アルゴリズム 50−6
(1996. 3. 15)

部分定義論理関数のホーン拡張について

牧野和久[†‡], 畑中健一[§], 茨木俊秀[†]
[†] 606-01 京都市左京区吉田本町 京都大学 工学部 数理工学教室
[§]554 大阪市此花区島屋 1-1-3 住友電気工業 (株)
[‡]日本学術振興会特別研究員

あらまし 本論文では, 部分定義論理関数 (T,F) に対するホーン拡張 $f:\{0,1\}^n \mapsto \{0,1\}$ について考察する. ただし, $T \subseteq \{0,1\}^n$ は正例の集合, $F \subseteq \{0,1\}^n$ は負例の集合を表すものとする. (T,F) が与えられたとき, ホーン拡張が存在するかどうか多項式時間決定可能であることが知られているが, 複数のホーン拡張が存在するので, 極大あるいは, 極小の真ベクトル集合 $T(f)$ をもつ拡張 f について考察する. 任意の (T,F) に対して, 唯一の極大 (すなわち, 最大) ホーン拡張が存在するが, 一般に, たくさんの極小ホーン拡張が存在する. 我々は, まず, 極大ホーン拡張に対して, たとえその DNF 表現が長くなるときでも, 多項式時間帰属性神託が構成可能であるを示す. さらに, 与えられたホーン DNF が極小ホーン拡張を示すかどうかの決定, 極小ホーン拡張のホーン DNF 表現の構成, また, (T,F) が唯一の極小ホーン拡張をもつかどうかの決定がすべて多項式時間可能であることを示す. 最後に, 最小の $|T(f)|$ をもつホーン拡張, および, リテラル数が最小の DNF 表現をもつホーン拡張を求める問題が NP-困難であるを示す.

和文キーワード: 部分定義論理関数, 拡張, ホーン関数, 知識獲得.

Horn Extensions of a Partially Defined Boolean Function

Kazuhisa MAKINO[†‡], Ken-ichi HATANAKA[§] and Toshihide IBARAKI[†]
[†]Department of Applied Mathematics and Physics, Graduate School of Engineering, Kyoto University, Kyoto 606-01, Japan.
[§]Sumitomo Electric Industries, Shimaya 1-1-3, Konohana-ku, Osaka, 554, Japan.
[‡]Research Fellow of the Japan Society for the Promotion of Science

abstract Given a partially defined Boolean function (pdBf in short) (T,F), we investigate in this paper how to find a Horn extension $f:\{0,1\}^n \mapsto \{0,1\}$, which is consistent with (T,F), where $T \subseteq \{0,1\}^n$ denotes a set of true Boolean vectors (or positive examples) and $F \subseteq \{0,1\}^n$ denotes a set of false Boolean vectors (or negative examples). Given a pdBf (T,F), it is known that the existence of a Horn extension can be checked in polynomial time. As there are many Horn extensions, however, we consider those extensions f which have maximal and minimal sets $T(f)$ of the true vectors of f, respectively. For a pdBf (T,F), there always exists the unique maximal (i.e., maximum) Horn extension, but there are in general many minimal Horn extensions. We first show that a polynomial time membership oracle can be constructed for the maximum extension, even if its DNF (disjunctive normal form) can be very long. Our main contribution is then to show that checking if a given Horn DNF represents a minimal extension, and generating a Horn DNF of a minimal Horn extension can both be done in polynomial time. We also can check in polynomial time if a pdBf (T,F) has the unique minimal Horn extension. However, the problems of finding a Horn extension f with the smallest $|T(f)|$, and of obtaining a Horn DNF, whose number of literals is smallest, are both NP-hard.

英文 **key words**: partially defined Boolean function, extension, Horn function, knowledge acquisition.

1 Introduction

A basic problem in knowledge acquisition in the form of Boolean logic (e.g., [3, 12]) can be stated as follows: Given a set of data, represented as a set $T \subseteq \{0,1\}^n$ of binary "true n-vectors" (or "positive examples") and a set $F \subseteq \{0,1\}^n$ of "false n-vectors" (or "negative examples"), establish a (fully defined) Boolean function (i.e., extension) $f : \{0,1\}^n \mapsto \{0,1\}$ in a specified class $\mathcal{C}$, such that $T \subseteq T(f)$ and $F \subseteq F(f)$, where $T(f)$ (resp. $F(f)$) denotes the set of true (resp. false) vectors of f. A pair of sets (T, F) is called a *partially defined Boolean function* (pdBf) throughout this paper.

For instance, a vector x may represent the symptoms to diagnose a disease; e.g., x_1 denotes whether temperature is high ($x_1 = 1$) or not ($x_1 = 0$), and x_2 denotes whether blood presure is high ($x_2 = 1$) or not ($x_2 = 0$), etc. Each vector x in T corresponds to a case of symptoms which caused the disease, while a vector in F describes a case with which the disease did not appear. Establishing an extension f, which is consistent with the given data, amounts to finding a logical diagnostic explanation of the given data.

In this paper, we consider the case in which f is a Horn function [10]. Horn functions play an important role in artificial intelligence, logic programming and so on, since the *non-tautology problem* of a Horn DNF (disjunctive normal form) (H-NON-TAU) in short) can be solved in polynomial time [4], whereas non-tautology problem (NON-TAU) of a general DNF is NP-complete. As problem NON-TAU, which is the dual formulation of *satisfiability problem* (SAT) of CNF (conjunctive normal form), is very fundamental, many problems related to Horn functions can be solved efficiently. In terms of sets $T(f)$ and $F(f)$, a Horn function has an elegant characterization: f is Horn if and only if $F(f)$ is closed under intersection of vectors (i.e., $v, w \in F(f)$ implies $v \bigwedge w \in F(f)$, where $\bigwedge$ denotes the componentwise AND operation).

As there are in general many Horn extensions f for a given pdBf (T, F), we shall mainly consider those extensions which have *maximal* and *minimal* $T(f)$, respectively. We note here that most of the papers on the representation by Horn theory (e.g., [3, 5, 11, 12]) are based on model theory, in which finding a Horn representation f of a given model $(T(g), F(g))$, where g is a Boolean function and the sets $T(g)$, $F(g)$ of vectors are explicitly given, is their primary target. For example, the problem of finding the best Horn approximation of model $(T(g), F(g))$, i.e., finding the Horn function with the minimum $F(f)$ under constraint $F(f) \supseteq F(g)$, has received some attention [12], and it is known [12, 13] that obtaining an irredundant DNF of such f is at least as difficult as computing the DNF of the dual h^d of a positive (i.e., monotone) Boolean function h. The latter problem is a well-known open problem [2, 6], for which the recent result of Fredman and Khachiyan [7] shows that there is an $O(m^{o(\log m)})$ time algorithm, where m is the total length of DNFs for both h and h^d. We emphasize that our problem setting is different from model theory in that the input (T, F) is only partially defined. However, the above problem of best Horn approximation is very close to the problem of finding a maximal Horn extension. We also note that, although the problem of finding a best approximation in terms of $T(g)$ is a bit artificial (since $T(g)$ is not closed under intersection), finding its counterpart (i.e., a minimal Horn extension) is quite a natural problem in our framework.

It is known [3] that the existence of at least one Horn extension of a given pdBf (T, F) can be checked in polynomial time. After preparing necessary notations and definitions in Section 2, and introducing canonical Horn DNF in Section 3, we proceed to maximal and minimal Horn extensions. In Section 4, by using an argument similar to the one used in model theory, we show that there exists the unique maximal Horn extension $f_{\max}$ (i.e., maximum), and provide a polynomial time membership oracle for $f_{\max}$. In Section 5, we investigate minimal Horn extensions. Contrary to the case of maximum Horn extension, there are in general many minimal Horn extensions. Our main contribution is to show that the minimality of f_φ, which denotes the function represented by a Horn DNF φ, can be checked in polynomial time. Based on this, a minimal Horn extension of a pdBf (T, F) can be generated in polynomial time, and the uniqueness of a minimal extension can also be checked in polynomial time.

To derive the above main results, we first show that any minimal Horn extension can be represented by a *canonical* Horn DNF, though the converse is not true. The nontriviality of finding a canonical DNF representing a minimal Horn extension may be exemplified by the existence of a canonical DNF that satisfies local minimality but does not represent a minimal Horn extension. To overcome this, we reduce the minimality condition to the equivalence $f_\varphi = f_{\varphi^*}$, where φ is a given Horn DNF and φ^* is a DNF derived from φ and (T, F). Although this does not immediately give a polynomial time algorithm, since φ^* is not Horn, we then derive a series of lemmas, with which condition $f_\varphi = f_{\varphi^*}$ can be eventually decomposed into a polynomial number of H-NON-TAU problems, each of which can then be solved in polynomial time.

Finally, we show in Section 6 that the problems of computing a Horn extension f with the minimum $|T(f)|$, and of finding the shortest Horn DNF (i.e., having the smallest number of literals) that represents a Horn extension are both NP-hard. It is still not known whether there is a polynomial total time algorithm to generate all minimal Horn extensions of a given pdBf (T, F).

2 Preliminaries

A *Boolean function* (or a *function* in short) is a mapping $f : \{0,1\}^n \mapsto \{0,1\}$. If $f(x) = 1$ (resp. 0), then x is called a *true* (resp. *false*) vector of f. The set of all true vectors (resp. false vectors) is denoted by $T(f)$ (resp. $F(f)$). Denote, for a vector $v \in \{0,1\}^n$, $ON(v) = \{j \mid v_j = 1, j = 1, 2, \ldots, n\}$ and $OFF(v) = \{j \mid v_j = 0, j = 1, 2, \ldots, n\}$. Two special functions with $T(f) = \emptyset$ and $F(f) = \emptyset$ are respectively denoted by $f = \perp$ and $f = \top$. For two functions f and g on the same set of variables, we write $f \leq g$ if $f(x) = 1$ implies $g(x) = 1$ for any $x \in \{0,1\}^n$, and $f < g$ if $f \leq g$ and $f \neq g$.

Boolean variables $x_1, \ldots, x_n$ and their negations $\bar{x}_1, \ldots, \bar{x}_n$ are called *literals*, where we call literals $x_1, \ldots, x_n$ *positive*, and literals $\bar{x}_1, \ldots, \bar{x}_n$ *negative*. A *term* t is a conjunction of literals such that at most one of x_i and $\bar{x}_i$ appears for each i. We call that a term t *absorbs* a term t' if $t \geq t'$, where terms t and t' are considered as functions here. For example, a term $x\bar{y}$ absorbs a term $x\bar{y}z$. A term t is called an *implicant* of a function f if $t \leq f$. An implicant t of a function is called *prime* if there is no implicant $t' > t$. A *disjunctive normal form* (DNF) φ is a disjunction of terms. It is well known that a DNF φ defines a function, which we denote f_φ, and any function can be represented by a DNF (however, such a representation may not be unique). Sometimes in this paper, we do not distinguish a DNF φ from the function f_φ it represents. For example, a term t is also considered as the function f_t. The number of literals in a DNF φ is denoted by $|\varphi|$. In this paper, we shall exclusively deal with DNF expressions but all the results can be translated into the results for CNFs by dualizing the involved concepts.

A term is called *positive* if it contains only positive literals, *Horn* if it contains at most one negative literal. A DNF is called *positive* if it contains only positive terms, and *Horn* if it contains only Horn terms. For example, a DNF $\varphi = 123 \vee 245 \vee 156$ is positive, and $\psi = 15\bar{7} \vee 24 \vee \bar{2}67$ is Horn. (Here, for simplicity, a positive literal x_i is denoted as i and a negative literal $\bar{x}_i$ as $\bar{i}$.) A Boolean function is called *positive* (or *monotone*) if it can be represented by a positive DNF, and *Horn* if it can be represented by a Horn DNF. We sometimes call a Horn DNF representing a Horn extension of a pdBf (T, F) as a Horn DNF of (T, F). It is known [9] that, if f is Horn, then all prime implicants of f are Horn. It is important to know that NON-TAU for a Horn DNF φ (H-NON-TAU) can be solved in linear time in $|\varphi|$ [4]. Based on this, conditions such as $f_\varphi = f_\psi$ and $f_\varphi < f_\psi$ can be checked in $O(|\varphi||\psi|)$ time for given Horn DNFs φ and ψ [9]. Also, for a term t (not necessarily Horn), condition $t \leq f_\varphi$ can be checked in $O(|\varphi|)$ time [9].

A *partially defined Boolean function* (*pdBf*) is defined by a pair of sets (T, F) satisfying $T \cap F = \emptyset$, where $T, F \subseteq \{0,1\}^n$. A function f is an *extension* (or theory) of pdBf(T, F) if $T \subseteq T(f)$ and $F \subseteq F(f)$, and a *Horn extension* if f is in addition Horn. A Horn extension f of a pdBf (T, F) is called *minimal* (resp. *maximal*) if there is no Horn extension f' satisfying $f' < f$

(resp. $f' > f$), that is, set $T(f)$ is minimal (resp. maximal). Furthermore, a Horn extension f of a pdBf (T, F) is *minimum* (resp. *maximum*) if there is no Horn extension f' such that $|T(f')| < |T(f)|$ (resp. $|T(f')| > |T(f)|$).

3 Canonical Horn DNF

Call the componentwise AND operation $\bigwedge$ of vectors v and w as the *intersection* of v and w. For example, if $v = (0101)$ and $w = (1001)$, then $v \bigwedge w = (0001)$. For a set $X \subseteq \{0,1\}^n$, the set of vectors $C(X)$ is called *intersection closure* if it is the minimal set that contains X and is closed under intersection.

Proposition 3.1 [5] A function f is Horn if and only if $F(f) = C(F(f))$. □

Now we consider the following problem, and note that it can be solved in polynomial time.

Problem H-EXTENSION
Input: A pdBf (T, F).
Question: Is there a Horn extension f of (T, F)?

Definition 3.1 For a pdBf (T, F) and a vector $v \in T$, the set of terms $R(v)$ is defined by $R(v) = \{\bigwedge_{j \in ON(v)} x_j\}$, if $OFF(v) = \emptyset$; $R(v) = \{(\bigwedge_{j \in ON(v)} x_j)\, \bar{x}_l \mid l \in I(v)\}$ if $OFF(v) \neq \emptyset$ and $I(v) \neq \emptyset$; otherwise, $R(v) = \emptyset$, where

$$F^+(v) = \{w \in F \mid w \geq v\}$$
$$I(v) = (\cap_{w \in F^+(v)} ON(w)) \cap OFF(v).$$

By convention, we define $I(v) = OFF(v)$ if $F^+(v) = \emptyset$. A DNF φ is called a *canonical* Horn DNF of (T, F) if φ is given by

$$\varphi = \bigvee_{v \in T} t_v, \text{ where } t_v \in R(v), \quad (1)$$

i.e., by selecting one term from each $R(v)$, $v \in T$. Remark that the canonical Horn DNF is not defined if $R(v) = \emptyset$ holds for some $v \in T$. □

For example, let $T = \{v^{(1)} = (11000), v^{(2)} = (10010)\}$ and $F = \{w^{(1)} = (11011), w^{(2)} = (10011)\}$. Then $F^+(v^{(1)}) = \{w^{(1)}\}$, $F^+(v^{(2)}) = \{w^{(1)}, w^{(2)}\}$, $I(v^{(1)}) = \{4, 5\}$ and $I(v^{(2)}) = \{5\}$. Thus we have $R(v^{(1)}) = \{x_1x_2\bar{x}_4, x_1x_2\bar{x}_5\}$ and $R(v^{(2)}) = \{x_1x_4\bar{x}_5\}$. Therefore, $\varphi^{(1)} = x_1x_2\bar{x}_4 \vee x_1x_4\bar{x}_5$ and $\varphi^{(2)} = x_1x_2\bar{x}_5 \vee x_1x_4\bar{x}_5$ are all canonical Horn DNFs of the above (T, F). Construction of Horn DNFs in this manner can be found in various literature in learning theory [1], model theory [11] and Horn approximation [15].

Lemma 3.1 [3] Any canonical Horn DNF φ of a given pdBf (T, F) represents a Horn extension of (T, F), and (T, F) has no Horn extension if there is no canonical Horn DNF. □

Since the existence of a canonical Horn DNF can be easily checked in polynomial time, we establish:

Theorem 3.1 Problem H-EXTENSION can be solved in $O(n|T||F|)$ time, and if a pdBf (T, F) has a Horn extension, one of its canonical Horn DNFs can be obtained in $O(n|T||F|)$ time. □

4 Maximum Horn Extension

Theorem 4.1 If a given pdBf (T, F) has a Horn extension, its maximal Horn extension is unique.

Proof. By Proposition 3.1, $F(f)$ of any Horn extension f of (T, F) is closed under intersection. Let us define $f_{\max}$ by $F(f_{\max}) = C(F)$. Since $C(F)$ is the unique minimal set that contains F and is closed under intersection, this $f_{\max}$ is the unique maximal Horn extension of (T, F). □

Unfortunately, there may not be any compact DNF representation of $f_{\max}$, since it is known [11] that there is a pdBf (T, F) for which the size of any DNF φ of $f_{\max}$ is exponential in n, $|T|$ and $|F|$. However, we can do better if we do not stick to the DNF representation. Note that $f_{\max}$ is defined by set $C(F)$, and hence $v \in C(F)$ holds if and only if

$$\bigwedge_{w \in F^+(v)} w = v.$$

As this condition can be checked in polynomial time in n and $|F|$ for a given v, we can build an oracle that answers membership queries for $f_{\max}$ in polynomial time.

5 Minimal Horn Extensions

5.1 Checking the minimality of a Horn DNF

Problem MINIMAL-H-EXTENSION
Input: A pdBf (T, F) and a Horn DNF φ.
Question: Is φ a minimal Horn DNF of (T, F) ?

We show via a series of lemmas that this problem can be solved in polynomial time. The proofs for some results are omitted due to the space constraint; see [14] for details.

For a pdBf (T, F), a vector $v \in T$ and a Horn DNF φ of (T, F), define

$$\begin{aligned} I(\varphi; v) &= \{l \in I(v) \mid (\bigwedge_{j \in ON(v)} x_j)\, \bar{x}_l \le f_\varphi\}, \\ R(\varphi; v) &= \{t \in R(v) \mid t \le f_\varphi\} \\ &(= \{(\bigwedge_{j \in ON(v)} x_j)\, \bar{x}_l \mid l \in I(\varphi; v)\}). \end{aligned}$$

Since it is easy to see that every minimal Horn extension φ can be represented by a canonical Horn DNF, $I(\varphi; v) \ne \emptyset$ and $R(\varphi; v) \ne \emptyset$ hold for all $v \in T$. The set $I(\varphi; v)$ can be constructed in linear time in $|\varphi|$, by using the forward chaining procedure [8]. We start with the next necessary and sufficient condition.

Lemma 5.1 Let φ be a Horn DNF of a given pdBf (T, F). Then φ is minimal if and only if $f_\varphi = f_\psi$ holds for every canonical Horn DNF ψ given by

$$\psi = \bigvee_{v \in T} t_v; \quad t_v \in R(\varphi; v). \qquad \square$$

It is known that there is a canonical Horn DNF φ that is not minimal but satisfies the *local minimality* (i.e., every DNF ψ obtained from φ by replacing one of its terms t_v by $t'_v \in R(\varphi; v) \setminus \{t_v\}$ for any $v \in T$ with $|R(\varphi; v)| > 1$ satisfies $f_\psi = f_\varphi$). This may suggest that problem MINIMAL-H-EXTENSION is not trivial. The condition in Lemma 5.1 can be rewritten as follows.

Lemma 5.2 Let φ be a Horn DNF of a given pdBf (T, F). Define

$$\begin{aligned} &\varphi^* = \bigvee_{v \in T} t^*_v, \\ &t^*_v = \bigwedge_{j \in ON(v)} x_j \bigwedge_{j \in I(\varphi; v)} \bar{x}_j, \end{aligned} \qquad (2)$$

where $t^*_v = t_v$ if $I(\varphi; v) = \emptyset$ (i.e., $OFF(v) = \emptyset$). Then φ is minimal if and only if $f_\varphi = f_{\varphi^*}$ holds. $\square$

Note that φ^* is not Horn, and therefore checking if $f_\varphi = f_{\varphi^*}$ may not be obvious. For a term t_v in a canonical Horn DNF $\varphi = \bigvee_{v \in T} t_v$, define

$$\hat{I}(\varphi; v) = I(\varphi; v) \setminus \{k\}, \qquad (3)$$

where $\bar{x}_k$ is the negative literal in t_v (if t_v has no negative literal, let $\hat{I}(\varphi; v) = I(\varphi; v) = \emptyset$). Then we define a term $d_i(t_v)$ for $i \in \hat{I}(\varphi; v)$, and a formula $d(t_v)$ as follows.

$$\begin{aligned} d_i(t_v) &= x_i t_v \qquad (4) \\ d(t_v) &= \begin{cases} \bigvee_{i \in \hat{I}(\varphi; v)} d_i(t_v) & \text{if } \hat{I}(\varphi; v) \ne \emptyset \\ \bot & \text{otherwise.} \end{cases} \end{aligned}$$

For example, if $t_v = 123\bar{4}$ and $I(\varphi; v) = \{4, 5, 6, 7\}$, then $\hat{I}(\varphi; v) = \{5, 6, 7\}$ and $d(t_v) = 123\bar{4}(5 \vee 6 \vee 7)$. It is easy to see that $T(d(t_v)) = T(t_v) \setminus T(t^*_v)$, since $t^*_v = \bigwedge_{j \in ON(v)} x_j \bigwedge_{j \in I(\varphi; v)} \bar{x}_j$. Hence for any t_v in φ, $d(t_v) \le f_{\varphi^*}$ holds if and only if $t_v \le f_{\varphi^*}$ holds. Thus $d(t_v) \le f_{\varphi^*}$ holds for all t_v in φ if and only if $f_\varphi \le f_{\varphi^*}$ holds. The last condition is equivalent to $f_\varphi = f_{\varphi^*}$ since $f_\varphi \ge f_{\varphi^*}$ is obvious from definition. This proves the following lemma.

Lemma 5.3 Let $\varphi = \bigvee_{v \in T} t_v$ be a canonical Horn DNF of a given pdBf. Then $f_\varphi = f_{\varphi^*}$ holds if and only if

$$d_i(t_v) \le f_{\varphi^*} \qquad (5)$$

holds for all t_v in φ and $i \in \hat{I}(\varphi; v)$. $\square$

Now we consider how to check condition (5) in polynomial time. Let $\varphi = \bigvee_{v \in T} t_v$ be a canonical Horn DNF. For a term t_v in φ and $i \in \hat{I}(\varphi; v)$, define

$$\begin{aligned} d^*_i(t_v) &= x_i t_v \Big(\bigwedge_{j \in OFF(v) \setminus I(\varphi; v)} \bar{x}_j \Big) \\ W_i(\varphi; v) &= \{w \in T \mid T(d^*_i(t_v)) \cap T(t^*_w) \ne \emptyset\}. \qquad (6) \end{aligned}$$

Lemma 5.4 Let $\varphi = \bigvee_{v \in T} t_v$ be a canonical Horn DNF of a pdBf (T, F). Then for every $w \in W_i(\varphi; v)$, where $v \in T$ and $i \in \hat{I}(\varphi; v)$, the following two properties hold:

(i) $ON(w) \subseteq ON(v) \cup \hat{I}(\varphi; v)$,
(ii) $I(\varphi; w) \subseteq I(\varphi; v) \setminus \{i\}$.

Proof. Recall that

$$d_i^*(t_v) = x_i t_v \, (\bigwedge_{j\in OFF(v)\setminus I(\varphi;v)} \bar{x}_j)$$
$$= \bigwedge_{j\in ON(v)\cup\{i\}} x_j \bigwedge_{j\in (OFF(v)\setminus I(\varphi;v))\cup\{k\}} \bar{x}_j \quad (7)$$
$$t_w^* = \bigwedge_{j\in ON(w)} x_j \bigwedge_{j\in I(\varphi;w)} \bar{x}_j.$$

If there exists an $l \in ON(w)\setminus(ON(v)\cup\hat{I}(\varphi;v))$ $(= ON(w)\cap(OFF(v)\setminus\hat{I}(\varphi;v)))$, then the negative literal $\bar{x}_l$ appears in $d_i^*(t_v)$ and the positive literal x_l appears in t_w^*; hence $T(d_i^*(t_v))\cap T(t_w^*) = \emptyset$, contradicting the definition of $W_i(\varphi;v)$. This proves property (i).

Next, to prove (ii), assume the contrary, i.e., there exists an index $l \in (I(\varphi;w)\setminus I(\varphi;v))\cup\{i\}$. The following three cases are possible.

(a) $l = i$: Then $T(d_i^*(t_v))\cap T(t_w^*) = \emptyset$ holds, since $d_i^*(t_v)$ contains x_i and t_w^* contains $\bar{x}_i$, which is a contradiction.

(b) $l \in ON(v)\cap I(\varphi;w)$: Then $T(d_i^*(t_v))\cap T(t_w^*) = \emptyset$ again holds, which is a contradiction.

(c) $l \in OFF(v)\cap I(\varphi;w)$: Clearly, $l \in (OFF(v)\setminus I(\varphi;v))\cap I(\varphi;w)$. Then $l \in I(\varphi;w)$ implies $(\bigwedge_{j\in ON(w)} x_j)\,\bar{x}_l \le f_\varphi$, and therefore, by property (i),

$$(\bigwedge_{j\in ON(v)\cup I(\varphi;v)} x_j)\,\bar{x}_l \le f_\varphi. \quad (8)$$

Now $l \in OFF(v)\setminus I(\varphi;v)$ implies $(\bigwedge_{j\in ON(v)} x_j)$ $\bar{x}_l \not\le f_\varphi$. However, since $(\bigwedge_{j\in ON(v)} x_j)\,\bar{x}_h \le f_\varphi$ for all $h \in I(\varphi;v)$, and

$$T((\bigwedge_{j\in ON(v)\cup I(\varphi;v)} x_j)\,\bar{x}_l) = T((\bigwedge_{j\in ON(v)} x_j)\,\bar{x}_l) \setminus T((\bigwedge_{j\in ON(v)} x_j)(\bigvee_{h\in I(\varphi;v)} \bar{x}_h)\,\bar{x}_l),$$

we have $(\bigwedge_{j\in ON(v)\cup I(\varphi;v)} x_j)\,\bar{x}_l \not\le f_\varphi$, which is a contradiction to (8). □

Lemma 5.5 Let $\varphi = \bigvee_{v\in T} t_v$ be a canonical Horn DNF of a pdBf (T,F). Then for every $v \in T$ and $i \in \hat{I}(\varphi;v)$, $d_i(t_v) \le f_{\varphi^*}$ holds if and only if $d_i^*(t_v) \le f_{\varphi^*}$ holds.

Proof. The only-if-part is immediate since $d_i^*(t_v) \le d_i(t_v)$ by definition. To prove the if-part, note that $d_i^*(t_v) \le f_{\varphi^*}$ holds if and only if

$$d_i^*(t_v) \le \bigvee_{w\in W_i(\varphi;v)} t_w^* \quad (9)$$

holds. This implies that the index set of $t_w^* = \bigwedge_{j\in ON(w)} x_j \bigwedge_{j\in I(\varphi;w)} \bar{x}_j$ satisfies

$$(ON(w)\cup I(\varphi;w))\cap(OFF(v)\setminus I(\varphi;v)) = \emptyset.$$

Thus by the definition (7) of $d_i^*(t_v)$ and t_w^*, we have

$$d_i^*(t_v) \le \bigvee_{w\in W_i(\varphi;v)} t_w^* \text{ if and only if } d_i(t_v) \le \bigvee_{w\in W_i(\varphi;v)} t_w^*. \quad (10)$$

This proves the if-part. □

By property (ii) of Lemma 5.4, checking condition $d_i^*(t_v) \le f_{\varphi^*}$ can be simplified, i.e., $d_i^*(t_v) \le f_{\varphi^*}$ is equivalent to $d_i^*(t_v) \le \bigvee_{w\in Q_v} t_w^*$, where $Q_v = \{w \in T \mid |I(\varphi;w)| < |I(\varphi;v)|\}$. In other words, it is not necessary to know $W_i(\varphi;v)$ of (9). Thus, combining Lemmas 5.2, 5.3 and 5.5, we have the following lemma.

Lemma 5.6 Let $\varphi = \bigvee_{v\in T} t_v$ be a canonical Horn DNF of a pdBf (T,F). Then φ is a minimal Horn DNF of (T,F) if and only if

$$d_i^*(t_v) \le \bigvee_{w\in Q_v} t_w^* \quad (11)$$

holds for all $v \in T$ and $i \in \hat{I}(\varphi;v)$, where $Q_v = \{w \in T \mid |I(\varphi;w)| < |I(\varphi;v)|\}$. □

Theorem 5.1 Let $\varphi = \bigvee_{v\in T} t_v$ be a canonical Horn DNF of a pdBf (T,F) with $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(|T|)}\}$, where $|I(\varphi;v^{(i)})| \le |I(\varphi;v^{(j)})|$ for $i < j$. Then φ is a minimal Horn DNF of (T,F) if and only if

$$d_i^*(t_{v^{(l)}}) \le \bigvee_{j\in\{1,2,\ldots,l-1\}} t_{v^{(j)}}, \quad i \in \hat{I}(\varphi;v^{(l)}),\ l = 1,2,\ldots,|T|. \quad (12)$$

Proof. By Lemma 5.6, φ is a minimal Horn DNF of (T,F) if and only if (11) holds for all $v^{(l)} \in T$ and $i \in \hat{I}(\varphi;v^{(l)})$. Then, since $\{v^{(1)}, v^{(2)}, \ldots, v^{(l-1)}\} \supseteq Q_{v^{(l)}}$ and $t_{v^{(l)}} \ge t_{v^{(l)}}^*$, we can conclude that (12) holds for all $v^{(l)}$ and i.

Conversely, assuming that (12) holds for all $v^{(l)}$ and i, we prove (11) by induction on l. In case of $l = 1$, (12) implies $d_i^*(t_{v^{(1)}}) = \bot$ for all $i \in \hat{I}(\varphi;v^{(1)})$, and hence (11) holds. Let $d_i^*(t_{v^{(l)}}) \le \bigvee_{w\in Q_{v^{(l)}}} t_w^*$ hold for $l = 1,2,\ldots,l^*-1$ and all $i \in \hat{I}(\varphi;v^{(l)})$. By (ii) of Lemma 5.4, this is equivalent to $d_i^*(t_{v^{(l)}}) \le \bigvee_{w\in W_i(\varphi;v^{(l)})} t_w^*$. Furthermore, by (10), we have $d_i(t_{v^{(l)}}) \le \bigvee_{w\in W_i(\varphi;v^{(l)})} t_w^*$. This is also equivalent to $d_i(t_{v^{(l)}}) \le \bigvee_{w\in Q_{v^{(l)}}} t_w^*$, and hence $\bigvee_{i\in\hat{I}(\varphi;v^{(l)})} d_i(t_{v^{(l)}}) \le \bigvee_{w\in Q_{v^{(l)}}} t_w^*$. Thus

$$t_{v^{(l)}} = t^*_{v^{(l)}} \vee (\bigvee_{i\in \hat{I}(\varphi;v^{(l)})} d_i(t_{v^{(l)}}))$$
$$\leq t^*_{v^{(l)}} \vee (\bigvee_{w\in Q_{v^{(l)}}} t^*_w) \leq \bigvee_{j\in\{1,2,\ldots,l\}} t^*_{v^{(j)}}.$$

Hence, (12) implies

$$d^*_i(v^{(l^*)}) \leq \bigvee_{j\in\{1,2,\ldots,l^*-1\}} t_{v^{(j)}} \leq \bigvee_{j\in\{1,2,\ldots,l^*-1\}} t^*_{v^{(j)}},$$

and (11) holds for $v = v^{(l^*)}$ by $W_i(\varphi; v^{(l^*)}) \subseteq Q_{v^{(l^*)}} \subseteq \{1,2,\ldots,l^*-1\}$. □

Note that relation (12) can be checked in polynomial time, since the right-hand side of (12) is a Horn DNF. In other words, MINIMAL-H-EXTENSION can now be solved in polynomial time by the following algorithm.

Algorithm CHECK-MINIMAL
Input: A pdBf (T,F) and a Horn DNF φ.
Question: Is φ a minimal Horn DNF of (T,F)?

Step 1: Check if f_φ is a Horn extension of (T,F). If not, output "no" and halt.

Step 2: Construct a canonical Horn DNF $\psi = \bigvee_{v\in T} t_v$ such that $f_\psi \leq f_\varphi$. If $f_\psi < f_\varphi$, then output "no" and halt; otherwise (i.e., $f_\psi = f_\varphi$), rewrite ψ as φ.

Step 3: Sort all vectors in T to have $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(|T|)}\}$, where $|I(\varphi; v^{(i)})| \leq |I(\varphi; v^{(j)})|$ for $i < j$.

Step 4: Output "yes" if (12) holds for φ; otherwise, "no". Then halt. □

By analyzing the time complexity of each step, it is not difficult to obtain the next theorem.

Theorem 5.2 Given a pdBf (T,F) and a Horn DNF φ, problem MINIMAL-H-EXTENSION can be solved in $O(|F||\varphi| + n|T||\varphi| + n|T|^2)$ time by algorithm CHECK-MINIMAL, where $T,F \subseteq \{0,1\}^n$ and $|\varphi|$ denotes the number of literals in φ. □

5.2 Unique minimal Horn extension

Lemma 5.7 Let $\varphi = \bigvee_{v\in T} t_v$ be a minimal canonical Horn DNF of a pdBf (T,F). Define

$$\varphi^\dagger = \bigvee_{v\in T} t^\dagger_v,$$
$$t^\dagger_v = \bigwedge_{j\in ON(v)} x_j \bigwedge_{j\in I(v)} \bar{x}_j, \quad v\in T,$$

where $t^\dagger_v = t_v$ if $I(v) = \emptyset$. Then (T,F) has the unique minimal Horn extension (which is f_φ) if and only if $f_\varphi = f_{\varphi^\dagger}$ holds. □

Note that this lemma corresponds to lemma 5.2, and all other lemmas, theorems and algorithms in Subsection 5.1 are valid, even if $*$ and $I(\varphi; v)$ are replaced by $\dagger$ and $I(v)$, respectively. (Recall that $t^\dagger_v$ and $I(v)$ become t^*_v and $I(\varphi;v)$, respectively, if φ represents $f_{\max}$.) Therefore, we have the following theorem.

Theorem 5.3 Deciding if a pdBf (T,F) has an unique minimal Horn extension can be done in $O(n|T|(|F| + n|T|^2))$ time. □

5.3 Generating a minimal Horn extension

To generate a minimal canonical Horn DNF of a given pdBf (T,F), we first construct a canonical DNF φ, and then recursively check if (12) holds for φ or not. If yes, output φ and halt. Otherwise, find a counterexample to condition (12); i.e.,

$$w \in T(d^*_i(t_{v^{(l^*)}})) \setminus T(\bigvee_{l\in\{1,2,\ldots,l^*-1\}} t_{v^{(l)}}) \qquad (13)$$

for $i \in \hat{I}(\varphi; v^{(l^*)})$, and update φ to φ' such that $\varphi'(w) = 0$ and $\varphi' < \varphi$. Formally, it can be written as follows.

Algorithm FIND-MINIMAL
Input: A pdBf (T,F).
Output: A minimal canonical Horn DNF φ of (T,F) if (T,F) has a Horn extension; otherwise, "no".

Step 1: If (T,F) has a Horn extension, construct a canonical DNF $\varphi = \bigvee_{v\in T} t_v$; otherwise, output "no" and halt.

Step 2: Sort all vectors in T to have $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(|T|)}\}$, where $|I(\varphi; v^{(i)})| \leq |I(\varphi; v^{(j)})|$ for $i < j$.

Step 3: Check if condition (12) holds for the current φ. If yes, output φ and halt. Otherwise, take the minimum $l = l^*$ for which

(12) does not hold, and find a counterexample w satisfying (13). Based on this w, define $R_{v^{(l)}} = \{t_{v^{(l)}}\}$ for $l = 1, 2, \ldots, l^* - 1$; $R_{v^{(l)}} = \{t \in R(\varphi; v^{(l)}) \mid t(w) = 0\}$ for $l = l^*, l^* + 1, \ldots, |T|$, and reconstruct a Horn DNF φ by

$$\varphi := \bigvee_{v^{(l)} \in T} t_{v^{(l)}}; \quad t_{v^{(l)}} \in R_{v^{(l)}}, \qquad (14)$$

where $t_{v^{(l)}} \in R_{v^{(l)}}$ is chosen arbitrarily if $|R_{v^{(l)}}| \geq 2$. Return to Step 2. □

Theorem 5.4 Given a pdBf (T, F), where T, $F \subseteq \{0,1\}^n$, a minimal canonical Horn DNF φ of (T, F) can be generated in $O(n|T|(|F| + n|T|^2))$ time if (T, F) has a Horn extension. □

6 NP-hardness results

Theorem 6.1 Computing a minimum Horn extension (i.e., with the smallest $|T(f)|$) of a pdBf (T, F) is NP-hard, even if $F = \emptyset$. □

Theorem 6.2 Computing a shortest Horn DNF φ (i.e., with the smallest $|\varphi|$) of a pdBf (T, F) is NP-hard □

References

[1] D. Angluin, M. Frazier, and L. Pitt, Learning conjunctions of Horn clauses, *31st Annual Symposium on Foundations of Computer Science*, (1990) 186-192.

[2] J. C. Bioch and T. Ibaraki, Complexity of identification and dualization of positive Boolean functions, *Information and Computation*, 123 (1995) 50-63.

[3] E. Boros, T. Ibaraki and K. Makino, Error-free and best-fit extensions of partially defined Boolean functions, RUTCOR Research Report RRR 14-95, Rutgers University, 1995.

[4] D. W. Dowling and J. H. Gallier, Linear-time algorithms for testing the satisfying of propositional Horn formulae, *Journal of Logic Programming*, 3 (1984) 267-284.

[5] R. Dechter and J. Pearl, Structure identification in relational data, *Artificial Intelligence*, 58 (1992) 237-270.

[6] T. Eiter and G. Gottlob, Identifying the minimal transversals of a hypergraph and related problems, *SIAM J. on Computing*, 24 (1995) 1278-1304.

[7] M. Fredman and L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, Technical Report LCSR-TR-225, Department of Computer Science, Rutgers University, 1994.

[8] P. L. Hammer and A. Kogan, Optimal compression of propositional Horn knowledge bases: Complexity and approximation, *Artificial Intelligence*, 64 (1993) 131-145.

[9] P. L. Hammer and A. Kogan, Horn functions and their DNFs, *Information Processing Letters*, 44 (1992) 23-29.

[10] A. Horn, On sentences which are true of direct unions of algebras, *Journal of Symbolic Logic*, 16 (1951) 14-21.

[11] H. A. Kautz, M. J. Kearns, and B. Selman, Horn approximations of empirical data, *Artificial Intelligence*, 74 (1995) 129-145.

[12] D. Kavvadias, C. H. Papadimitriou, and M. Sideri, On Horn envelopes and hypergraph transversals, *ISAAC'93* Algorithms and Computation, edited by K. W. Ng et al., Springer Lecture Notes in Computer Science, 762 (1993) 399-405.

[13] R. Khardon, Translating between Horn representations and their characteristic models, Technical Report TR-03-95, Center for Research in Computing Technology, Harva- rd University, February 1995.

[14] K. Makino, K. Hatanaka and T Ibaraki, Horn extensions of a partially defined Boolean function, RUTCOR Research Report RRR 27-95, Rutgers University, 1995.

[15] B. Selman and H. Kautz, Knowledge compilation using Horn approximations, *Proceedings of the Ninth National Conference on Artificial Intelligence*, (1991) 904-909.