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マトロイド的特性に関する点除去問題のプライマル・デュアル法による近似

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概要：本稿は、グラフの遺伝的特性に関する点除去問題に対する多項式時間近似可能性について論じる。そのような点除去問題は一般に NP-困難であるが、極小禁止グラフが有限個しか存在しない特性の場合、最小値の高々ある定数倍の値をもつ近似解が効率良く求められることが知られている。これに対し、極小禁止グラフが無限個存在する場合は、帰還点集合問題を唯一の例外として、そのような多項式時間アルゴリズムは知られていない。

ここでは、どのようなグラフの辺集合の上でも定義できるようなマトロイドによって導入される遺伝的特性に着目する。まず初めに、そのような特性に関する点除去問題全てが一様に、簡単かつ新しい整数計画として定式化されることを示す。そして、この整数計画ならびにその線形計画緩和の双対から、そのような点除去問題全てに対するプライマル・デュアル近似アルゴリズムを設計する。このアルゴリズムを解析し、保証できる近似比の一般式を与える。

次に、このプライマル・デュアル近似アプローチの応用の一つとして、帰還点集合問題が唯一の例外ではないことを示す。即ち、極小禁止グラフを無限個もちながら、対応する点除去問題において近似比２の解が効率良く計算できるような遺伝的特性が他にも、しかも無限個存在することを示す。グラフのマトロイダル・ファミリーという概念とその定義の緩和から、そのようなグラフ特性を導く。しかも、そのような特性の無限列は、点被覆問題や帰還点集合問題におけるグラフ特性の一般化となっている。

A Primal–Dual Approximation of Node–Deletion Problems for Matroidal Properties

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Abstract : This paper is concerned with polynomial time approximability of node–deletion problems for hereditary properties. It has been known that when such a property has only a finite number of minimal forbidden graphs the corresponding node–deletion problem can be efficiently approximated to within some constant factor of the optimum. On the other hand when there exist infinitely many minimal forbidden graphs no constant factor approximation has been known except for the case of the Feedback Vertex Set (FVS) problem in undirected graphs.

We will focus our attention to such properties that are derived from matroids definable on the edge set of any graph. It will be shown first that all the node–deletion problem for such properties can be uniformly formulated by a simple but non–standard form of the integer programming. The primal–dual approximation algorithm for all such problems is then presented based on this and the dual of its linear programming relaxation. The performance ratios implied by this approach will be analyzed and given in a general form.

We will show next, as an application of the primal–dual approach to the node–deletion problems, that the FVS problem is not the sole exceptional case; i.e., there exist other graph (hereditary) properties with an infinite number of minimal forbidden graphs, for which the node–deletion problems are efficiently approximable to within a factor of 2. In fact, we will show, there are infinitely many of them. Such properties are derived from the notion of matroidal families of graphs and relaxing the definitions for them. It will be observed at the same time that an infinite sequence of such properties is constituted with those for both Vertex Cover and FVS problems at its basis and thus providing a proper generalization of them.

1 Introduction

This paper is concerned with polynomial time approximability of node–deletion problems for hereditary properties. The *node–deletion problem for a graph property* π (denoted ND(π) throughout the paper) is a typical graph optimization problem; that is, given a graph G with weights on nodes, find a node set of the minimum weight sum whose deletion from G leaves a subgraph satisfying the property π. Many well known graph problems fall into this class of problems when desired graph properties are specified appropriately. Lewis and Yannakakis proved a general result that whenever π is nontrivial and hereditary on induced subgraphs ND(π) is *NP*–hard [12]. Here a property π is *nontrivial* if infinitely many graphs satisfy π and infinitely many graphs fail to satisfy it. It is *hereditary* on induced subgraphs if, in any graph satisfying π, every node–induced subgraph also satisfies π. When this general result of *NP*–hardness was established in 1980, almost nothing was known about approximability of ND(π) with a sole exception of good approximation algorithms for the Vertex Cover (VC) problem. Thus it was natural for them to pose a question: can other node–deletion problems be approximated well ? This question is largely unanswered even today, and more specific one is still open: what (of graph properties) separate the problems with constant approximation ratios from others ?

There also exist some general results on approximability of node–deletion problems within the recent framework of complexity of approximating *NP*–hard optimization problems. As pointed out in [13] the result cited above with those of [15] and [2] imply that ND(π) for every nontrivial hereditary π is *MAX SNP*–hard, and hence, no polynomial time approximation scheme exists for any of them unless $P = NP$. There exists, however, a stronger "lower bound" for an approximation ratio achievable in polynomial time for any ND(π); namely the best possible approximation ratio for the VC problem because no ND(π) can have one better than the one for VC (this is due to the generic reductions of [12]). The currently known best lower bound (1.068 of [7]) for this ratio appears to be still quite weak. On the other hand the situation for the upper bounds is not much better. A better approximation of the VC problem has been a subject of extensive research over the years, yet the best constant bound has remained same at 2, which a simple maximal matching heuristic [11] can achieve, while the best known heuristics can accomplish only slightly better ($2 - \frac{\log \log n}{2 \log n}$ of [4, 14]).

Another observation presented in [13] is that whenever hereditary π has only a finite number of minimal forbidden graphs ND(π) can be efficiently approximated to within some constant factor of the optimum. Lund and Yannakakis in fact conjectured that those with finitely many forbidden graphs are the only hereditary properties which yield such node–deletion problems that are amenable to constant factor approximation (see also [19]). It was found later, however, that this conjecture does not hold as is when the (unweighted) Feedback Vertex Set (FVS) problem in undirected graphs was shown to be approximable to within a constant factor [5]. This approximation ratio was later on extended to the one for the weighted FVS and was further improved to 2 in [3, 6], matching the best constant factor known for the VC problem. Recently Chudak et al. [8] gave a primal–dual interpretation of those 2–approximation algorithms of [3, 6] for the FVS problem. They also provided a new primal–dual algorithm for the problem, which has the same performance ratio but is slightly simpler than the previous two.

In this paper we shall concentrate on such hereditary properties that can be derived from (independent sets of) matroids definable on the edge set of any graph (details given later). The class of ND(π)'s for such properties includes VC, FVS, and many others. It will be shown then that all ND(π)'s in this class can be formulated by a simple and identical form of the integer programming using matroid rank functions. Moreover the primal–dual approximation algorithm for ND(π) designed based on such a formulation and the dual of its linear programming relaxation becomes even simpler than those for the FVS problem given in [8]. An analysis of this algorithm reveals that its performance ratio bound can be reduced to the one arising from the combinatorial structure underlying problems of interest.

We will show next, as an application of the primal–dual approach to ND(π), that the FVS problem is not the sole exceptional case; i.e., there exist other graph (hereditary) properties with an infinite number of minimal forbidden graphs, for which the node–deletion problems are efficiently approximable to within a factor of 2. In fact, we will show, there are infinitely many of them (at least countably many). Such properties are derived from the notion of matroidal families of graphs and relaxing the definitions for them (details later). It will be observed at the same time that an infinite sequence of such properties is constituted with those for both Vertex Cover and FVS problems at its basis and thus providing a proper generalization of them.

2 Preliminaries

2.1 Notation and Definitions

A subgraph of $G = (V, E)$ induced by $X \subseteq V$ is denoted by $G[X]$. For $X \subseteq V$ let $E[X]$ denote the set of edges induced by X, and conversely, let $V[F]$ for $F \subseteq E$ denote the set of vertices incident to some edge in F. $E[X, Y]$ is the set of edges with one end in X and the other in Y. For any graph G let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, of G. The set of edges incident to a node u is denoted $\delta(u)$, and when those edges are restricted to the ones in a subgraph $G[X]$ we denote it by $\delta_X(u)(= \delta(u) \cap E[X])$.

For a hereditary property π any graph which does not satisfy π is called a *forbidden* graph for π, and it is a *minimal* one if, additionally, every "proper" induced subgraph of it satisfies π. Any hereditary property π is equivalently characterized by the set of all minimal forbidden graphs for π.

It is customary to measure the quality of an approximation algorithm by its *performance ratio*, which is the worst case ratio of the optimal solution value to the value of an approximate solution returned by the algorithm.

2.2 Matroidal Properties

One way to represent a *matroid* M is by a pair of a *ground set* E and a *rank function* r defined on 2^E. A set $F \subseteq E$ is called *independent* in M iff $r(F) = |F|$, and conversely, $r(F)$ is the cardinality of a largest independent subset of F for an arbitrary $F \subseteq E$. A set which is not independent is called *dependent.* A maximal (and hence, maximum in any matroid) independent set is called a *base*, whereas a minimal dependent set is called a *circuit*, of the matroid. For any matroid $M = (E, r)$ there is a *dual matroid* $M^d = (E, r^d)$ defined on the same ground set E. The rank functions r and r^d are related s.t.

$$r^d(E - F) = (|E| - r(E)) - (|F| - r(F))$$

for any $F \subseteq E$ (For more on matroid theory see, for instance, [18]).

Let M be a matroid which can be defined on the edge set of any graph (called an *edge set matroid*) and denote by $M(G)$ the matroid defined by M on the edge set of G. We say that a graph property π is *matroidal* if for some edge set matroid M a (sub)graph G satisfies π iff its edge set is independent in $M(G)$ (Such a property is said to be *derived from* the matroid M). Such a property is hereditary on induced subgraphs because a subset of an independent set is independent in any matroid. Therefore, node-deletion problems for any nontrivial matroidal properties are *NP*–hard and *MAX SNP*–hard according to the results of [12] and [13]. Also note that the family of minimal forbidden graphs for such a property π corresponds to the family of circuits of the corresponding matroid $M(G)$ for all possible G.

2.3 Matroidal Families of Graphs

A *matroidal family of graphs* is a non–empty collection P of finite, *connected* graphs with the following property: given an arbitrary graph G, the edge sets of the subgraphs of G that are isomorphic to some member of P are the circuits of a matroid on $E(G)$. The matroid defined this way by the matroidal family P on the edge set of graph G will be denoted by $P(G)$.

The following four matroidal families, $P_0, P_1, P_2,$ and P_3, are those that were discovered first [16, 17]. The family P_0 consists of one graph only, namely two node with one edge in between. This is also the only finite matroidal family. The family P_1 consists of all the cycles; thus, $P_1(G)$ is the cycle matroid defined on $E(G)$. The family P_2 consists of all the bicycles, where a bicycle is a graph formed by minimally connecting two independent cycles. These two cycles can be joined together by either (1) sharing only a single node, (2) sharing only a connected path, or (3) having a simple path attached only at each end of it. The family P_3 consists of all the even cycles (i.e. cycles of even length) and the bicycles with no even cycle. The matroidal properties derived from these families thus correspond, respectively, to "a graph has no edge" (P_0), "a graph contains no cycle" (P_1), "every connected component contains at most one cycle" (P_2), and "every connected component contains at most one odd cycle and no even cycle" (P_3). Therefore, ND(π) is actually the VC (FVS, respectively) problem when π is the matroidal property derived from P_0 (P_1, respectively).

It has been known that in fact there exist infinitely many (uncountably many) matroidal families of graphs, and the first description of them (countably many matroidal families) was obtained by Andreae [1]:

Proposition 1 *Let s and t be integers, $s \geq 0$ and $-2s+1 \leq t \leq 1$. Let $P_{s,t}$ be the set of all graphs G s.t.*

(i) *$s|V(G)| + t = |E(G)|$, and*

(ii) *G is minimal with respect to property (i); i.e., no graph isomorphic to a proper subgraph of G satisfies property (i).*

Then $P_{s,t}$ is a matroidal family.

It is not so hard to verify that $P_1 = P_{1,0}$, $P_2 = P_{1,1}$, and $P_0 = P_{s,-2s+1}$ (P_3 is not of the form $P_{s,t}$).

3 Primal–Dual Approximation for Matroidal Properties

Chudak et al. gave new primal–dual formulations and the algorithms based on them for the FVS problem in undirected graphs [8]. These algorithms are not new ones but actually are primal–dual "interpretations" of the previously known algorithms from [3, 6]. We shall show below that in fact every ND(π) with matroidal π has simple and identical primal–dual formulation as well as algorithm based on it. Chudak et al. also gave a new algorithm for the FVS problem which is a modification and slight simplification of the previous algorithms cited above. Our algorithm for ND(π) is even simpler than theirs.

We claim that ND(π) on graph $G = (V, E)$ can be formulated by the following integer programming when π is a matroidal property derived from $M = (E(G), r_G)$:

$$\text{(IP)} \qquad \begin{array}{lll} \text{Min} & \sum_{u \in V} w_u x_u & \\ \text{subject to:} & & \\ & \sum_{u \in S} r^d_{G[S]}(\delta_S(u)) x_u \geq r^d_{G[S]}(E[S]) & S \subseteq V \\ & x_u \in \{0,1\} & u \in V \end{array}$$

Theorem 2 *When π is a matroidal property F is a solution of ND(π) iff $x^F \in \{0,1\}^V$ (incidence vector of F) is a feasible solution to (IP).*

Consider now the dual of the linear programming relaxation of (IP):

$$\text{(D)} \qquad \begin{array}{lll} \text{Max} & \sum_{S \subseteq V} r^d_{G[S]}(E[S]) y_S & \\ \text{subject to:} & & \\ & \sum_{S: u \in S} r^d_{G[S]}(\delta_S(u)) y_S \leq w_u & u \in V \\ & y_S \geq 0 & S \subseteq V \end{array}$$

The primal–dual approximation algorithm, based on (IP) and (D) above, for ND(π) with matroidal π is presented in Figure 1. We elaborate more on it below. The algorithm starts with $F = \emptyset$, the original graph $G[S'] = (V, E)$ and the dual feasible solution $y = 0$. Given F, if it is not yet a solution of ND(π) there must exist some set $S \subseteq V$ corresponding to a violated constraint of (IP). In particular the set of all the remaining nodes $S'(= V - F)$ must be always such a set, and thus we can always choose S' as a "violated set". The algorithm then increases the dual variable $y_{S'}$ as much as possible until for some vertex u in S' the dual constraint for u becomes tight; i.e., $\sum_{S: u \in S} r^d_{G[S]}(\delta_S(u)) y_S = w_u$. The algorithm adds u into a solution set F and at the same time removes it from S'. Clearly F eventually becomes a solution of ND(π) (and to (IP)) while y is kept feasible to (D). Lastly, the nodes in F are examined one by one, in the reverse order of their inclusion to F, and whenever any of them is found to be extraneous it is thrown out of F.
The analysis of the performance ratio of this algorithm can be reduced to the following combinatorial arguments.

Initialize $F = \emptyset, S' = V, y = 0, l = 0$.
While F is not a solution of ND(π) do
  $l \leftarrow l + 1$.
  Increase $y_{S'}$ until for some $u \in S'$ the dual constraint corresponding to u becomes tight.
  Let $u_l \leftarrow u$.
  Add u_l into F and remove u from S'.
For $j = l$ downto 1 do
  if $F - \{u_j\}$ is a solution of ND(π) in G then remove u_j from F.
Output F.

Figure 1: Primal–Dual Approximation Algorithm for ND(π)

Theorem 3 *Let π be a matroidal property derived from $M = (E(G), r_G)$. Then the performance ratio of the primal–dual algorithm is bounded by*

$$\max\{\frac{\sum_{u \in F} r_G^d(\delta(u))}{r_G^d(E)}\}$$

where max *is taken over any minimal solution F of ND(π) in any $G = (V, E)$.*

4 Uniformly Sparse Graph Properties

It was shown in [9] that when π is derived either from P_0 or P_1 (i.e., the VC or FVS property) (an essentially same algorithm as) the primal–dual algorithm delivers a solution with approximation ratio of 2. We add here one more to this list:

Theorem 4 *When π is derived from P_3 the primal–dual algorithm for ND(π) has performance ratio of 2.*

The case of $P_2 \equiv P_{1,1}$ will be subsumed by the general result given below.

We now turn our attention to a "relaxation" of the matroidal families of graphs, dropping the connectivity requirement on graphs in the families. Recall the countably many matroidal families $P_{s,t}$ ($s \geq 0, -2s + 1 \leq t \leq 1$) of graphs from Proposition 1. Fix s to 1, let t be any integer $\geq -2s + 1 = -1$, and consider the sets of graphs, that are no longer necessary to be connected, using the same set of the definitions for $P_{s,t}$; i.e., $P_{1,t}$ is the set of all graphs G s.t.

(i) $|V(G)| + t = |E(G)|$, and

(ii) G is minimal with respect to property (i); i.e., no graph isomorphic to a proper subgraph of G satisfies property (i).

Let $Q_k \stackrel{\text{def}}{=} P_{1,k+1}, k \geq -2$. It is useful to observe here what graph properties are actually derived from Q_k's. A graph $G = (V, E)$ satisfies such a property iff for every $F \subseteq E$, $|F| - |V[F]| \leq k$, and thus we may call such a property *uniformly k-sparse.*

Proposition 5 *Q_k defines the set of circuits of a matroid on any (edge set of) graph for all k.*

We should also note:

Proposition 6 *Q_k consists of an infinite number of distinct graphs for all $k \geq -1$.*

The next is a key lemma of the present paper:

Lemma 7 *Let π be a property derived from $Q_k = (E(G), r_G)$. Suppose X is any minimal solution of ND(π) in any G. Then,*

$$\sum_{u \in X} r_G^d(\delta(u)) \leq 2 \cdot r_G^d(E).$$

Proof. See Appendix. □

Finally, observe that given $G = (V, E)$ we can compute efficiently the rank $r(F)$ of any $F \subseteq E$ (and thus $r^d(\delta(u))$ for each $u \in V$) under $Q_k(G)$ (for instance, using the formula (1)). Therefore, our primal–dual algorithm runs in polynomial time for every Q_k. Now from Lemma 7 and Theorem 3 it easily follows that

Corollary 8 *When π is such a property that is derived from Q_k for any fixed k the primal–dual algorithm computes a solution of ND(π) in polynomial time; its performance ratio is bounded above by 2.*

And hence, there exist at least countable many nontrivial hereditary properties with an infinite number of minimal forbidden graphs, for which the node–deletion problems are efficiently approximable to within a factor of 2.

Appendix

Definitions. Let C be a connected component. Define the *surplus sp* of C by $sp(C) = |E(C)| - |V(C)|$ and the *bounded surplus* $\bar{sp}$ of C by $\bar{sp}(C) = \min\{k, sp(C)\}$. Let $\mathcal{C}^+(F)$ ($\mathcal{C}^-(F)$) denote the set of components, induced by an edge set F, with a positive bounded surplus (with a negative bounded surplus, respectively). When E' is an edge subset of E define $sp(E')$ to be the surplus of the graph induced by E'.

Notice that $\mathcal{C}^-(F)$ consists of all the acyclic components, each with a (bounded) surplus of -1, induced by F. Also notice that for any component C and for any $E' \subseteq E(C)$, $sp(E') \leq sp(C)$. The rank function of the matroid $Q_k(G)$ defined on $G = (V, E)$ can be given by

$$r(F) = |V[F]| + \min\{k, \sum_{C \in \mathcal{C}^+(F)} \bar{sp}(C)\} - |\mathcal{C}^-(F)| \tag{1}$$

for any $F \subseteq E$.

Proof of Lemma 7. Assume throughout that $k \geq 0$ (the case of $k \leq -1$ is no harder). Consider first the edge set $E[V - X]$, which must be an independent set of $Q_k(G)$, since X is a solution of ND(π). Using (1) we have

$$|E[V - X]| = r(E[V - X]) = |V - X| - (\text{\# of acyclic components in } G[V - X]) + l \tag{2}$$

for some $0 \leq l \leq k$. We shall use the following auxiliary lemma in proving Lemma 7.

Lemma 9 *Assume (2). If X is a minimal solution of ND(π) then*

$$|E[X, V - X]| \geq (k - l + 1)|X| + \sum_{u \in X} (\text{\# of acyclic components, in } G[V - X], \text{ adjacent to } u) \tag{3}$$

Proof. For every acyclic component T in $G[V - X]$ adjacent to $u \in X$, pick one edge e connecting u and T, and color e red so that u and its adjacent acyclic components form a single tree via these red edges.

Let $D \subseteq E$ be any circuit of $Q_k(G)$ induced by u and $V - X$. We shall examine how many edges necessarily exist in $E[\{u\}, V - X]$ besides those colored red. Since D is a circuit it belongs to Q_k, and by the definition of Q_k we know $sp(D) = k + 1$. Also recall that $E[V - X]$ is an independent set of $Q_k(G)$, which implies that every subgraph of $G[V - X]$ has surplus of at most l. Addition of u and all the red edges to such a subgraph results in no increase in its surplus. Further addition of any single edge of no color between u and $V - X$ can increase its surplus by at most one. Therefore, D must contain at least $k + 1 - l$ edges from $E[\{u\}, V - X]$ other than those red edges, and this means the existence of that many edges in $E[\{u\}, V - X]$. Thus,

$$|E[\{u\}, V - X]| \geq k - l + 1 + (\text{\# of acyclic components, in } G[V - X], \text{ adjacent to } u)$$

for each u in X. Summing up over all the nodes in X, we obtain (3). □

Suppose G contains an acyclic component T. Then since X contains no node of T, due to its minimality, we can restrict ourselves w.l.o.g. to G without T. So assume that G contains no acyclic component. Now suppose $r(E) < |V| + k$. Then E must be independent in $Q_k(G)$ and G satisfies π. But then a solution X minimal in G must be empty and the inequality in question trivially holds.

So assume that $r(E) = |V| + k$, and using (2) we can write

$$\begin{aligned} r^d(E) &= |E| - r(E) \\ &= |E[X]| + |E[X, V - X]| - (r(E) - |E[V - X]|) \\ &= |E[X]| + |E[X, V - X]| \\ &\quad -(|X| + k - l + (\#\text{ of acyclic components in } G[V - X])) \end{aligned} \quad (4)$$

Consider first the case when X consists of a singleton and let $X = \{u\}$. In this case $E[X] = \emptyset$ and $\delta(u) = E[X, V - X]$. Hence, using the general formula for dual rank,

$$\begin{aligned} r^d(\delta(u)) &= (|E| - r(E)) - (|E[V - X]| - r(E[V - X])) \\ &= (|E| - (|V| + k)) - (|E[V - X]| \\ &\quad -(|V - X| - (\#\text{ of acyclic components in } G[V - X]) + l)) \\ &= |E[X, V - X]| - |X| - (k - l) - (\#\text{ of acyclic components in } G[V - X]) \end{aligned}$$

Applying (3) with $|X| = 1$ we get

$$\begin{aligned} & 2r^d(E) - \sum_{u \in X} r^d(\delta(u)) \\ \geq\ & |E[X, V - X]| - 2(|X| + k - l + (\#\text{ of acyclic components})) + |X| + (k - l) \\ & +(\#\text{ of acyclic components}) \\ =\ & |E[X, V - X]| - 1 - (k - l) - (\#\text{ of acyclic components}) \\ \geq\ & 0 \end{aligned}$$

Next assume that $|X| \geq 2$. Call such a component in $G[V - X]$ that is adjacent to a single node in X as a *leaf* component. Recall that the dual rank of any $E' \subseteq E$ under a matroid M can be equivalently defined as

$$r^d(E') = \max\{|E' - B| : B \text{ is a base of } M\}$$

Take any node u of X. To estimate the value of $r^d(\delta(u))$ we observe how many edges incident to u must belong to a base of $Q_k(G)$. Let $I, J \subseteq E$ be mutually disjoint sets s.t. I is an independent set and $sp(J) \leq 0$. And then, $I \cup J$ in general is an independent set of $Q_k(G)$. This observation allows us to argue that, for every acyclic leaf component T which is adjacent only to u, any base B of $Q_k(G)$ must use all the edges in T and (at least) one edge connecting u and T. Besides them B must use at least one more edge from $\delta(u)$. To see why notice first that if no other edges of $\delta(u)$ belong to B the component of B containing u is a tree at best. As observed above, however, it is always possible to extend this component by one more edge incident to it *if it exists*. So it remains to see that at least one more edge is incident to u, and this is easy to do for, otherwise, u belongs to an acyclic component of G, which we've excluded at the beginning of the current analysis. Therefore, we can write

$$r^d(\delta(u)) \leq |\delta(u)| - ((\#\text{ of acyclic leaf components adjacent to } u) + 1)$$

and hence,

$$\sum_{u \in X} r^d(\delta(u)) \leq 2|E[X]| + |E[X, V - X]| - |X| - (\#\text{ of acyclic leaf components}) \quad (5)$$

Notice that, since there is no isolated acyclic component in G, we can reduce (3) to

$$\begin{aligned} |E[X, V - X]| \ \geq\ & (k - l + 1)|X| + 2(\#\text{ of acyclic non–leaf components}) \\ & +(\#\text{ of acyclic leaf components}) \end{aligned} \quad (6)$$

Combining (4), (5) and (6),

$$\begin{aligned}
&2r^d(E) - \sum_{u\in X} r^d(\delta(u)) \\
&\geq |E[X, V-X]| - 2(|X| + k - l + (\#\text{ of acyclic components in } G[V-X])) \\
&\quad +|X| + (\#\text{ of acyclic leaf components in } G[V-X]) \\
&= |E[X, V-X]| - (|X| + 2(k-l)) \\
&\quad -(2(\#\text{ of acyclic non–leaf components}) + (\#\text{ of acyclic leaf components})) \\
&\geq (k-l)(|X|-2) \\
&\geq 0
\end{aligned}$$

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