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Polybasic Polyhedra: Structure of Polyhedra with Edge Vectors of Support Size at Most 2

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Abstract We consider a class of pointed convex polyhedra in $\mathbf{R}^V$ whose edge vectors have the support of size at most 2. We call such a convex polyhedron a polybasic polyhedron and show that for a pointed polyhedron $P \subseteq \mathbf{R}^V$ the following three statements are equivalent:
(1) P is a polybasic polyhedron.
(2) Each face of P with a normal vector of the full support V is obtained from a base polyhedron by a reflection and scalings along axes.
(3) The support function of P is a submodular function on each orthant of $\mathbf{R}^V$.

複基多面体: 台の大きさが 2 以下の辺ベクトルを有する多面体の構造

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概要 各辺ベクトルの台の大きさが 2 以下である多面体のクラスについて考える. このような凸多面体を複基多面体と呼び, 頂点をもつ任意の多面体 $P \subseteq \mathbf{R}^V$ について, 次の 3 つが等価であることを示す.
(1) P が複基多面体である.
(2) 台 V の法線ベクトルをもつ P の各面は, ある基多面体の反転および軸方向のスケーリングによって得られる.
(3) P の支持関数は, $\mathbf{R}^V$ の各象限上の劣モジュラ関数である.

1. Introduction

We consider a class of pointed convex polyhedra in $\mathbf{R}^V$ whose edge vectors have the support of size at most 2. We call such a convex polyhedron a polybasic polyhedron. Polybasic polyhedra are closely related to base polyhedra associated with submodular set functions (see [7]).

In Section 2 we give some definitions and preliminaries together with some examples of polybasic polyhedra. In Sections 3 and 4 we show that for a pointed polyhedron $P \subseteq \mathbf{R}^V$ the following three statements are equivalent:
(1) P is a polybasic polyhedron.
(2) Each face of P with a normal vector of the full support V is obtained from a base polyhedron by a reflection and scalings along axes.
(3) The support function of P is a submodular function on each orthant of $\mathbf{R}^V$.

2. Definitions and Preliminaries

Let V be a finite set and P be a pointed polyhedron in $\mathbf{R}^V$. For any extreme point x of P

denote by $T(x)$ the tangent cone of P at x. We call an extreme vector of the tangent cone $T(x)$ for some extreme point x of P an *edge vector* of P. The *support* of a vector x in $\mathbf{R}^V$ is defined by $\mathrm{supp}(x) = \{v \mid v \in V,\ x(v) \neq 0\}$. We also define $\mathrm{supp}^+(x) = \{v \mid v \in V,\ x(v) > 0\}$ and $\mathrm{supp}^-(x) = \{v \mid v \in V,\ x(v) < 0\}$. If each edge vector x of P satisfies $|\mathrm{supp}(x)| \leq 2$, then we call such a polyhedron P a *polybasic* polyhedron. (The meaning of this term will be made clear later.)

Let $\mathcal{D} \subseteq 2^V$ be a distributive lattice, i.e., for any $X, Y \in \mathcal{D}$ we have $X \cup Y, X \cap Y \in \mathcal{D}$. Also let $f : \mathcal{D} \to \mathbf{R}$ be a submodular function, i.e.,

$$f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y) \quad (X, Y \in \mathcal{D}).$$

We assume $\emptyset, V \in \mathcal{D}$ and $f(\emptyset) = 0$. The *base polyhedron* $\mathrm{B}(f)$ associated with such a submodular function f is defined by

$$\mathrm{B}(f) = \{x \mid x \in \mathbf{R}^V,\ \forall X \in \mathcal{D} : x(X) \leq f(X),\ x(V) = f(V)\}, \quad (2.1)$$

where for any $X \subseteq V$ we define $x(X) = \sum_{v \in X} x(v)$ (see [7] for more details about submodular functions). Note that $\mathrm{B}(f)$ is pointed if and only if $\mathcal{D}$ is simple, i.e., the length of a maximal chain of $\mathcal{D}$ is equal to $|V|$ ([7, Th. 3.11]).

For each $X \subseteq V$ denote by χ_X the characteristic vector of X in $\mathbf{R}^V$, i.e., $\chi_X(v) = 1$ $(v \in X)$ and $\chi_X(v) = 0$ $(v \in V \setminus X)$. For each $u \in V$ we also denote $\chi_{\{u\}}$ by χ_u.

We have

Theorem 2.1 (see [7]): *For the base polyhedron* $\mathrm{B}(f)$ *given by* (2.1) *let* $\hat{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be the support function of* $\mathrm{B}(f)$, *i.e.,*

$$\hat{f}(p) = \sup\{\langle p, x\rangle \mid x \in \mathrm{B}(f)\},$$

where $\langle p, x\rangle = \sum_{v \in V} p(v)x(v)$. *Then,*

$$\mathcal{D} = \{X \mid X \subseteq V,\ \hat{f}(\chi_X) < +\infty\},$$
$$\hat{f}(\chi_X) = f(X) \qquad (X \in \mathcal{D}).$$

□

Base polyhedra are characterized by the following.

Theorem 2.2 (Tomizawa[12]): *For any pointed polyhedron* $Q \subseteq \mathbf{R}^V$, *Q is a base polyhedron if and only if each edge vector of Q is a multiple of one of the forms* $\chi_u - \chi_v$ $(u, v \in V, u \neq v)$. □

For any disjoint $S, T \subseteq V$ with $S \cup T = V$ we call the ordered pair (S, T) an *orthant* of $\mathbf{R}^V$. For any polyhedron Q and any $T \subseteq V$ the *reflection of Q by T* is defined by

$$Q{:}T = \{y \mid x \in Q,\ y \in \mathbf{R}^V, \forall v \in T{:}y(v) = -x(v),\ \forall v \in V \setminus T : y(v) = x(v)\}.$$

A hyperplane $H : \langle p, x\rangle = d$ is called a *supporting hyperplane* of a polyhedron P if P is included in the half space $\langle p, x\rangle \leq d$ and $P \cap H$ is nonempty. We call $F = P \cap H$ the *face of P with a normal vector p*.

3. Polybasic Polyhedra

The face structure of polybasic polyhedra is revealed by the following theorem.

Theorem 3.1: *Suppose that $P \subseteq \mathbf{R}^V$ is a polybasic polyhedron. Let F be a nonempty face of P with a normal vector p having the full support V. Then there uniquely exists a base polyhedron $\mathrm{B}(f)$ associated with a submodular function $f : \mathcal{D} \to \mathbf{R}$ such that*

$$F = \{x \mid x \in \mathbf{R}^V, y \in \mathrm{B}(f), \forall v \in V{:}y(v) = p(v)x(v)\}.$$

(Proof) Define

$$B = \{y \mid y \in \mathbf{R}^V,\ x \in F,\ \forall v \in V : y(v) = p(v)x(v)\}. \quad (3.1)$$

Then, since F lies on a hyperplane $\sum_{v \in V} p(v)x(v) = d(= \text{const.})$, the polyhedron B lies on the hyperplane

$$y(V) = d. \quad (3.2)$$

Moreover, since P is a polybasic polyhedron, its face F is also a polybasic polyhedron. It follows from (3.1) that B is also a polybasic polyhedron. This, together with (3.2), implies that each edge vector of B is a multiple of one of the forms $\chi_u - \chi_v$ $(u, v \in V, u \neq v)$. Hence B is a base polyhedron, due to Theorem 2.2. □

From this theorem we have

Theorem 3.2: *For a pointed polyhedron $P \subseteq \mathbf{R}^V$, P is a polybasic polyhedron if and only if each face of P with a normal vector having the full support V is obtained from a base polyhedron by a reflection and scalings along axes.*

(Proof) The "if part" easily follows from Theorem 2.2. So, we show the "only if" part. It follows from the proof of Theorem 3.1 that for each face F of a polybasic polyhedron $P \subseteq \mathbf{R}^V$ with a normal vector p having the full support V there uniquely exists a base polyhedron $\mathrm{B}(f)$ associated with a submodular function $f : \mathcal{D} \to \mathbf{R}$ such that

$$F = \{x \mid x \in \mathbf{R}^V,\ y \in \mathrm{B}(f),\ \forall v \in V : y(v) = p(v)x(v)\}.$$

This means that F is obtained from the base polyhedron $\mathrm{B}(f)$ by means of the reflection by $T = \{v \mid v \in V,\ p(v) < 0\}$ and of scalings along axes by $1/|p(v)|$ $(v \in V)$. □

Remark 3.1: It should be noted that if a vector $x \in \mathbf{R}^V$ satisfies $\langle p, x\rangle \neq 0$ for each $p \in \mathbf{R}^V$ of full support V, then we have $|\mathrm{supp}(x)| = 1$. Hence each edge of a (not necessarily polybasic) polyhedron that does not have any normal vector of the full support V has the edge vector of the form $\pm\chi_v$ $(v \in V)$. □

Examples of polybasic polyhedra are given as follows.

Example 1: Base polyhedra

See Theorem 2.2. □

Example 2: Generalized polymatroids [6]

Each edge vector is a multiple of one of the forms $\chi_u - \chi_v$ $(u, v \in V, u \neq v)$ and $\pm\chi_v$ $(v \in V)$. □

Example 3: Bisubmodular polyhedra [2], [3], [4], [5], [7, Sec. 3.5(b)]

Each edge vector is a multiple of one of the forms $\pm\chi_u \pm \chi_v$ $(u, v \in V, u \neq v)$ and $\pm\chi_v$ $(v \in V)$. □

Example 4: Extended submodular polyhedra [8]

An extended submodular polyhedron P is considered by Kashiwabara and Takabatake [8]. A pointed extended submodular polyhedron is a polybasic polyhedron that is lower-hereditary, i.e., $x \leq y \in P$ implies $x \in P$. For each edge vector x with $|\mathrm{supp}(x)| = 2$, we have $|\mathrm{supp}^+(x)| = |\mathrm{supp}^-(x)| = 1$. □

Example 5: Polybasic zonotopes

Let $D : V \times A \to \mathbf{R}$ be a $|V| \times |A|$ real matrix such that each column vector $D(\cdot, a)$ $(a \in A)$ satisfies $|\mathrm{supp}(D(\cdot, a))| \leq 2$. For a positive vector $u \in \mathbf{R}^A$ consider the polyhedron $P(D, u)$ defined by

$$P(D, u) = \{y \mid y = Dx,\ \mathbf{0} \leq x \leq u\}. \qquad (3.3)$$

Here, $P(D, u)$ is a zonotope and is a polybasic polyhedron, as can be seen from (3.3). Note that edge vectors of $P(D, u)$ are given by $\pm D(\cdot, a)$ for column vectors $D(\cdot, a)$ $(a \in A)$ of D. $P(D, u)$ can be regarded as the set of boundaries of flows in a bidirected network determined by D and u.

If each column vector $D(\cdot, a)$ of D has exactly two nonzero components, one being equal to 1 and the other to $-\gamma(a)$ with $\gamma(a) > 0$, then D can be regarded as the incidence matrix of a generalized network $\mathcal{N}$ with the vertex set V, the arc set A, and a gain $\gamma(a)$ and a capacity $u(a)$ for each arc $a \in A$ (see, e.g., [1, Chapt. 15]). The corresponding $P(D, u)$ is the set of the boundaries Dx of all flows x in $\mathcal{N}$ such that $\mathbf{0} \leq x \leq u$.

For a polybasic zonotope $P(D, u)$ we can easily compute

$$h(p) = \max\{\langle p, y\rangle \mid y \in P(D, u)\}$$

for each coefficient vector $p \in \mathbf{R}^V$. We can also easily show that the support function h is submodular on each orthant of $\mathbf{R}^V$. □

4. Submodular Functions Associated with Polybasic Polyhedra

Let $P \subseteq \mathbf{R}^V$ be a polybasic polyhedron and let $h : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be the support function of P, i.e.,

$$h(p) = \sup\{\langle p, x\rangle \mid x \in P\}.$$

As is well known, the support function h is positively homogeneous and convex, and hence h is subadditive, i.e., $h(p) + h(q) \geq h(p + q)$.

When p is a normal vector of P that determines a face F, we simply say that p is a normal vector of F, in the following.

Based on Theorem 3.1, we have

Lemma 4.1: *Let F be a face of a polybasic polyhedron P in $\mathbf{R}^V$ with a positive normal vector $p \in \mathbf{R}^V$. Then, for a sufficiently small α with $0 < \alpha \leq 1$, there uniquely exists a simple distributive lattice $\mathcal{D} \subseteq 2^V$ with $\emptyset, V \in \mathcal{D}$ such that F is the set of vectors x satisfying*

$$\begin{aligned}&\langle \alpha p^X + (1-\alpha)p, x\rangle \\ &\qquad \leq h(\alpha p^X + (1-\alpha)p) \quad (X \in \mathcal{D}),\\ &\langle p, x\rangle = h(p),\end{aligned}$$

where p^X is a vector in $\mathbf{R}^V$ such that $p^X(v) = p(v)$ $(v \in X)$ and $p^X(v) = 0$ $(v \in V \setminus X)$. Moreover, $h(\alpha p^X + (1-\alpha)p)$ as a set function in $X \in \mathcal{D}$ is submodular on $\mathcal{D}$.

(Proof) It follows from Theorem 3.1 that for a uniquely determined submodular set function $f : \mathcal{D} \to \mathbf{R}$ on a simple distributive lattice $\mathcal{D}$ with $\emptyset, V \in \mathcal{D}$ the face F is the set of vectors $x \in \mathbf{R}^V$ satisfying

$$\langle p^X, x\rangle \leq f(X) \quad (X \in \mathcal{D}), \tag{4.1}$$
$$\langle p, x\rangle = f(V). \tag{4.2}$$

For any α with $0 < \alpha \leq 1$, (4.1) and (4.2) are equivalently written as

$$\begin{aligned}&\langle \alpha p^X + (1-\alpha)p, x\rangle \\ &\qquad \leq \alpha f(X) + (1-\alpha) f(V) \quad (X \in \mathcal{D}), \quad (4.3)\\ &\langle p, x\rangle = f(V).\end{aligned}$$

Recall that P is a polyhedron. Because of the finiteness characteristic of P, for a sufficiently small $\alpha > 0$ the equation obtained from (4.3) for each $X \in \mathcal{D}$ is a supporting hyperplane of P and we have from Theorem 2.1

$$h(\alpha p^X + (1-\alpha)p) = \alpha f(X) + (1-\alpha) f(V).$$

This completes the proof. □

We also have

Lemma 4.2: *Let α be a positive real such that Lemma 4.1 holds. Also let $C(F, p, \alpha)$ be the cone generated by $\{\alpha p^X + (1-\alpha)p \mid X \in \mathcal{D}\}$. Then for any $q \in C(F, p, \alpha)$ there uniquely exist a chain $\mathcal{C}$ of $\mathcal{D}$:*

$$\mathcal{C} : \ (\emptyset \neq) S_0 \subset S_1 \subset \cdots \subset S_k \tag{4.4}$$

and positive reals $\lambda_i > 0$ $(i = 0, 1, \cdots, k)$ such that

$$q = \sum_{i=0}^{k} \lambda_i (\alpha p^{S_i} + (1-\alpha)p), \tag{4.5}$$

and we have

$$h(q) = \sum_{i=0}^{k} \lambda_i h(\alpha p^{S_i} + (1-\alpha)p). \tag{4.6}$$

Moreover, for any $q_1, q_2 \in C(F, p, \alpha)$ we have

$$h(q_1) + h(q_2) \geq h(q_1 \vee q_2) + h(q_1 \wedge q_2), \tag{4.7}$$

where $(q_1 \vee q_2)(v) = \max\{q_1(v), q_2(v)\}$ and $(q_1 \wedge q_2)(v) = \min\{q_1(v), q_2(v)\}$ for each $v \in V$.

(Proof) We can easily show (4.4)~(4.6) by the greedy algorithm for base polyhedra (see [7]; this is essentially the same as the Lovász extension of a submodular set function [9]).

We show (4.7). Suppose that q_1 and q_2 are expressed as

$$q_1 = \sum_{i=0}^{k_1} \lambda_i (\alpha p^{S_i} + (1-\alpha)p),$$
$$q_2 = \sum_{i=0}^{k_2} \mu_i (\alpha p^{T_i} + (1-\alpha)p)$$

for some chains of $\mathcal{D}$

$$S_0 \subset S_1 \subset \cdots \subset S_{k_1}, \quad T_0 \subset T_1 \subset \cdots \subset T_{k_2}. \tag{4.8}$$

Also suppose without loss of generality that

$$\sum_{i=0}^{k_1} \lambda_i \leq \sum_{i=0}^{k_2} \mu_i, \qquad \sum_{i=0}^{k_1} \lambda_i > \sum_{i=0}^{k_2} \mu_i (1-\alpha). \tag{4.9}$$

Note that if the latter inequality in (4.9) does not hold, we have $q_1 \leq q_2$ and hence (4.7) trivially holds.

Now, put

$$L_1 = \sum_{i=0}^{k_1} \lambda_i (1-\alpha), \qquad L_2 = \sum_{i=0}^{k_2} \mu_i (1-\alpha)$$

and let

$$\eta_0 > \eta_1 > \cdots > \eta_{k_3+1}$$

be the distinct values of

$$L_1, \ L_2, \ \sum_{i=r}^{k_1} \lambda_i \alpha + L_1 \quad (r = 0, 1, \cdots, k_1),$$
$$\sum_{i=r}^{k_2} \mu_i \alpha + L_2 \quad (r = 0, 1, \cdots, k_2).$$

Suppose that $\eta_{t+1} = L_2$ and $\eta_s = \sum_{i=0}^{k_1} \lambda_i \alpha + L_1$, where $s \le t$ from (4.9). Define

$$\nu_i = (\eta_i - \eta_{i+1})/\alpha \ (>0) \quad (i = 0, 1, \cdots, k_3), \tag{4.10}$$

$$S'_i = \mathrm{supp}(q_1 - (q_1 \wedge \eta_{i+1} p)) \quad (i = s, s+1, \cdots, k_3), \tag{4.11}$$

$$T'_i = \mathrm{supp}(q_2 - (q_2 \wedge \eta_{i+1} p)) \quad (i = 0, 1, \cdots, t). \tag{4.12}$$

Then q_1 and q_2 are rewritten as

$$q_1 = \sum_{i=s}^{k_3} \nu_i (\alpha p^{S'_i} + (1-\alpha) p), \tag{4.13}$$

$$q_2 = \sum_{i=0}^{t} \nu_i (\alpha p^{T'_i} + (1-\alpha) p), \tag{4.14}$$

where note that the two monotone nondecreasing sequences of subsets of V

$$(S_0 =) S'_s \subseteq S'_{s+1} \subseteq \cdots \subseteq S'_{k_3} (= S_{k_1}),$$
$$(T_0 =) T'_0 \subseteq T'_1 \subseteq \cdots \subseteq T'_t (= T_{k_2})$$

are obtained from those in (4.8) by possibly repeating some elements in (4.8). It follows from (4.10)~(4.14) that

$$\begin{aligned} & h(q_1) + h(q_2) \\ &= \sum_{i=s}^{k_3} \nu_i \tilde{h}(S'_i) + \sum_{i=0}^{t} \nu_i \tilde{h}(T'_i) \\ &= \sum_{i=t+1}^{k_3} \nu_i \tilde{h}(S'_i) + \sum_{i=0}^{s-1} \nu_i \tilde{h}(T'_i) \\ &\quad + \sum_{i=s}^{t} \nu_i \{\tilde{h}(S'_i) + \tilde{h}(T'_i)\} \\ &\ge \sum_{i=t+1}^{k_3} \nu_i \tilde{h}(S'_i) + \sum_{i=0}^{s-1} \nu_i \tilde{h}(T'_i) \\ &\quad + \sum_{i=s}^{t} \nu_i \{\tilde{h}(S'_i \cup T'_i) + \tilde{h}(S'_i \cap T'_i)\} \\ &= \sum_{i=0}^{s-1} \nu_i \tilde{h}(T'_i) + \sum_{i=s}^{t} \nu_i \tilde{h}(S'_i \cup T'_i) \\ &\quad + \sum_{i=s}^{t} \nu_i \tilde{h}(S'_i \cap T'_i) + \sum_{i=t+1}^{k_3} \nu_i \tilde{h}(S'_i) \\ &= h(q_1 \vee q_2) + h(q_1 \wedge q_2), \end{aligned}$$

where $\tilde{h}(X)$ denotes $h(\alpha p^X + (1-\alpha) p)$ for $X \subseteq V$. This completes the proof of the lemma. □

For any orthant (S, T) define

$$\mathbf{R}^{(S,T)} = \{q \mid q \in \mathbf{R}^V, \mathrm{supp}^+(q) \subseteq S, \mathrm{supp}^-(q) \subseteq T\}.$$

Also, for any $q_1, q_2 \in \mathbf{R}^{(S,T)}$ define

$$(q_1 \vee_{(S,T)} q_2)(v) = \begin{cases} \max\{q_1(v), q_2(v)\} & (v \in S) \\ \min\{q_1(v), q_2(v)\} & (v \in T), \end{cases}$$

$$(q_1 \wedge_{(S,T)} q_2)(v) = \begin{cases} \min\{q_1(v), q_2(v)\} & (v \in S) \\ \max\{q_1(v), q_2(v)\} & (v \in T). \end{cases}$$

Lemma 4.3: *For any orthant (S, T) and any $q_1, q_2 \in \mathbf{R}^{(S,T)}$, if $h(q_1), h(q_2) < +\infty$, then $h(q_1 \vee_{(S,T)} q_2), h(q_1 \wedge_{(S,T)} q_2) < +\infty$.*

(Proof) We consider the positive orthant $(V, \emptyset)$. Suppose to the contrary that $h(q_1), h(q_2) < +\infty$ and $h(q_1 \vee q_2) = +\infty$. Then there exists a facet F of P such that

$$\sup\{\langle q_1 \vee q_2, x\rangle \mid x \in F\} = +\infty. \tag{4.15}$$

On the other hand, from the assumption

$$\begin{aligned} &\sup\{\langle q_1, x\rangle \mid x \in F\} < +\infty, \\ &\sup\{\langle q_2, x\rangle \mid x \in F\} < +\infty. \end{aligned} \tag{4.16}$$

Since F is obtained from a base polyhedron by scalings and the present lemma for $(S, T) = (V, \emptyset)$ holds for a base polyhedron (see [9], [11]), (4.15) contradicts (4.16). Hence, $h(q_1 \vee q_2) < +\infty$. Similarly, we have $h(q_1 \wedge q_2) < +\infty$. □

From Lemmas 4.2 and 4.3 we can show the following. Denote by F_p the face of P having a normal vector p.

Theorem 4.4: *For any orthant (S, T) and any $q_1, q_2 \in \mathbf{R}^{(S,T)}$ there holds*

$$h(q_1) + h(q_2) \ge h(q_1 \vee_{(S,T)} q_2) + h(q_1 \wedge_{(S,T)} q_2). \tag{4.17}$$

(Proof) It suffices to consider the positive orthant $(V, \emptyset)$. We can suppose that $h(p) < +\infty$ for some positive vector $p \in \mathbf{R}^V$. Consider any positive vector $q \in \mathbf{R}^{(V,\emptyset)}$, any positive reals $\beta_1, \beta_2 > 0$, and any distinct $u_1, u_2 \in V$ such that $h(q + \beta_1 \chi_{u_1})$, $h(q + \beta_2 \chi_{u_2}) < +\infty$. Then, there exist (sufficiently small) positive numbers $\delta_1, \delta_2 > 0$ such that $\beta_i = N_i \delta_i$ ($i = 1, 2$) for some positive integers N_i ($i = 1, 2$) and that for each j_i with $0 \le j_i < N_i$ ($i = 1, 2$), putting

$$q_0 = q + j_1 \delta_1 \chi_{u_1} + j_2 \delta_2 \chi_{u_2}, \tag{4.18}$$

$$q_1 = q + (j_1 + 1) \delta_1 \chi_{u_1} + j_2 \delta_2 \chi_{u_2}, \tag{4.19}$$

$$q_2 = q + j_1 \delta_1 \chi_{u_1} + (j_2 + 1) \delta_2 \chi_{u_2}, \tag{4.20}$$

$$q_3 = q + (j_1 + 1) \delta_1 \chi_{u_1} + (j_2 + 1) \delta_2 \chi_{u_2}, \tag{4.21}$$

either (i) we have $F_{q_0} = F_{q_1} = F_{q_2} = F_{q_3}$ or (ii) for a unique minimal face F^* that contains F_{q_i} $(i = 0,1,2,3)$, we have $q_i \in C(F^*, p_{F^*}, \alpha_{p_{F^*}})$ $(i = 0,1,2,3)$ for an appropriate positive normal vector p_{F^*} of F^* belonging to the convex hull of $\{q_i \mid i = 0,1,2,3\}$ and for a positive real $\alpha_{p_{F^*}} > 0$. Here, note that due to Lemma 4.3 and the convexity of h the values of h for all the vectors given by (4.18)~(4.21) are finite. In Case (i),

$$h(q_1) + h(q_2) = h(q_3) + h(q_0) \tag{4.22}$$

and in Case (ii),

$$h(q_1) + h(q_2) \geq h(q_3) + h(q_0) \tag{4.23}$$

due to Lemma 4.2, where note that $q_1 \vee q_2 = q_3$ and $q_1 \wedge q_2 = q_0$. From (4.22) and (4.23) for $0 \leq j_i < N_i$ $(i = 1,2)$ we get

$$\begin{aligned} h(q + \beta_1\chi_{u_1}) + h(q + \beta_2\chi_{u_2}) \\ \geq h(q + \beta_1\chi_{u_1} + \beta_2\chi_{u_2}) + h(q). \end{aligned}$$

This implies the submodularity inequality (4.17) with $(S,T) = (V,\emptyset)$ for any positive vectors $q_1, q_2 \in \mathbf{R}^{(V,\emptyset)}$. Hence, (4.17) with $(S,T) = (V,\emptyset)$ holds for any nonnegative vectors $q_1, q_2 \in \mathbf{R}^{(V,\emptyset)}$ since h is a closed convex function. □

For a function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ define $\mathrm{dom}(g) = \{x \mid x \in \mathbf{R}^V,\ g(x) < +\infty\}$, which is called the effective domain of g. Also, for any orthant (S,T), if g satisfies the following condition:

(*) For any $q_1, q_2 \in \mathbf{R}^{(S,T)}$,
$g(q_1) + g(q_2) \geq g(q_1 \vee_{(S,T)} q_2) + g(q_1 \wedge_{(S,T)} q_2)$,

then we say that g is *submodular on the orthant* (S,T) *of* $\mathbf{R}^V$.

Lemma 4.5: *Let $P \subseteq \mathbf{R}^V$ be a pointed polyhedron and $h : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be the support function of P. Suppose that h is submodular on the positive orthant $(V,\emptyset)$ of $\mathbf{R}^V$. Also let F be a face of P with a positive normal vector p and define*

$$\begin{aligned} \mathcal{D} = \{X \mid X \subseteq V, \\ h(\alpha p^X + (1-\alpha)p) < +\infty\}, \end{aligned}$$

where α is a sufficiently small positive real so that $\alpha p^X + (1-\alpha)p$ for any $X \in \mathcal{D}$ is a normal vector of a face of F (of P). Then, $\mathcal{D}$ is a simple distributive lattice with $\emptyset, V \in \mathcal{D}$.

(Proof) It easily follows from the submodularity of h on orthant $(V,\emptyset)$ that $X, Y \in \mathcal{D}$ implies $X \cup Y, X \cap Y \in \mathcal{D}$. Hence $\mathcal{D}$ is a distributive lattice and we easily see that $\emptyset, V \in \mathcal{D}$. So, let us show that $\mathcal{D}$ is simple.

Suppose to the contrary that $\mathcal{D}$ is not simple. Let $u, v \in V$ be a pair of distinct elements of V such that for any $X \in \mathcal{D}$ we have $|X \cap \{u,v\}| \neq 1$. Then, we claim that for any $x \in F$ and any $\beta > 0$

$$x + \beta((1/p(u))\chi_u - (1/p(v))\chi_v) \in F. \tag{4.24}$$

For, otherwise there exists $\beta^* = \max\{\beta \geq 0 \mid x + \beta((1/p(u))\chi_u - (1/p(v))\chi_v) \in F\}$ for some $x \in F$ and let q be a positive normal vector of the unique minimal face of P that contains $x + \beta^*((1/p(u))\chi_u - (1/p(v))\chi_v)$. We can choose such a q satisfying

$$q(u)/p(u) > q(v)/p(v), \qquad h(q) < +\infty. \tag{4.25}$$

Let $\gamma_0 < \gamma_1 < \cdots < \gamma_l$ be the distinct values of $q(w)/p(w)$ $(w \in V)$ and suppose that $\gamma_k = q(u)/p(u)$ and $\gamma_{k-1} = q(z)/p(z)$ for some $z \in V$. Note that from (4.25) $q(z)/p(z) \geq q(v)/p(v)$. Defining $Z = \{w \mid w \in V,\ \gamma_k \leq q(w)/p(w)\}$, since $q, \gamma_k p, \gamma_{k-1}p \in \mathrm{dom}(h)$, we have

$$\begin{aligned} (q \wedge \gamma_k p) \vee \gamma_{k-1}p \\ = (\gamma_k - \gamma_{k-1})p^Z + \gamma_{k-1}p \in \mathrm{dom}(h). \end{aligned} \tag{4.26}$$

Since we can choose q close enough to p so that $(1-\alpha)\gamma_k \leq \gamma_{k-1}$ and hence $(\gamma_k - \gamma_{k-1})p^Z + \gamma_{k-1}p = \lambda(\alpha p^Z + (1-\alpha)p) + \delta p$ for some $\lambda > 0$ and $\delta \geq 0$, it follows from (4.26) that $Z \in \mathcal{D}$. From the definition of Z we have $u \in Z$ and $v \notin Z$, which contradicts the assumption on u, v. Hence this completes the proof of the claim (4.24).

Now, because of the symmetry in u, v in the claim of (4.24) we see that F contains an affine space of dimension greater than or equal to one. Hence F is not pointed. This contradicts the pointedness of P. □

Lemma 4.6: *Under the same assumption as in Lemma 4.5, for any nonempty face F' of F there exists a normal vector q of F' that is close enough to p so that $q \in C(F,p,\alpha)$.*

(Proof) By the same argument as in the proof of Lemma 4.5 from (4.24)~(4.26) q can be expressed as

$$q=\sum_{i=1}^{l}(\gamma_i-\gamma_{i-1})p^{Z_i}+\gamma_0 p, \qquad (4.27)$$

where $Z_i \in \mathcal{D}$ for $i=1,2,\cdots,l$. We can choose q close enough to p so that $(1-\alpha)\gamma_l \leq \gamma_0$. Then it follows from (4.27) that $q \in C(F, p, \alpha)$. □

Lemma 4.7: *Under the same assumption as in Lemma 4.5, let F be a face of P with a positive normal vector p. Then, for a sufficiently small positive real $\alpha > 0$ the face F is expressed by*

$$\langle \alpha p^X+(1-\alpha)p, x\rangle \leq h(\alpha p^X+(1-\alpha)p) \quad (X\in\mathcal{D}), \qquad (4.28)$$

$$\langle p, x\rangle = h(p) \qquad (4.29)$$

where $\mathcal{D}=\{X \mid X\subseteq V,\ h(\alpha p^X+(1-\alpha)p)<+\infty\}$ is a simple distributive lattice with $\emptyset, V\in\mathcal{D}$, due to Lemma 4.5.

(Proof) It follows from Lemma 4.6 that for any nonempty face F' of F there exists a normal vector q of F' that is close enough to p so that $q\in C(F,p,\alpha)$. For such a normal vector q there uniquely exist a chain $\mathcal{C}$ of $\mathcal{D}$:

$$\mathcal{C}:\ S_0\subset S_1\subset\cdots\subset S_k \qquad (4.30)$$

and positive reals $\lambda_i>0$ $(i=0,1,\cdots,k)$ such that

$$q=\sum_{i=0}^{k}\lambda_i(\alpha p^{S_i}+(1-\alpha)p). \qquad (4.31)$$

Note that the support function h of P is positively homogeneous and convex, and hence subadditive. It follows from (4.30), (4.31), and the submodularity assumption on h that

$$\begin{aligned}
&h(\sum_{i=0}^{k}\lambda_i(\alpha p^{S_i}+(1-\alpha)p))+(\sum_{i=0}^{k}\lambda_i)h(p)\\
&\geq h(\sum_{i=0}^{k-1}\lambda_i(\alpha p^{S_i}+(1-\alpha)p)+\lambda_k p)\\
&\quad +h(\lambda_k(\alpha p^{S_k}+(1-\alpha)p)\\
&\quad +(1-\alpha)(\sum_{i=0}^{k-1}\lambda_i)p)+\alpha(\sum_{i=0}^{k-1}\lambda_i)h(p)\\
&= h(\sum_{i=0}^{k-1}\lambda_i(\alpha p^{S_i}+(1-\alpha)p))+(\sum_{i=0}^{k}\lambda_i)h(p)\\
&\quad +h(\lambda_k(\alpha p^{S_k}+(1-\alpha)p))\\
&\geq h(\sum_{i=0}^{k-2}\lambda_i(\alpha p^{S_i}+(1-\alpha)p))+(\sum_{i=0}^{k}\lambda_i)h(p)\\
&\quad +\sum_{i=k-1}^{k}h(\lambda_i(\alpha p^{S_i}+(1-\alpha)p))\\
&\geq\\
&\cdots\\
&\geq h(\lambda_0(\alpha p^{S_0}+(1-\alpha)p))+(\sum_{i=0}^{k}\lambda_i)h(p)\\
&\quad +\sum_{i=1}^{k}h(\lambda_i(\alpha p^{S_i}+(1-\alpha)p))\\
&=\sum_{i=0}^{k}h(\lambda_i(\alpha p^{S_i}+(1-\alpha)p))+(\sum_{i=0}^{k}\lambda_i)h(p),
\end{aligned}$$

where note that $h(q'+\lambda p)=h(q')+\lambda h(p)$ for any $q'\in C(F,p,\alpha)$ due to (4.29) (recall that we choose α sufficiently small). From this we have

$$h(q)\geq\sum_{i=0}^{k}\lambda_i h(\alpha p^{S_i}+(1-\alpha)p).$$

(In fact, this holds with equality, due to the homogeneity and convexity of h.) It follows that (4.28) and (4.29) imply the following inequality.

$$\langle q, x\rangle\leq h(q).$$

Hence F is expressed by (4.28) and (4.29). □

The following theorem gives another characterization of polybasic polyhedra.

Theorem 4.8: *Let $P\subseteq \mathbf{R}^V$ be a pointed polyhedron and $h:\mathbf{R}^V\to\mathbf{R}\cup\{+\infty\}$ be the support function of P. Then, P is a polybasic polyhedron if and only if h is submodular on each orthant of $\mathbf{R}^V$.*

(Proof) *The "only if" part*: This follows from Theorem 4.4.

The "if" part: We show that each edge vector of P has the support of size at most 2. Let F be an edge of P and $z\in\mathbf{R}^V$ be an associated edge vector of F. Suppose $|\mathrm{supp}(z)|\geq 2$ and let p be a normal vector of F in P. Then we can perturb p to get a new normal vector p' of F with $\mathrm{supp}^+(p)\subseteq\mathrm{supp}^+(p')$, $\mathrm{supp}^-(p)\subseteq\mathrm{supp}^-(p')$, and $\mathrm{supp}(p')=V$. So, we assume that F has a normal vector p with $\mathrm{supp}(p)=V$. Suppose without loss of generality that p is a positive vector. Then we see from Lemma 4.7 that for a sufficiently small positive real $\alpha > 0$ the edge F is expressed by

$$\langle \alpha p^X + (1-\alpha)p, x\rangle \le h(\alpha p^X + (1-\alpha)p) \quad (X \in \mathcal{D}), \tag{4.32}$$

$$\langle p, x\rangle = h(p) \tag{4.33}$$

where $\mathcal{D} = \{X \mid X \subseteq V,\ h(\alpha p^X + (1-\alpha)p) < +\infty\}$ is a simple distributive lattice with $\emptyset, V \in \mathcal{D}$. Hence F is obtained from a base polyhedron by scalings along axes, and we thus have $|\mathrm{supp}(z)| = 2$. □

Remark 4.1: It should be noted that the class of polybasic polyhedra in $\mathbf{R}^V$ is closed with respect to the following operations:
(1) taking the Minkowski sum (or the vector sum) of two polybasic polyhedra,
(2) taking the intersection of a polybasic polyhedron P and a box $B = \{x \mid x \in \mathbf{R}^V, a \le x \le b\}$ with $P \cap B \neq \emptyset$, where $a \in (\mathbf{R} \cup \{-\infty\})^V$ and $b \in (\mathbf{R} \cup \{+\infty\})^V$.

Note that the nonempty intersection of a base polyhedron and a box is again a base polyhedron and that of a submodular polyhedron and a bounded box is a translation of a polymatroid polytope (see [7]). □

Remark 4.2: There is some similarity between the class of polybasic polyhedra and that of M-convex functions considered by Murota ([10], [11]). For each face F, which is not parallel to the axis of the function values, of the epigraph of an M-convex function from $\mathbf{R}^V$ to $\mathbf{R}$, F is a projection, along the axis of the function values, of a base polyhedron in $\mathbf{R}^V$. Here each such face is related to a base polyhedron by a projection while each face of a polybasic polyhedron is related to a base polyhedron by scalings and a reflection. □

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