## 1 Introduction

For a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is a graph with the vertex set $V_1 \times V_2 = \{(x,y) \,|\, x \in V_1, y \in V_2\}$ and the edge set such that two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$, or $u_2 = v_2$ and $u_1v_1 \in E_1$. The definition of the product of two graphs can be generalized to the product of $n$ graphs in the natural way. That is, $G_1 \times G_2 \times G_3$ is $(G_1 \times G_2) \times G_3$ or $G_1 \times (G_2 \times G_3)$. Note that $(G_1 \times G_2) \times G_3$ and $G_1 \times (G_2 \times G_3)$ are isomorphic. The product of $n$ graphs $G_1 \times G_2 \times \cdots \times G_n$ means $(G_1 \times \cdots \times G_k) \times (G_{k+1} \times \cdots \times G_n)$ for some $k$ $(1 \le k \le n-1)$. Each $G_i$ $(1 \le i \le n)$ is called a component of $G_1 \times G_2 \times \cdots \times G_n$.

Some of popular interconnection networks are product graphs. For example, the $n$-dimensional hypercube $Q_n$ is $Q_{n-1} \times K_2 = Q_{n-2} \times K_2 \times K_2 = \cdots = K_2 \times K_2 \times \cdots \times K_2$, and an $n$-dimensional generalized hypercube $Q_n^t$ is $Q_{n-1}^t \times K_t = Q_{n-2}^t \times K_t \times K_t = \cdots = K_t \times K_t \times \cdots \times K_t$, where $K_t$ is the complete graph of order $t$. The $(m_1 \times \cdots \times m_n)$-mesh is $L_{m_1} \times \cdots \times L_{m_n}$, and the $(m_1 \times \cdots \times m_n)$-torus is $R_{m_1} \times \cdots \times R_{m_n}$, where $L_i$ and $R_i$ are a linearly linked graph of order $i$ and a ring of order $i$, respectively. The hyper de Bruijn graph $HD(m,n)$ is $Q_m \times D_n$, and the hyper Petersen graph $HP_n$ is $Q_{n-3} \times P$, where $D_n$ and $P$ are the binary de Bruijn graph of order $2^n$ and the Petersen graph, respectively.

Denote by $d_G(x,y)$ the distance between $x$ and $y$ in $G$, by $D(G)$ the diameter of $G$, by $d_{avg}(G)$ the average distance between vertices in $G$, and by $c(G)$ the vertex connectivity. Youssef [4] showed that for a pair of graphs $G_1$ and $G_2$, $d_{G_1 \times G_2}((x_1,x_2),(y_1,y_2)) = d_{G_1}(x_1,y_1) + d_{G_2}(x_2,y_2)$, $D(G_1 \times G_2) = D(G_1) + D(G_2)$, $d_{avg}(G_1 \times G_2) = d_{avg}(G_1) + d_{avg}(G_2)$, and $c(G_1 \times G_2) = c(G_1) + c(G_2)$.

Two spanning trees of a graph $G = (V,E)$ are called independent if they are rooted at the same root $r$, and for each vertex $v$ in $V$, the two paths from $r$ to $v$, one path in each tree, are internally vertex disjoint. A graph $G$ is called an $n$-channel graph at vertex $r$ if there are $n$ independent spanning trees of $G$ rooted at $r$. If $G$ is an $n$-channel graph at every vertex of $G$, we call $G$ an $n$-channel graph. For example, $R_3 \times R_3$ is an 4-channel graph, and 4 independent spanning trees of $R_3 \times R_3$ are shown in Figure 1. Itai and Rodeh [3] gave a linear time algorithm

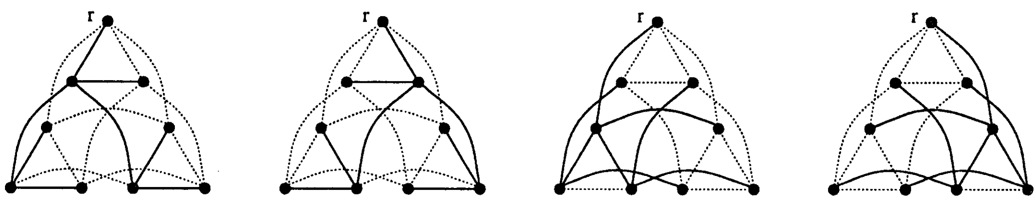


Figure 1: 4 Independent spanning trees of $R_3 \times R_3$.

for finding two independent spanning trees in a biconnected graph. Cheriyan and Maheshwari [2] showed how to find three independent spanning trees of $G = (V,E)$ in $O(|V||E|)$ time. Zehavi and Itai [5] also showed that for any 3-connected graph $G$ and any vertex $r$ there are three independent spanning trees rooted at $r$. They conjectured in [5] that any $k$-vertex connected graph has $k$ independent spanning trees rooted at an arbitrary vertex $r$. This conjecture is still open for any $k > 3$.

It has been shown that broadcasting through independent spanning trees are efficient and reliable [1] [3]. In fact, if $G$ is an $n$-channel graph and the source vertex is not faulty, then there exists a broadcasting scheme that tolerates up to $n-1$ faults of fail-stop type and up to $\lfloor (n-1)/2 \rfloor$ faults of Byzantine type even in the worst case. All transmissions by such a broadcasting scheme contribute to the majority voting to obtain the correct message, and its communication complexity is optimal to tolerate up to $\lfloor (n-1)/2 \rfloor$ faults of Byzantine type [1].

A set of independent spanning trees rooted at the same vertex is said to be well-formed if for each pair of distinct independent spanning trees in the set, $T_1$ and $T_2$, any son of the root in $T_1$ is a leaf of $T_2$, and any son of the root in $T_2$ is a leaf of $T_1$. If for each vertex $u$ of $G$ there are $n$ well-formed independent spanning trees, $G$ is called a well-formed $n$-channel graph. It is open whether for any pair of an $n_1$-channel graph and an $n_2$- channel graph, the product of these two graphs is an $(n_1+n_2)$-channel graph.

In this paper we show that if $G_1$ is a well-formed $n_1$-channel graph and $G_2$ is a well-formed $n_2$-channel graph, then $G_1 \times G_2$ is a well-formed $(n_1+n_2)$-channel graph. The proof of this fact is by constructing $n_1+n_2$ well-formed independent spanning trees of $G_1 \times G_2$ from $n_1$ well-formed independent spanning trees of $G_1$ and $n_2$ well-formed independent spanning trees of $G_2$. It is also not known whether any $n$-channel graph is a well-formed $n$-channel graph. That is, we do not know far whether the problem solved in this paper is equivalent to the open problem mentioned above. From the result solved in this paper we can say that if for each component graph $G_i$ $(1 \leq i \leq n)$, the vertex connectivity of $G_i$ coincides with the number of well-formed independent spanning trees rooted at the same vertex of $G_i$, then the vertex connectivity and the number of well-formed independent spanning trees rooted at the same vertex of $G_1 \times G_2 \times \cdots \times G_n$ coincide.

## 2 Well-Formed Independent Spanning Trees

A graph $G$ is $n$-regular, or regular of degree $n$, if every vertex of $G$ has degree $n$. A rooted tree is called a broom if its root has just one son.

**Theorem 1** *Independent spanning trees at the same root of a graph are well-formed, if each of the independent spanning trees is a broom.*

**Proof:** Let $T_1$ and $T_2$ be independent spanning trees rooted at $r$ of $G$. Suppose that both $T_1$ and $T_2$ are brooms. Assume, to the contrary, that the son $s$ of $r$ in $T_1$ is not a leaf of $T_2$. Then there is a son $v$ of $s$ in $T_2$. The path from $r$ to $v$ in $T_1$ and the path from $r$ to $v$ in $T_2$ have a common internal vertex $s$. This is contrary to the assumption that $T_1$ and $T_2$ are independent. Hence, both $T_1$ and $T_2$ should be well-formed. □

If an $n$-regular graph is $n$-channel, then for any set of $n$ independent spanning trees at the same root of the graph, each tree of the set should be a broom. We therefore have the next theorem.

**Theorem 2** *If $G$ is $n$-regular and $n$-channel, then $G$ is a well-formed $n$-channel graph.*

We show some independent spanning trees of $L_2 \times L_3$ in Figure 2, where trees shown in (a) are independent spanning trees but they are not well-formed, trees shown in (b) are well-formed independent spanning trees but they are not brooms, and trees shown in (c) are well-formed independent spanning trees and they are brooms. So far we cannot find any example such that a graph $G$ is $n$-channel but there are no $n$ well-formed independent spanning trees rooted at the same vertex of $G$. However, we do not know at present whether any $n$-channel graph is well-formed. In the following section, the condition that independent spanning trees of component

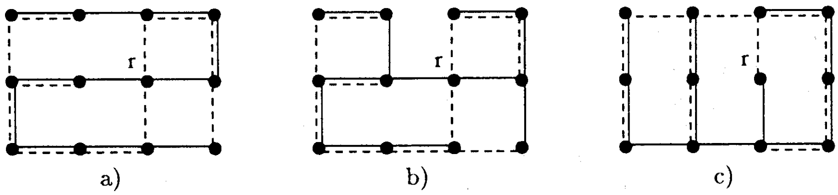


Figure 2: Some independent spanning trees of $L_2 \times L_3$.

graphs are well-formed, is required for the construction of $n_1 + n_2$ independent spanning trees of a product graph from $n_1$ independent spanning trees and $n_2$ independent spanning trees of its component graphs.

# 3 Construction of Independent Spanning Trees

An interesting question is whether $G = G_1 \times \cdots \times G_n$ is an $(n_1 + n_2 + \cdots + n_n)$-channel graph if for each $i$ $(1 \leq i \leq n)$, $G_i$ is an $n_i$-channel graph. To solve this problem, it is sufficient to show that for any pair of an $n_1$-channel graph $G_1$ and an $n_2$-channel graph $G_2$, $G = G_1 \times G_2$ is an $(n_1 + n_2)$-channel graph. Unfortunately we have not succeeded yet in solving this problem. In this section we show how to construct $n_1 + n_2$ well-formed independent spanning trees rooted at $(r_1, r_2)$ of $G$ from $n_1$ well-formed independent spanning trees rooted at $r_1$ of $G_1$ and $n_2$ well-formed independent spanning trees rooted at $r_2$ of $G_2$.

We first define two operations "$\cdot$" and "$\circ$" on spanning trees. We can regard these operations as constructions of spanning trees of a product network. Let $G_1$ and $G_2$ be two graphs, $r_1$ be a vertex of $G_1$, and $r_2$ be a vertex of $G_2$. Let $T_1$ and $T_2$ be spanning trees rooted at $r_1$ of $G_1$ and rooted at $r_2$ of $G_2$, respectively. Let $s$ be a son of $r_1$ in $T_1$. We construct a spanning tree rooted at $(r_1, r_2)$ of $G_1 \times G_2$, denoted by $\tilde{T_1} \cdot T_2(s)$ as follows:

(1) From $(r_1, r_2)$ we develop the first component along $T_1$. That is, we connect $(x, r_2)$ and $(y, r_2)$ by an edge if there is an edge between $x$ and $y$ in $T_1$. Note that by this construction all the vertices in $\{(x, r_2) | x \in V(G_1)\}$ are connected.

(2) For any $x \in V(G_1) - \{r_1\}$, from $(x, r_2)$ we develop the second component along $T_2$. That is, for each $x \in V(G_1) - \{r_1\}$ we connect $(x, y)$ and $(x, z)$ by an edge if there is an edge between $y$ and $z$ in $T_2$. Note that by the construction of (1) and (2), all the vertices in $G_1 \times G_2 - \{(r_1, y) | y \in G_2 - \{r_2\}\}$ are now connected.

(3) For each $y \in V(G_2) - \{r_2\}$, we connect $(s, y)$ and $(r_1, y)$ by an edge.

It is not difficult to verify that $\tilde{T_1} \cdot T_2(s)$ constructed above is a spanning tree of $G_1 \times G_2$. This is the natural way to construct a spanning tree from $T_1$ and $T_2$. That is, we first develop the first component, and then develop the second component. The only exception is the connection from $(r_1, r_2)$ to $(r_1, y)$ for each $y \in V(G_2) - \{r_2\}$. This path in $\tilde{T_1} \cdot T_2(s)$ is exhibited by $(r_1, r_2) \rightarrow (s, r_2) \longrightarrow^{T_2} (s, y) \rightarrow (r_1, y)$ instead of $(r_1, r_2) \longrightarrow^{T_2} (r_1, y)$. For a vertex $(x, y) \in (V(G_1) - \{r_1\}) \times (V(G_2) - \{r_2\})$, the path from $(r_1, r_2)$ to $(x, y)$ in $\tilde{T_1} \cdot T_2(s)$ is exhibited by $(r_1, r_2) \longrightarrow^{T_1} (x, r_2) \longrightarrow^{T_2} (x, y)$.

We can symmetrically construct a spanning tree $T_1 \cdot \tilde{T_2}(t)$ rooted at $(r_1, r_2)$ of $G_1 \times G_2$ as follows:

(1) From $(r_1, r_2)$ we develop the second component along $T_2$.

(2) For each $y \in V(G_2) - \{r_2\}$, from $(r_1, y)$ we develop the first component along $T_1$.

(3) For each $x \in V(G_1) - \{r_1\}$, we connect $(x, t)$ and $(x, r_2)$ by an edge, where $t$ is a son of $r_2$ in $T_2$.

From the construction described above the next lemma is immediate.

**Lemma 1** *For $i = 1, 2$, let $r_i$ be a vertex of $G_i$, $T_i$ be a spanning tree rooted at $r_i$ of $G_i$, and $s_i$ be a son of $r_i$ in $T_i$. Then $\tilde{T_1} \cdot T_2(s_1)$ and $T_1 \cdot \tilde{T_2}(s_2)$ are well-formed independent spanning trees rooted at $(r_1, r_2)$ of $G_1 \times G_2$.*

We next define another way of constructing a spanning tree $\tilde{T_1} \circ T_2(s)$ rooted at $(r_1, r_2)$ of $G_1 \times G_2$ from spanning trees $T_1$ and $T_2$ rooted at $r_1$ of $G_1$ and rooted at $r_2$ of $G_2$, respectively, where $s$ is a son of $r_1$ in $T_1$.

Suppose that $s_1, s_2, \cdots, s_k$ are all the sons of $r_1$ in $T_1$. Denote the subtree rooted at $s_i$ of $T_1$ by $ST_1^i$ for $i = 1, 2, ..., k$. We let $s$ be any one of $s_1, s_2, \cdots, s_k$. Without loss of generality, we may assume that $s = s_1$. The construction of $\tilde{T_1} \circ T_2(s_1)$ is as follows:

(1) For any $1 \leq i \leq k$, we connect $(r_1, r_2)$ and $(s_i, r_2)$ by an edge.

(2) For $1 \leq i \leq k$, from $(s_i, r_2)$ we develop the second component along $T_2$. From the construction of (1) and (2) all the vertices of $\{(r_1, r_2)\} \cup \{(s_i, y) | i = 1, 2, \cdots, k \text{ and } y \in G_2\}$ are now connected.

(3) For any $y \in V(G_2)$ and $1 \leq i \leq k$, we develop the first component from $(s_i, y)$ along $ST_1^i$. At this stage only the vertices of $\{(r_1, y) | y \in G_2 - \{r_2\}\}$ are not connected.

(4) For each $y \in V(G_2) - \{r_2\}$, we connect $(s_1, y)$ and $(r_1, y)$ by an edge.

It is not difficult to verify that $\tilde{T_1} \circ T_2(s_1)$ constructed above is a spanning tree of $G_1 \times G_2$. The strategy of the construction of $\tilde{T_1} \circ T_2(s_1)$ can be described as follows: We develop one step along first component, then develop the second component, and then develop the first component. Let $(x, y) \in V(G_1 \times G_2)$, $x \neq r_1$, $y \neq r_2$ and $x$ be a vertex in $ST_1^i$. The difference between $\tilde{T_1} \cdot T_2(s_1)$ and $\tilde{T_1} \circ T_2(s_1)$ is clear from the following contrast. The path from $(r_1, r_2)$ to $(x, y)$ in $\tilde{T_1} \cdot T_2(s_1)$ is exhibited by $(r_1, r_2) \longrightarrow^{T_1} (x, r_2) \longrightarrow^{T_2} (x, y)$ while the path from $(r_1, r_2)$ to $(x, y)$ in $\tilde{T_1} \circ T_2(s_1)$ is exhibited by $(r_1, r_2) \rightarrow (s_i, r_2) \longrightarrow^{T_2} (s_i, y) \longrightarrow^{T_1} (x, y)$.

We can symmetrically construct a spanning tree $T_1 \circ \tilde{T_2}(t)$ rooted at $(r_1, r_2)$ of $G_1 \times G_2$, where $t$ is a son of $r_2$ in $T_2$.

From the construction of the spanning trees described above we have the following two lemmas.

**Lemma 2** *Let $T_1, T_2, \cdots, T_k$ be $k$ well-formed independent spanning trees rooted at $r_1$ of $G_1$, and let $s_1, s_2, \cdots, s_k$ be sons of $r_1$ in $T_1, T_2, \cdots, T_k$, respectively. Let $S$ be a spanning tree rooted at $r_2$ of $G_2$. Then $\tilde{T_1} \circ S(s_1)$, $\tilde{T_2} \circ S(s_2)$, $\cdots$, $\tilde{T_k} \circ S(s_k)$ are well-formed independent spanning trees rooted at $(r_1, r_2)$ of $G_1 \times G_2$.*

**Lemma 3** *Let $T_1$ and $T_2$ be well-formed independent spanning trees rooted at $r_1$ of $G_1$, and let $S_1$ and $S_2$ be well-formed independent spanning trees rooted at $r_2$ of $G_2$. Then $\tilde{T_1} \circ S_1(t_1)$ and $T_2 \circ \tilde{S_2}(s_2)$ are well-formed independent spanning trees rooted at $(r_1, r_2)$ of $G_1 \times G_2$, where $t_1$ is a son of $r_1$ in $T_1$ and $s_2$ is a son of $r_2$ in $S_2$.*

We are now ready to describe how we can construct $(n_1 + n_2)$ well-formed independent spanning trees in $G_1 \times G_2$ from $n_1$ well-formed independent spanning trees of $G_1$ and $n_2$ well-formed independent spanning trees of $G_2$. Let $T_1, T_2, \cdots, T_{n_1}$ be $n_1$ well-formed independent spanning trees rooted at $r_1$ of $G_1$, and let $S_1, S_2, \cdots, S_{n_2}$ be $n_2$ well-formed independent spanning trees rooted at $r_2$ of $G_2$. Let $A_i$ be the set of sons of $r_1$ in $T_i$ $(1 \leq i \leq n_1)$, and let $B_i$ be the set of sons of $r_2$ in $S_i$ $(1 \leq i \leq n_2)$. Since we assume that the $n_1$ independent spanning trees of $G_1$ and the $n_2$ independent spanning trees of $G_2$ are well-formed, for each $i$ $(2 \leq i \leq n_1)$ any element of $A_i$ is a leaf of $T_1$, and for each $i$ $(2 \leq i \leq n_2)$ any element of $B_i$ is a leaf of $S_1$. We choose one element, say $t_i$, from each $A_i$ $(1 \leq i \leq n_1)$, and one element, say $s_i$, from each $B_i$ $(1 \leq i \leq n_2)$.

Let $T_1[A_2, \cdots, A_{n_1}]$ denote the tree obtained by removing all the vertices in $A_2 \cup \cdots \cup A_{n_1}$ and their induced edges from $T_1$. Remember that every vertex in $A_2 \cup \cdots \cup A_{n_1}$ is a leaf of $T_1$. Similarly, let $S_1[B_2, \cdots, B_{n_2}]$ denote the tree obtained by removing all the vertices in $B_2 \cup \cdots \cup B_{n_2}$ and their induced edges from $S_1$. Furthermore, let $var(T_1)$ denote the spanning tree obtained by adding every edge $r_1 t$ and vertex $t$ to $T_1[A_2, \cdots, A_{n_1}]$ such that $t$ is in $A_2 \cup \cdots \cup A_{n_1}$. Note that $T_1$ is different from $var(T_1)$. That is, every $t$ in $A_2 \cup \cdots \cup A_{n_1}$ is directly connected to $r_1$ in $var(T_1)$. Similarly, let $var(S_1)$ denote the spanning tree obtained by adding every edge $r_2 s$ and vertex $s$ to $S_1[B_2, \cdots, B_{n_2}]$ such that $s$ is in $B_2 \cup \cdots \cup B_{n_2}$. Consider the following $n_1 + n_2$ trees:

$$\tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1) \ , \quad \tilde{T}_2 \circ var(S_1)(t_2) \ , \quad \cdots \ , \quad \tilde{T}_{n_1} \circ var(S_1)(t_{n_1}),$$

$$var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1) \ , \quad var(T_1) \circ \tilde{S}_2(s_2) \ , \quad \cdots \ , \quad var(T_1) \circ \tilde{S}_{n_2}(s_{n_2}).$$

Among the trees listed above, $\tilde{T}_i \circ var(S_1)(t_i)$ and $var(T_1) \circ \tilde{S}_j(s_j)$ $(2 \leq i \leq n_1,\ 2 \leq j \leq n_2)$ are spanning trees of $G_1 \times G_2$. However, $\tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$ and $var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$ are not spanning trees of $G_1 \times G_2$ since some vertices are missing. For example, vertices in $\{(t_i, x_2) | 2 \leq i \leq n_1 \text{ and } x_2 \in G_2\}$ are not contained in the vertex set of $\tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$. For $2 \leq i \leq n_1$, let $f(t_i)$ be the father of $t_i$ in $T_1$. We add the vertices and the edges in

$^{(t)}S = \{$ vertex $(t_i, x_2)$, edge $(f(t_i), x_2)(t_i, x_2) \mid 2 \leq i \leq n_1$ and $x_2 \in G_2\}$

to $\tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$. Then we can obtain a spanning tree of $G_1 \times G_2$, denoted by $^{(t)}S * \tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$. Similarly, we add the vertices and the edges in $T^{(s)}$ to $var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$ to make a spanning tree of $G_1 \times G_2$, denoted by $T^{(s)} * var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$, where

$T^{(s)} = \{$ vertex $(x_1, s_i)$, edge $(x_1, f(s_i))(x_1, s_i) \mid 2 \leq i \leq n_2$ and $x_1 \in G_1\}$.

**Theorem 3** *Let $T_1, \cdots, T_{n_1}$ be well-formed independent spanning trees rooted at $r_1$ of $G_1$, and let $S_1, \cdots, S_{n_2}$ be well-formed independent spanning trees rooted at $r_2$ of $G_2$. The following $n_1 + n_2$ trees are well-formed independent spanning trees rooted at $(r_1, r_2)$ of $G_1 \times G_2$:*

$$^{(t)}S * \tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1) \ , \quad \tilde{T}_2 \circ var(S_1)(t_2) \ , \quad \cdots \ , \quad \tilde{T}_{n_1} \circ var(S_1)(t_{n_1}),$$

$$T^{(s)} * var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1) \ , \quad var(T_1) \circ \tilde{S}_2(s_2) \ , \quad \cdots \ , \quad var(T_1) \circ \tilde{S}_{n_2}(s_{n_2}).$$

**Proof:** Let $(x_1, x_2)$ be an arbitrary vertex of $G_1 \times G_2$. We prove that the $n_1 + n_2$ paths from $(r_1, r_2)$ to $(x_1, x_2)$, each in one of the $n_1 + n_2$ spanning trees listed above, are internally vertex disjoint.

Case 1: $x_1 = r_1, x_2 \neq r_2$ or $x_1 \neq r_1, x_2 = r_2$.

Due to the symmetry, it is sufficient to consider only the case where $x_1 = r_1, x_2 \neq r_2$. The paths from $(r_1, r_2)$ to $(x_1, x_2)$ in $^{(t)}S * \tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$, $\tilde{T}_2 \circ var(S_1)(t_2)$, $\cdots$,

$\tilde{T}_{n_1} \circ var(S_1)(t_{n_1})$, $T^{(s)} * var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$, $var(T_1) \circ \tilde{S}_2(s_2)$, $\cdots$, $var(T_1) \circ \tilde{S}_{n_2}(s_{n_2})$ are $(r_1, r_2) \to (t_1, r_2) \longrightarrow^{var(S_1)} (t_1, x_2) \to (r_1, x_2)$, $(r_1, r_2) \to (t_2, r_2) \longrightarrow^{var(S_1)} (t_2, x_2) \to (r_1, x_2)$, $\cdots$, $(r_1, r_2) \to (t_{n_1}, r_2) \longrightarrow^{var(S_1)} (t_{n_1}, x_2) \to (r_1, x_2)$, $(r_1, r_2) \longrightarrow^{S_1} (r_1, x_2)$, $(r_1, r_2) \longrightarrow^{S_2} (r_1, x_2)$, $\cdots$, $(r_1, r_2) \longrightarrow^{S_{n_2}} (r_1, x_2)$, respectively. These paths are internally vertex disjoint.

Case 2: $x_1 \in V(G_1) - \{r_1, t_1, t_2, \cdots, t_{n_1}\}$ and $x_2 \in V(G_2) - \{r_2, s_1, s_2, \cdots, s_{n_2}\}$.

We can see that the paths from $(r_1, r_2)$ to $(x_1, x_2)$ in the $n_1+n_2$ spanning trees are internally vertex disjoint from Lemma 1, Lemma 2, Lemma 3 and the following facts:

The path from $(r_1, r_2)$ to $(x_1, x_2)$ in ${}^{(t)}S * \tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$, i.e., $(r_1, r_2) \longrightarrow^{T_1} (x_1, r_2) \longrightarrow^{S_1} (x_1, x_2)$, is internally vertex disjoint with the path from $(r_1, r_2)$ to $(x_1, x_2)$ in $\tilde{T}_i \circ var(S_1)(t_i)$, i.e., $(r_1, r_2) \to (t_i, r_2) \longrightarrow^{S_1} (t_i, x_2) \longrightarrow^{T_i} (x_1, x_2)$, for $2 \le i \le n_1$, and it is also internally vertex disjoint with the path in $var(T_1) \circ \tilde{S}_i(s_i)$, i.e., $(r_1, r_2) \to (r_1, s_i) \longrightarrow^{T_1} (x_1, s_i) \longrightarrow^{S_i} (x_1, x_2)$, for $2 \le i \le n_2$. Symmetrically, the path from $(r_1, r_2)$ to $(x_1, x_2)$ in $T^{(s)} * var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$ is internally vertex disjoint with the paths in $var(T_1) \circ \tilde{S}_i(s_i)$ and $\tilde{T}_i \circ var(S_1)(t_i)$.

Case 3: Either $x_1 \in \{t_1, t_2, \cdots, t_{n_1}\}$ or $x_2 \in \{s_1, s_2, \cdots, s_{n_2}\}$.

This case is more complicated than the previous two cases. Due to the symmetry, it is sufficient to consider only the case where $x_1 \in \{t_1, t_2, \cdots, t_{n_1}\}$ and $x_2 \in V(G_2) - \{r_2\}$. For the following four subcases we can verify the fact that path from $(r_1, r_2)$ to $(x_1, x_2)$ in the $n_1 + n_2$ spanning trees ${}^{(t)}S * \tilde{T}_1[A_2, \cdots, A_{n_1}] \cdot var(S_1)(t_1)$, $\tilde{T}_i \circ var(S_1)(t_i)$ for each $i$ $(2 \le i \le n_1)$, $T^{(s)} * var(T_1) \cdot \tilde{S}_1[B_2, \cdots, B_{n_2}](s_1)$, $var(T_1) \circ \tilde{S}_i(s_i)$ for each $i$ $(2 \le i \le n_2)$, are internally vertex disjoint.

(1) $x_1 = t_1, x_2 \in V(G_2) - \{r_2, s_1, \cdots, s_{n_2}\}$.

The $n_1 + n_2$ paths are

$(r_1, r_2) \to (t_1, r_2) \longrightarrow^{S_1} (t_1, x_2)$, $(r_1, r_2) \to (t_i, r_2) \longrightarrow^{S_1} (t_i, x_2) \longrightarrow^{T_i} (t_1, x_2)$ for each $i$ $(2 \le i \le n_1)$,

$(r_1, r_2) \longrightarrow^{S_1} (r_1, x_2) \to (t_1, x_2)$, $(r_1, r_2) \to (r_1, s_i) \to (t_1, s_{i},) \longrightarrow^{S_i} (t_1, x_2)$ for each $i$ $(2 \le i \le n_2)$.

(2) $x_1 = t_1, x_2 \in \{s_1, \cdots, s_{n_2}\}$.

The $n_1 + n_2$ paths are

$(r_1, r_2) \to (t_1, r_2) \to (t_1, x_2)$, $(r_1, r_2) \to (t_i, r_2) \to (t_i, x_2) \longrightarrow^{T_i} (t_1, x_2)$ for each $i$ $(2 \le i \le n_1)$,

$(r_1, r_2) \longrightarrow^{S_1} (r_1, f_1(x_2)) \to (t_1, f_1(x_2)) \to (t_1, x_2)$, where $f_1(x_2)$ denotes the father of $x_2$ in $S_1$,

$(r_1, r_2) \to (r_1, s_i) \to (t_1, s_i) \longrightarrow^{S_i} (t_1, x_2)$ for each $i$ satisfying $s_i \ne x_2$,

$(r_1, r_2) \to (r_1, s_i) \to (t_1, s_i)$ for $s_i = x_2$.

(3) $x_1 \in \{t_2, \cdots, t_{n_1}\}, x_2 \in V(G_2) - \{r_2, s_1, \cdots, s_{n_2}\}$.

The $n_1 + n_2$ paths are

$(r_1, r_2) \longrightarrow^{T_1} (f_1(x_1), r_2) \longrightarrow^{S_1} (f_1(x_1), x_2) \to (x_1, x_2)$, where $f_1(x_1)$ denotes the father of $x_1$ in $T_1$,

$(r_1, r_2) \to (t_i, r_2) \longrightarrow^{S_1} (t_i, x_2) \longrightarrow^{T_i} (x_1, x_2)$ for each $i$ satisfying $t_i \ne x_1$,

$(r_1, r_2) \to (t_i, r_2) \longrightarrow^{S_1} (t_i, x_2)$ for $t_i = x_1$,

$(r_1, r_2) \longrightarrow^{S_1} (r_1, x_2) \to (x_1, x_2)$,

$(r_1, r_2) \to (r_1, s_i) \to (x_1, s_i) \longrightarrow^{S_i} (x_1, x_2)$ for each $i$ $(2 \le i \le n_2)$.

(4) $x_1 \in \{t_2, \cdots, t_{n_1}\}, x_2 \in \{s_1, \cdots, s_{n_2}\}$.

The $n_1 + n_2$ paths are

$(r_1, r_2) \longrightarrow^{T_1} (f_1(x_1), r_2) \to (f_1(x_1), x_2) \to (x_1, x_2)$, $(r_1, r_2) \to (t_i, r_2) \to (t_i, x_2) \longrightarrow^{T_i} (x_1, x_2)$ for each $i$ satisfying $t_i \ne x_1$,

$(r_1, r_2) \to (t_i, r_2) \to (t_i, x_2)$ for $t_i = x_1$,

$(r_1, r_2) \longrightarrow^{S_1} (r_1, f_1(x_2)) \to (x_1, f_1(x_2)) \to (x_1, x_2)$,

$(r_1, r_2) \to (r_1, s_i) \to (x_1, s_i) \longrightarrow^{S_i} (x_1, x_2)$ for $s_i \ne x_2$,

$(r_1, r_2) \to (r_1, s_i) \to (x_1, s_i)$ for $s_i = x_2$.

From the construction described above, the set of these independent spanning trees rooted at $(r_1, r_2)$ of $G_1 \times G_2$ are well-formed. □

# 4 Concluding Remarks

We have shown how to construct $(n_1+n_2)$ well-formed independent spanning trees in the product graph of a well-formed $n_1$-channel graph and a well-formed $n_2$-channel graph. Hence, we can construct $(n_1+n_2+\cdots+n_m)$ well-formed independent spanning trees of $G = G_1 \times G_2 \times \cdots \times G_m$ by successively applying the construction given in this section if $G_i$ is a well-formed $n_i$-channel graph for each $i$ $(1 \leq i \leq m)$.

In practice shallow spanning trees are desirable. Generally speaking, the maximum hight of the $n$ independent spanning trees of a product network by the construction given in this section depends on the order of product as well as its component graphs. For example, the maximum hight of the $n$ independent spanning trees of the $n$-cube by our construction is $2n-1$ provided we take the product in the order

$$(\cdots(((G_1 \times G_2) \times G_3) \times G_4) \cdots \times G_{n-1}) \times G_n.$$

If we take the product in the order

$$(G_1 \times G_2) \times (G_3 \times G_4) \times \cdots \times (G_{n-3} \times G_{n-2}) \times (G_{n-1} \times G_n),$$

then the maximum hight of the $n$ independent spanning trees of the $n$-cube is about $\frac{3}{2}n$. This is optimal in the sense that the smallest maximum hight of $n$ independent spanning trees of the $n$-cube can be constructed by the method given in this section.

For the construction of independent spanning trees of product graphs, we do not know at present whether we can remove the condition that each component graph should be well-formed. A more interesting problem is how we construct independent spanning trees of an arbitrarily graph ? This is a very hard problem. In fact, it is open whether every $n$-connected graph has $n$ independent spanning trees with the same root. The problem has been solved only for $k$-connected graphs, $k \leq 3$. Furthermore, even if we know the constructions of independent spanning trees some graphs, it is still important how we can construct independent spanning trees with good properties, for example with low hights and regular structures.

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