劣モジュラ, 正モジュラシステムの多面体構造

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あらまし　Vを有限集合, $\Re$を実数集合とする. 交差劣モジュラ, 正モジュラ集合関数$f: 2^V \mapsto \Re$に対し, システム(V, f)の多面体$P = \{z \in \Re^V_- \mid \sum_{i \in X} z(i) \leq f(X),\ \forall X \in 2^V\}$を考える. ただし, $\Re^V_-$は$|V|$-次元非正ベクトルの集合とする. まず, Pが$\{z \in \Re^V_- \mid \sum_{i \in X} z(i) \leq f(X),\ \forall X \in \mathcal{X}\}$と等価となるようなラミナー族$\mathcal{X} \subseteq 2^V$が常に存在することを示す. この性質に基づけば, 新しく付加される辺に接続する節点数を最小にすることを目的とした辺連結度増大問題を多項式時間で解くことができる.

Polyhedral Structure of Submodular and Posi-modular Systems

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Abstract Let V be a finite set, and $\Re$ be the set of reals. We consider the polyhedron $P = \{z \in \Re^V_- \mid \sum_{i \in X} z(i) \leq f(X),\ \forall X \in 2^V\}$ for a system (V, f) with an intersecting submodular and posi-modular set function $f: 2^V \mapsto \Re$, where $\Re^V_-$ denotes the set of $|V|$-dimensional nonpositive vectors. We first prove that there is a laminar family $\mathcal{X} \subseteq 2^V$ such that P is characterized by $\{z \in \Re^V_- \mid \sum_{i \in X} z(i) \leq f(X),\ \forall X \in \mathcal{X}\}$. Based on this, we can solve in polynomial time the edge-connectivity augmentation problem with an additional constraint that the number of vertices to which new edges are incident is minimized.

1 Introduction

Let V be a finite set, where $|V|$ is denoted by n. A singleton set $\{v\}$ may be written as v. For two subsets $X, Y \subseteq V$, we say that X and Y *intersect* each other if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$ and $Y - X \neq \emptyset$. A family $\mathcal{X} \subset 2^V$ is called *laminar* if no two subsets in $\mathcal{X}$ intersect each other. Two intersecting sets X and Y are called *crossing* if $V - (X \cup Y) \neq \emptyset$ also holds.

Let $\Re$ (resp., $\Re_+$ and $\Re_-$) be the set of reals (resp., nonnegative reals and nonpositive reals), and let $\Re^V$ (resp., $\Re^V_+$ and $\Re^V_-$) be the set of n-dimensional real vectors (resp., nonnegative real vectors and nonpositive real vectors) on a ground set V. A *set function* f on V is a function $f: 2^V \mapsto \Re$. For a vector $z \in \Re^V$ and a subset $X \subseteq V$, we denote $\sum_{i \in X} z(i)$ by $z(X)$. Such a function $z: 2^V \mapsto \Re$ is called *modular*. A function f is called *fully (resp., intersecting, crossing) submodular* if it satisfies the following inequality

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \tag{1}$$

holds for every (resp., intersecting, crossing) pair of sets $X, Y \subseteq V$. A function f is called *fully (resp., intersecting, crossing) supermodular* if $-f$ is fully (resp., intersecting, crossing) submodular. An f is called *symmetric* if $f(X) = f(V - X)$ holds for all $X \subseteq V$. In this paper, we call a function f *fully (resp., intersecting, crossing) posi-modular* if

$$f(X) + f(Y) \geq f(X - Y) + f(Y - X) \tag{2}$$

holds for every (resp., intersecting, crossing) pair of sets $X, Y \subseteq V$ [6]. An f is called *fully (resp., intersecting, crossing) nega-modular* if $-f$ is fully (resp., intersecting, crossing) posi-modular. Any modular function z such that $z(i) \geq 0$ for all $i \in V$ is posi-modular. Also a symmetric fully submodular function f is fully posi-modular. However, the converse is not generally true.

A pair (V, f) of a finite set V and a set function f on V is called a *system.* The optimization in a system (V, f) has been much studied, such as:

$$\textbf{Problem 1 (primal type):} \quad \begin{array}{lll} \text{minimize} & \Phi(z) & \\ \text{subject to} & z(X) \leq f(X) & \text{for all } X \in 2^V \\ & 0 \leq z(i) \leq d(i) & \text{for all } i \in V \end{array}$$

(an additional constraint $z(V) = f(V)$ may also be imposed), where $\Phi(z) : R^V \mapsto \Re$ is an objective function and $d \in \Re^V_+$ is a given constant vector. For fully submodular functions f, Problem 1 appears in many applications [5]. Given a system (V, g), a dual type of this problem is stated as follows:

$$\textbf{Problem 2 (dual type):} \quad \begin{array}{lll} \text{minimize} & \Phi(t) & \\ \text{subject to} & g(X) \leq t(X) & \text{for all } X \in 2^V \\ & 0 \leq t(i) \leq d(i) & \text{for all } i \in V \end{array}$$

(where we may also impose an additional constraint $t(V) = g(V)$). Problem 2 with a certain supermodular function g appears in the edge-connectivity augmentation problem [1, 3] and the problem of computing the core of a convex game [8]. The above Problems 1 and 2 are generalized into the following common formulation. For two set functions g and f on V, and vectors $d_1, d_2 \in \Re^V_+$, we consider:

$$\textbf{Problem 3 (mixed type):} \quad \begin{array}{lll} \text{minimize} & \Phi(z) & \\ \text{subject to} & g(X) \leq z(X) \leq f(X) & \text{for all } X \in 2^V \\ & d_1(i) \leq z(i) \leq d_2(i) & \text{for all } i \in V. \end{array}$$

In this paper, we consider Problems 1-3 with intersecting submodular and posi-modular functions f and $-g$.

Before going into details, let us explain an application of Problem 2 to the edge-connectivity augmentation problem. Let $N = (V, E, c)$ be an undirected complete network with a vertex set V, an edge set $E = V \times V$ and an edge weight function $c : E \mapsto \Re_+$. The *cut function* $f_N : 2^V \mapsto \Re_+$ is defined by $f_N(X) = \sum\{c(e) \mid e = \{u, v\} \in E,\ u \in X,\ v \in V - X\}$ (where $f_N(\emptyset) = f_N(V) = 0$). It is known (and easy to see) that the cut function f_N is symmetric and fully submodular. The *edge-connectivity augmentation problem* asks to increase edge weights c to obtain a k-edge-connected network N' (i.e., $f_{N'}(X) \geq k$ holds for all $X \in 2^V - \{\emptyset, V\}$). Frank [3] introduced an additional constraint to this problem, *the degree constraint*: Given a vector $d \in \Re^V_+$, the output k-edge-connected network N' is required to satisfy $\sum_{e \in E(i)}(c'(e) - c(e)) \leq d(i)$ for all $i \in V$, where $E(i)$ denotes the set of edges incident to a vertex i. The problem can be formulated as Problem 2 by using the following result.

Lemma 1 [1, 3] *Given a network $N = (V, E, c)$, a constant $k \geq 0$, and a vector $t \in \Re^V_+$ such that*

$$f_N(X) + t(X) \geq k \text{ for all } X \in 2^V - \{\emptyset, V\}, \tag{3}$$

there is a k-edge-connected network $N' = (V, E, c')$ satisfying $\sum_{e \in E(i)}(c'(e) - c(e)) = t(i)$ for all $i \in V$. Also, the c' can be chosen as integers if c, t and $k \geq 2$ are integers and $t(V) = \sum_{i \in V} t(i)$ is an even integer. □

Notice that the total increase $\sum_{e \in E}(c'(e) - c(e))$ of weights is $\frac{1}{2}t(V)$. Therefore, in order to solve the edge-connectivity augmentation problem, we only need to find a vector $t \in \Re^V_+$ that minimizes $t(V) = \sum_{i \in V} t(i)$ among all vectors t satisfying (3) (and $t(i) \leq d(i)$, $i \in V$ if the degree constraint is imposed). Hence, by defining $\Phi(t) = \frac{1}{2}t(V)$ and a fully supermodular set function g by $g(X) = k - f_N(X)$ for all $X \in 2^V$, we see that the smallest amount $\alpha(N, k)$ of new weights to be added to obtain a k-edge-connected network N' is given by the minimum value of $\Phi(t)$ over all $t \in \Re^V_+$ satisfying $g(X) \leq t(X)$ for all $X \in 2^V - \{\emptyset, V\}$ (and $t(i) \leq d(i)$, $i \in V$ if the degree constraint is imposed). In the case of integer version, $\alpha(N, k)$ is given by $\lceil \frac{1}{2}\Phi(t) \rceil$. In any case, the problem of finding such a vector t can be formulated as Problem 2 with these Φ, g and d.

In this paper, we first characterize the polyhedra of Problems 1-3 with intersecting submodular and posi-modular functions f and $-g$, and then present a combinatorial algorithm for solving Problem 3 with a linear function $\Phi(t)$ (and assuming a further restriction on g). Note that Problem 3 is more general than Problem 2 in the sense that it allows additional constraints $z(X) \leq f(X)$, $X \in 2^V$. This enables us to solve in polynomial time the edge-connectivity augmentation problem with a more general degree constraint that, for each subset $X \subset V$, the total increase of degrees in X in the resulting network N' is bounded by a given constant $f(X)$.

We also show that Problem 2 can be solved for an objective function $\Phi(t) = |\{i \in V \mid t(i) > 0\}|$ in $O(n^3)$ function value oracle calls. Based on this, we can solve in polynomial time the problem of augmenting edge-connectivity of a network so as to minimize the number of vertices having edges whose weights are increased.

The paper is organized as follows. In Section 2, we characterize the polyhedron of Problem 2, and give algorithms for solving Problems 1-3. In Section 3, we characterize all the extreme points of the base polyhedron of Problem 2, and discuss a relation to the core of a convex game.

2 Polyhedral Structures and Problems 1 - 3

A *polyhedron* of a system (V, f) is defined by

$$P(f) = \{z \in \Re^V \mid z(X) \leq f(X),\ \forall X \in 2^V - \{\emptyset, V\}\}, \tag{1}$$

where $X = \emptyset$ and V are not considered in the definition, and a *base polyhedron* of (V, f) by

$$B(f) = \{z \in P(f) \mid z(V) = f(V)\}, \tag{2}$$

where possibly $B(f) = \emptyset$. Let $P_-(f)$ and $B_-(f)$ denote $P(f) \cap \Re^V_-$ and $B(f) \cap \Re^V_-$, respectively, and let $P_+(f)$ and $B_+(f)$ denote $P(f) \cap \Re^V_+$ and $B(f) \cap \Re^V_+$, respectively. For two set functions f_1 and f_2 on V, we denote by $(f_1 - f_2)$ the set function f' with $f'(X) = f_1(X) - f_2(X)$ for all $X \in 2^V$.

We say that a subset $X \subset V$ *separates* x and y if $|\{x, y\} \cap X| = 1$. For a submodular system (V, f), an ordered pair (x, y) of elements in V is called a *pendant pair* if $f(x) \leq f(X)$ holds for all sets $X \subset V$ that separate x and y.

Lemma 2 [6] *For an intersecting submodular and posi-modular set function f on V (where $n = |V| \geq 2$), there exists a pendant pair (x, y). Furthermore, such a pendant pair can be obtained by using $O(n^2)$ function value oracle calls.* □

2.1 Polyhedral Structure of $P_-(f)$

In this section, we first consider the set of all feasible vectors to Problem 2, where we assume that g is an intersecting supermodular and nega-modular set function and a vector $d \in \Re^V_+$ is given by $d(i) = +\infty$ $(i \in V)$. In this case, $f = -g$ is intersecting submodular and posi-modular. Then a vector t is feasible to Problem 2 if and only if $-t \in P_-(f)$ holds for a system (V, f).

We now prove that, given a system (V, f) with an intersecting submodular and posi-modular set function f, there is a laminar family $\mathcal{X} \subseteq 2^V - \{\emptyset, V\}$ that characterizes $P_-(f)$ as follows.

$$P_-(f) = P_-(f; \mathcal{X}), \tag{3}$$

where we use notations $P(f;\mathcal{X}) = \{z \in \Re^V \mid z(X) \leq f(X)$ for all $X \in \mathcal{X}\}$ and $P_-(f;\mathcal{X}) = P(f;\mathcal{X}) \cap \Re^V_-$.

Given an intersecting submodular and posi-modular set function f on V, we compute the above laminar family $\mathcal{X}$ as follows. Initially we set $\mathcal{X} := \emptyset$ and $z(i) := 0$ for all $i \in V$. Then, for each $i \in V$, we check whether $z(i) \leq f(i)$ (i.e., $f(i) \geq 0$) holds or not. If $f(i) < 0$ then we reset $z(i)$ by $z(i) := f(i)$ and add $\{i\}$ to $\mathcal{X}$. Now $z(i) \leq f(i)$ (i.e., $(f-z)(i) \geq 0$) holds for all $i \in V$. Note that $f - z$ remains to be intersecting submodular and posi-modular. Hence there is a pendant pair (x, y) in system $(V, f - z)$ by Lemma 2, for which any cut X separating x and y satisfies $z(X) \leq f(X)$. Then we can contract x and y into a single element x^* without losing any cut X that satisfies $z(X) > f(X)$. After this contraction, we check whether the new element x^* satisfies $(f-z)(x^*) \geq 0$. If $(f-z)(x^*) < 0$, then we add to $\mathcal{X}$ the set X^* of all elements which have been contracted into x^* so far, and decrease $z(i)$ of some $i \in X^*$ so that $z(X^*) = f(X^*)$ holds (where more than one $z(i)$ may be decreased as long as $z(X^*) = f(X^*)$ is satisfied). (If $(f-z)(x^*) \geq 0$, no $z(i)$ is changed.) Then we repeat finding a new pendant pair and contracting them into a single element in the resulting system, until the system has only one element. The entire algorithm is described as follows.

Algorithm LAMINAR
Input: A system (V, f) with an intersecting submodular and posi-modular set function f on V, where $n = |V| \geq 2$.
Output: A vector $z \in P_-(f)$, and a laminar $\mathcal{X}$ of V satisfying (3).

1 $\mathcal{X} := \emptyset$; $z(i) := 0$ for all $i \in V$;
2 For each $i \in V$, if $f(i) < 0$ then $z(i) := f(i)$ and $\mathcal{X} := \mathcal{X} \cup \{\{i\}\}$;
3 **for** $i := 1$ to $n-1$ **do**
4   Find a pendant pair (x, y) in $(V', f-z)$;
5   Let $(V', f-z)$ again denote the system obtained from the current $(V', f-z)$
      by contracting x and y into a single element x^*;
6   **if** $(f-z)(x^*) < 0$ **then**
7   Let X^* be the set of all elements of V which have been contracted to x^*;
8   Decrease $z(i)$, $i \in X^*$ arbitrarily so that the resulting $f - z$ satisfies
         $(f-z)(X^*) = 0$ in (V, f);
9       $\mathcal{X} := \mathcal{X} \cup \{X^*\}$
10   **end** /* if */
11 **end**. /* for */

Clearly, LAMINAR runs in $O(n^3)$ function value oracle calls, as the pendant pair in line 4 can be found in $O(n^2)$ function value oracle calls by Lemma 2. Note that the vector z output by LAMINAR may not be unique because there are many ways of decreasing $z(i)$, $i \in X^*$ in line 8. Let $OUTPUT(f)$ denote the set of all vectors z that can be output by LAMINAR for a given input f.

For a laminar family $\mathcal{X} \subseteq 2^V$ on V, a subset $Y \in \mathcal{X}$ is called a *child* of a subset $X \in \mathcal{X}$ (and the X is called the *parent* of Y) if $Y \subset X$ and there is no other subset $Y' \in \mathcal{X}$ with $Y \subset Y' \subset X$. For a subset $X \in \mathcal{X}$, let $ch(X)$ denote the set of children of X, and $pa(X)$ denote the parent of X (possibly $pa(X) = \emptyset$).

Let $\mathcal{X}$ be a family of subsets of V output by LAMINAR, which is clearly laminar. We represent $\mathcal{X}$ by a rooted tree as follows. Define the laminar family $\mathcal{V} = \mathcal{X} \cup \{V\} \cup \{\{i\} \mid i \in V\}$ and define the tree $T = (\mathcal{V}, \mathcal{E})$ on $\mathcal{V}$, where the parent-child relation in the tree is given by $pa(X)$ and $ch(X)$. Clearly V is the root of T. Define $f' : \mathcal{V} \mapsto \Re$ by $f'(X) = 0$ if $X = \{i\}$ and $f(i) \geq 0$; $f'(X) = f(X)$ otherwise. From the behavior of LAMINAR, the next properties are observed.

Lemma 3 *For a system (V, f) with an intersecting submodular and posi-modular set function f on V with $|V| \geq 2$, let z and $\mathcal{X}$ be the vector and the laminar family output by algorithm* LAMINAR. *Let the tree T be defined as above. Then:*

(i) $z \in P_-(f)$ *(hence* $OUTPUT(f) \subseteq P_-(f)$*).*

(ii) *For each non-root vertex* X *in* T, $f'(X) < \sum_{Y \in ch(X)} f'(Y)$ *holds.* □

From this lemma, we can prove (3). Clearly, $P_-(f;\mathcal{X}) \supseteq P_-(f)$. The converse is shown as follows. Given a vector $z' \in P_-(f)$, there is a way of decreasing $\{z(i) \mid i \in X^*\}$ in line 8 of LAMINAR so that the algorithm outputs a vector $z(\in OUTPUT(f))$ such that $z \geq z'$. Hence $z' \in P_-(f)$ follows (further detail is omitted).

Theorem 1 *For a system* (V, f) *with an intersecting submodular and posi-modular set function* f *on* V *with* $|V| \geq 2$, *let* $\mathcal{X}$ *be the laminar family output by algorithm* LAMINAR. *Then:*

(i) $P_-(f) = P_-(f;\mathcal{X})$.

(ii) $B_-(f) \neq \emptyset$ *if and only if* $f'(V) \leq \sum_{Y \in ch(V)} f'(Y)$ *holds.* □

2.2 Problem 2

Based on Theorem 1, we can solve Problem 2, if it has a linear objective function $\Phi(t) = \sum_{i \in V} w(i)t(i)$ defined by a cost vector $w \in \Re^V$. The proof for this case will be given in Section 2.4 under a more general setting of Problem 3. We consider another special case of Problem 2 when the objective function is given by

$$\Phi(t) = |\{i \in V \mid t(i) > 0\}| \text{ (i.e., the number of nonzero entries).}$$

By Theorem 1, we prove the next property (the proof is omitted).

Theorem 2 *For an intersecting supermodular and nega-modular set function* g *on* V, *and a vector* $d \in \Re^V_+$, *Problem 2 with an objective function* $\Phi(t) = |\{i \in V \mid t(i) > 0\}|$ *can be solved by using* $O(n^3)$ *function value oracle calls. If Problem 2 is feasible, then there is a feasible vector* t *which minimizes* $\Phi(t)$ *and* $\Phi'(t) = \sum_{i \in V} t(i)$ *at the same time. Such a solution* t *also can be found by using* $O(n^3)$ *function value oracle calls.* □

By applying this and Lemma 1 to the degree constrained edge-connectivity augmentation problem of a graph to minimize the number of vertices whose incident edges have increased weights, we obtain the next (the proof is omitted).

Theorem 3 *For a complete network* $N = (V, E = V \times V, c)$, $k \geq 2$ *and* $d \in \Re^V_+$, *where all* $c(e)$, k *and* $d(i)$ *are integers, let* $c'(\geq c)$ *be a new integer-valued edge weight function such that* $N' = (V, E, c')$ *is* k*-edge-connected under the degree constraint that new degree of each vertex* $i \in V$ *is at most* $d(i)$. *There is a* c' *that simultaneously minimizes* (i) *the number of vertices to which an edge with increased weight is incident and* (ii) *the total amount of increase* $\sum_{e \in E}(c'(e) - c(e))$. *Such a* c' *can be found in* $O((nm + n^2 \log n) \log n)$ *time, where* $n = |V|$ *and* m *is the number of edges of positive weights in* N. □

2.3 Problem 1

In this subsection, we consider polyhedra $P(f)$ and $P_+(f)$ for an intersecting submodular and posi-modular function f on V, which appear in Problem 1. However, we do not consider the constraint $z \leq d$, as this more general case will be considered in the next subsection as Problem 3.

To generalize Theorem 1 to this case, we further assume that the set function $\hat{f}$ defined by

$$\hat{f}(X) = f(X) - m_f(X) \text{ for } X \in 2^V$$

is intersecting submodular and posi-modular, where m_f denotes the modular function on V defined from f by $m_f(i) = f(i)$ for all $i \in V$.

Now we discuss how to compute a vector $z \in P_+(f)$. Let us consider $y = z - m_f$. Then $0 \leq z(X) \leq f(X)$ holds if and only if $0 \leq y(X) + m_f(X) \leq f(X) = \hat{f}(X) - m_f(X)$. Thus the problem is equivalent to finding a vector $y \in P(\hat{f}) \cap \{y \in \Re^V \mid y + m_f \geq 0\}$.

We first consider $P(f)$. Note that $P(\hat{f}) = P_-(\hat{f})$ since $\hat{f}(i) = 0$ holds for all $i \in V$. Therefore, $P(f) = \{z = y + m_f \mid y \in P_-(\hat{f})\}$. By applying Theorem 1 to system $(V, \hat{f})$, we obtain a laminar family $\mathcal{X}$ such that $P_-(\hat{f}) = P_-(\hat{f}; \mathcal{X})$, by using $O(n^3)$ function value oracle calls. Clearly, for any $z \in P(f; \mathcal{X})$, we have $y = z - m_f \in P_-(\hat{f}; \mathcal{X}) = P_-(\hat{f})$, and hence $z \in P(f)$ (the converse is also clear). Thus, $P(f) = P(f; \mathcal{X})$ holds.

Next consider a vector $z \in P_+(f)$. From the above argument, it holds $P_+(f) = P(f) \cap \Re^V_+ = \{z = y + m_f \mid y \in P_-(\hat{f}),\ y \geq -m_f\}$. A vector $y \in P_-(\hat{f})$ satisfying constraint $y \geq -m_f$ (if any) can be easily computed. From these, we establish the next result.

Theorem 4 *For a system (V, f) with a set function f on V with $n = |V| \geq 2$ such that $f - m_f$ is intersecting submodular and posi-modular, there is a laminar family $\mathcal{X}$ such that $P(f) = P(f; \mathcal{X})$. Such a family $\mathcal{X}$ and a vector $z \in P_+(f)$ (if any) can be found by $O(n^3)$ function value oracle calls.* □

2.4 Problem 3

In this subsection, we solve Problem 3 with a linear objective function $\Phi(z) = \sum_{i \in V} w(i)z(i)$ for a given vector $w \in \Re^V$.

Theorem 5 *Let g and f be set functions on V, and $d_1, d_2 \in \Re^V_+$ and $w \in \Re^V$. If $-g$ and $f - m_f$ are both intersecting submodular and posi-modular, then an optimal solution z to Problem 3 with objective function $\Phi(z) = \sum_{i \in V} w(i)z(i)$ can be found by using $O(n^3)$ function value oracle calls and by solving a minimum cost flow problem with $O(n)$ vertices and arcs.*

Proof Sketch: By Theorems 1 and 4, there are laminar families $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\{z \in \Re^V_+ \mid g(X) \leq z(X) \text{ for all } X \in \mathcal{X}_1\} = \{z \in \Re^V_+ \mid g(X) \leq z(X) \text{ for all } X \in 2^V - \{\emptyset, V\}\}$ and $\{z \in \Re^V \mid z(X) \leq f(X) \text{ for all } X \in \mathcal{X}_2\} = \{z \in \Re^V \mid z(X) \leq f(X) \text{ for all } X \in 2^V - \{\emptyset, V\}\}$. Thus, the problem is restated as

$$
\begin{array}{lll}
\text{minimize} & \Phi(z) = \sum_{i \in V} w(i)z(i) & \\
\text{subject to} & g(X) \leq z(X) & \text{for all } X \in \mathcal{X}'_1 \\
 & z(X) \leq f(X) & \text{for all } X \in \mathcal{X}'_2 \\
 & d_1(i) \leq z(i) \leq d_2(i) & \text{for all } i \in V,
\end{array}
$$

where $\mathcal{X}'_i = \mathcal{X}_i \cup \{\emptyset, V\}$ for $i = 1, 2$. Then it is not difficult to see that the problem can be formulated as the minimum cost flow problem in a directed network (further detail is omitted). □

3 Extreme points of base polyhedron

In this section, we characterize all extreme points of a base polyhedron $B_-(f)$ defined for an intersecting submodular and posi-modular set function f. We then show some relation of the result to a core in a convex game.

3.1 All extreme points of $B_-(f)$

Let Π_n be the set of all permutations of $(1, 2, \ldots, n)$. For a subset $P \subseteq \Re^V_-$ and a permutation $\pi \in \Pi_{|V|}$, a vector $z \in \Re^V_-$ is called *lexicographically π-minimal* (*π-minimal*, for short) in P if there is no other vector $z' \in P$ which is lexicographically smaller than z with respect to π; i.e., there is no j such that $z'(\pi(i)) = z(\pi(i))$ for $i = 1, 2, \ldots, j-1$ and $z'(\pi(j)) < z(\pi(j))$. Let $L(f)$ be the set of π-minimal vectors in $B_-(f)$ for all $\pi \in \Pi_n$, and $EP(f)$ be the set of all extreme points in $B_-(f)$. Based on Theorem 1, we can show the next result.

Theorem 6 *Let (V, f) be a system with an intersecting submodular and posi-modular set function f on V with $n = |V| \geq 2$. If $B_-(f) \neq \emptyset$, then $L(f) = EP(f)$ holds.* □

If $B_-(f) \neq \emptyset$ and $L(f) = EP(f)$, then we define the *mean vector* ψ_f of all π-minimal vectors z_π, $\pi \in \Pi_n$ by $\psi_f = \frac{1}{n!}\sum\{z_\pi \mid \pi \in \Pi_n\}$, where possibly $z_\pi = z_{\pi'}$ holds for two permutations $\pi, \pi' \in \Pi_n$. Again by Theorem 1, we can show that, for an intersecting submodular and posi-modular set function f, the mean vector ψ_f can be efficiently computed from the laminar family $\mathcal{X}$.

Theorem 7 *For a system (V, f) with an intersecting submodular and posi-modular set function f on V with $n = |V| \geq 2$, let $\mathcal{X}$ be the laminar family output by algorithm* LAMINAR. *Then the mean vector ψ_f of all lexicographically minimal vectors in $B_-(f)$ can be computed from $\mathcal{X}$ in $O(n^2)$ time.* □

3.2 A relation to a convex game

A cooperative game in the game theory is defined by a pair (V, g) of a set V of players and a nonnegative set function g on V, where g is called the characteristic function of the game and satisfies $g(\emptyset) = 0$. Several solution concepts such as core, Shapley value, τ-value and others have been proposed. The core of a game (V, g) is the set $CORE(g)$ of nonnegative vectors $z \in \Re^V_+$ such that $z(X) \geq g(X)$ for all $X \in 2^V$ and $z(V) = g(V)$. In other words, it can be defined by $CORE(g) = \{-z' \mid z' \in B_-(-g)\}$ for a system $(V, -g)$. Note that $CORE(g)$ is always a convex set. The problem of testing whether a convex game (V, g) has a nonempty $CORE(g)$ can be solved by computing the minimum value $\Phi(z) = \sum_{i \in V} z(i)$ in Problem 2 with $d = +\infty$.

The Shapley value $\phi_g \in \Re^V$ is a solution concept proposed by Shapley [7], which is defined by

$$\phi_g(i) = \sum_{S \subseteq V:\, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}[g(S) - g(S-i) \text{ for each } i \in V.$$

A game (V, g) is called *convex* if g is a *fully supermodular* set function on V. Several structural properties have been studied for a convex game. In this section, we consider a game (V, h) with an intersecting supermodular and nega-modular function h on V, which is slightly different from a convex game (V, g). We show that a game (V, h) has a considerably different structure from that of a convex game. Let us review some structural properties of a convex game.

Theorem 8 [2, 8] *For a convex game (V, g), $CORE(g)$ is always nonempty. For any permutation $\pi \in \Pi_n$, the π-minimal vector $z_\pi \in \Re^V_+$ belongs to $CORE(g)$ and is given by $z_\pi(\pi(i)) = g(\{\pi(1), \pi(2), \ldots, \pi(i)\}) - g(\{\pi(1), \pi(2), \ldots, \pi(i-1)\})$ for $i = 1, 2, \ldots, n$. Moreover, the set of all extreme points of $CORE(g)$ is given by the set of π-minimal vectors z_π, $\pi \in \Pi_n$.* □

We denote by ψ_g the mean vector of all π-minimal vectors z_π, $\pi \in \Pi_n$, for a convex game (V, g).

Theorem 9 [7] *For a convex game (V, g), the Shapley value $\phi_g \in \Re^V$ is given by $\phi_g = \psi_g$.* □

As to computing the Shapley value, we easily observe the following intractability.

Lemma 4 *For a convex game (V, g), there is no algorithm that computes the Shapley value $\phi_g \in \Re^V$ by using less than $2^n - 1$ function value oracle calls, where $n = |V|$.* □

Now let us consider the counter part of the above results in a game (V, h) with an intersecting supermodular and nega-modular function h. By applying Theorems 1 and 6 to system $(V, -h)$, we have the next result.

Theorem 10 *For a game (V, h) with an intersecting supermodular and nega-modular function $h : 2^V \mapsto \Re_+$, $CORE(h)$ is nonempty if and only if $h(V) \geq \sum_{Y \in ch(V)} h(Y)$ holds, where $ch(V)$ is the set of maximal subsets X in the laminar family $\mathcal{X}$ obtained from $(V, -h)$ by algorithm* LAMINAR. *Moreover, the set of all extreme points of $CORE(h)$ is given by the set of π-minimal vectors z_π, $\pi \in \Pi_n$.* □

From Theorem 7, we obtain the following.

Theorem 11 *For a game (V, h) with an intersecting supermodular and nega-modular function $h : 2^V \mapsto \Re_+$, assume that $CORE(h) \neq \emptyset$. Then the mean vector ψ_h of all π-minimal vectors z_π, $\pi \in \Pi_n$, can be computed by using $O(n^3)$ function value oracle calls, where $n = |V|$.* □

Moreover, the mean vector ψ_g is no longer equal to the Shapley value ϕ_h, as shown by the next lemma.

Lemma 5 *For a game (V, h) with an intersecting supermodular and nega-modular function $h : 2^V \mapsto \Re_+$, there is no algorithm that computes the Shapley value $\phi_h \in \Re^V$ by using less than $2^n - 1$ function value oracle calls, where $n = |V|$.* □

4 Conclusion

In this paper, we showed that, for an intersecting submodular and posi-modular set function f on V, its polyhedron $P_-(f)$ is described by a set of inequalities $z(X) \leq f(X)$ such that X is in a laminar family $\mathcal{X} \subseteq 2^V$. Furthermore, such a laminar family can be obtained combinatorially by $O(|V|^3)$ function value oracle calls. This significantly reduces the complexity of finding a vector z in the polyhedron $P_-(f)$. As a result, we show that several optimization problems over the polyhedron have efficient combinatorial algorithms, and that the core and its mean vector of some cooperative game can be efficiently computed. It is left for the future research to widen the class of set functions to which similar algorithms are applicable.

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