# 2 方向量子 1 カウンタオートマトン

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要旨 1997 年に Kondacs と Watrous は 2 方向有限オートマトンを量子化したモデル (2 方向量子有限オートマトン, 2QFA) を提案した. さらに 1999 年には Kravtsev によって 1 方向 1 カウンタオートマトンを量子化したモデル (1 方向量子 1 カウンタオートマトン, 1Q1CA) が提案された. これらのモデルを統一的に扱うため, 我々は 2 方向量子 1 カウンタオートマトン (2Q1CA) というモデルを提案する. 以下では 2Q1CA の定義を与え, このモデルにおいて $\{a^n b^{n^2} \mid n \geq 1\}$ and $\{a^m b^n c^{mn} \mid m, n \geq 1\}$ などの言語が受理できることを証明する。

# Two-way quantum one-counter automata

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**Abstract** In 1997, Kondacs and Watrous introduced the quantum counterpart of two-way finite state automata (two-way quantum finite state automata, 2QFAs). In 1999, Kravtsev introduced the quantum counterpart of one-way one-counter automata (one-way quantum one-counter automata, 1Q1CAs). To deal with these two models all together, we introduce two-way quantum one-counter automata (2Q1CA). We give the definition of 2Q1CAs and investigate the languages recognized by automata on this model. We prove that $\{a^n b^{n^2} \mid n \geq 1\}$ and $\{a^m b^n c^{mn} \mid m, n \geq 1\}$ are recognizable.

## 1 Introduction

In 1997, Kondacs and Watrous [1] introduced the quantum counterpart of one-way and two-way finite state automata (one-way and two-way quantum finite state automata, 1QFAs and 2QFAs). In the classical case, it is well known that both of the classes of languages recognized by one-way and two-way deterministic finite state automata are the class of regular languages. However, Kondacs and Watrous [1] showed that the class of languages recognized by 1QFAs is properly contained in the class of regular languages, while that of 2QFAs properly contains the class of regular languages.

In 1999, Kravtsev [2] introduced the quantum counterpart of one-way one-counter automata (one-way quantum one-counter automata, 1Q1CAs). In the classical case, we can easily see that the languages recognized by one-way deterministic one-counter automata (1D1CAs) are properly contained in the class of context-free languages (CFL), because 1D1CAs are the special case of one-way pushdown automata (1PDAs) and the languages recognized by 1PDAs are CFL. Kravtsev [2] showed that some non-regular languages and some non-context-free languages are recognized by 1Q1CAs, that is, in some case, 1Q1CAs are more powerful than classical counterparts. In 2000, Yamasaki et al. [4] showed that these non-regular or non-context-free languages are also recognized by one-way probabilistic reversible one-counter automata (1PR1CAs), which are the special case of 1Q1CAs. On the other hand, [4] also showed that there exists a regular language which cannot be recognized by 1Q1CAs. This result means that, in some cases, 1Q1CAs are less powerful than classical counterparts.

In one-way case, the automata can read each symbol of inputs only once. This restriction is quite strong for a quantum case, because each step of evolution of quantum automata should be unitary. If we remove this restriction of one-way, how much power is added to quantum one-counter automata? In this paper, we define two-way quantum one-counter automata (2Q1CAs) and prove that 2Q1CAs can recognize some languages such as $\{a^n b^{n^2} \mid n \geq 1\}$ or $\{a^m b^n c^{mn} \mid m, n \geq 1\}$.

## 2 Definition

**Definition 1** *A two-way quantum one-counter automaton (2Q1CA) is defined as* $M = (Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}})$, *where* $Q$ *is a finite set of states,* $\Sigma$ *is a finite input alphabet,* $q_0$ *is an initial state,* $Q_{\mathrm{acc}} \subset Q$ *is the set of accepting states,* $Q_{\mathrm{rej}} \subset Q$ *is the set of rejecting states, and* $\delta : Q \times \Gamma \times S \times Q \times \{-1, 0, +1\} \times \{\leftarrow, \downarrow, \rightarrow\}$ *is a transition function, where* $\Gamma = \Sigma \cup \{¢, \$\}$, *symbol* $¢ \notin \Sigma$ *is the left end-marker, symbol* $\$ \notin \Sigma$ *is the right end-marker, and* $S = \{0, 1\}$.

For any $q_1, q_2 \in Q$, $\sigma, \sigma_1, \sigma_2 \in \Gamma$, $c \in \{-1, 0, +1\}$, $d \in \{\leftarrow, \downarrow, \rightarrow\}$, $\delta$ satisfies the following conditions (well-formedness conditions):

$$\sum_{q',c,d} \delta^*(q_1, \sigma, s, q', c, d)\delta(q_2, \sigma, s, q', c, d) = \begin{cases} 1 & (q_1 = q_2) \\ 0 & (q_1 \neq q_2) \end{cases}, \quad (1)$$

$$\sum_{q',c} \delta^*(q_1, \sigma_1, s, q', c, \leftarrow)\delta(q_2, \sigma_2, s, q', c, \downarrow) + \sum_{q',c} \delta^*(q_1, \sigma_1, s, q', c, \downarrow)\delta(q_2, \sigma_2, s, q', c, \rightarrow) = 0, \quad (2)$$

$$\sum_{q',c} \delta^*(q_1, \sigma_1, s, q', c, \leftarrow)\delta(q_2, \sigma_2, s, q', c, \rightarrow) = 0, \quad (3)$$

$$\sum_{q',d} \delta^*(q_1, \sigma, s_1, q', -1, d)\delta(q_2, \sigma, s_2, q', 0, d) + \sum_{q',d} \delta^*(q_1, \sigma, s_1, q', 0, d)\delta(q_2, \sigma, s_2, q', +1, d) = 0, \quad (4)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \leftarrow)\delta(q_2, \sigma_2, s_2, q', 0, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \downarrow)\delta(q_2, \sigma_2, s_2, q', 0, \rightarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \leftarrow)\delta(q_2, \sigma_2, s_2, q', +1, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \downarrow)\delta(q_2, \sigma_2, s_2, q', +1, \rightarrow) = 0, \quad (5)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \rightarrow)\delta(q_2, \sigma_2, s_2, q', 0, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \downarrow)\delta(q_2, \sigma_2, s_2, q', 0, \leftarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \rightarrow)\delta(q_2, \sigma_2, s_2, q', +1, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \downarrow)\delta(q_2, \sigma_2, s_2, q', +1, \leftarrow) = 0, \quad (6)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \leftarrow)\delta(q_2, \sigma_2, s_2, q', 0, \rightarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \leftarrow)\delta(q_2, \sigma_2, s_2, q', +1, \rightarrow) = 0, \quad (7)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \rightarrow)\delta(q_2, \sigma_2, s_2, q', 0, \leftarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', 0, \rightarrow)\delta(q_2, \sigma_2, s_2, q', +1, \leftarrow) = 0, \quad (8)$$

$$\sum_{q',d} \delta^*(q_1, \sigma, s_1, q', -1, d)\delta(q_2, \sigma, s_2, q', +1, d) = 0, \quad (9)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \leftarrow)\delta(q_2, \sigma_2, s_2, q', +1, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \downarrow)\delta(q_2, \sigma_2, s_2, q', +1, \rightarrow) = 0, (10)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \rightarrow)\delta(q_2, \sigma_2, s_2, q', +1, \downarrow) + \sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \downarrow)\delta(q_2, \sigma_2, s_2, q', +1, \leftarrow) = 0, (11)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \leftarrow)\delta(q_2, \sigma_2, s_2, q', +1, \rightarrow) = 0, (12)$$

$$\sum_{q'} \delta^*(q_1, \sigma_1, s_1, q', -1, \rightarrow)\delta(q_2, \sigma_2, s_2, q', +1, \leftarrow) = 0. (13)$$

We assume that 2Q1CAs have a counter which can hold an arbitrary integer and the counter value is 0 at the start of computation. According to the fourth element of $\delta$, $-1, 0, +1$ respectively, corresponds to decrease of the counter value by 1, retainment the same, and increase by 1. Let $s = \operatorname{sign} k$, where $k$ is the counter value and $\operatorname{sign} k = 0$ if $k = 0$, otherwise 1.

We also assume that all inputs written on the tape are started by ¢ and terminated by \$, and that such a tape, of length $|x| + 2$, is circular. According to the fifth element of $\delta$, $\leftarrow, \downarrow, \rightarrow$ respectively, corresponds to left move of the tape head by one square, retainment the same, and right move by one square.

A computation on an input $x$ of length $n$ corresponds to a unitary evolution in the Hilbert space $\mathcal{H}_n = l_2(C_n)$. For each $(q, a, b) \in C_n, q \in Q, a \in \mathbb{Z}, b \in [0, n+1]$, let $|q, a, b\rangle$ denote the basis vector in $l_2(C_n)$. An operator $U_x^\delta$ for an input $x$ on $\mathcal{H}_n$ is defined as follows:

$$U_x^\delta |q, a, b\rangle = \sum_{q',c,d} \delta(q, w_x(b), \operatorname{sign} a, q', c, d) |q', a + c, b + \mu(d)\rangle,$$

where $w_x(b)$ is the $b$th symbol of $w_x = ¢x\$$ and $\mu(d) = -1(0)[+1]$ if $d = \leftarrow (\downarrow)[\rightarrow]$. We assume that this operator is unitary, that is, $(U_x^\delta)^* U_x^\delta = I$.

After each transition, a state of 2Q1CAs is observed. A computational observable $O$ corresponds to the orthogonal decomposition $l_2(C_n) = E_{\rm acc} \oplus E_{\rm rej} \oplus E_{\rm non}$. The outcome of any observation will be either "accept"($E_{\rm acc}$), "reject"($E_{\rm rej}$) or "non-halting"($E_{\rm non}$). The probability of acceptance, rejection, and non-halting at each step is equal to the sum of the squared amplitude of each basis state in the new state corresponding subspace. After the measurement, the state collapses to the projection to $E_{\rm acc}, E_{\rm rej}, E_{\rm non}$.

**Lemma 1** *A 2Q1CA satisfies the well-formedness conditions if and only if $U_x^\delta$ is a unitary operator.*

*Proof.* By the definition, operating $U_x^\delta$ transforms such two quantum states as $|q_1, a_1, b_1\rangle, |q_2, a_2, b_2\rangle$ into the following states:

$$\begin{aligned} &U_x^\delta |q_1, a_1, b_1\rangle \\ &= \sum_{q_1', c_1, d_1} \delta(q_1, \sigma_1, s_1, q_1', c_1, d_1) \left|q_1', a_1 + c_1, b_1 + \mu(d_1)\right\rangle, \\ &U_x^\delta |q_2, a_2, b_2\rangle \\ &= \sum_{q_2', c_2, d_2} \delta(q_2, \sigma_2, s_2, q_2', c_2, d_2) \left|q_2', a_2 + c_2, b_2 + \mu(d_2)\right\rangle. \end{aligned}$$

First, we assume that $U_x^\delta$ is a unitary operator, that is, $\left(U_x^\delta\right)^* U_x^\delta = I$. Then we have the inner product of the previous two vectors as follows:

$$\begin{aligned} &\langle q_1, a_1, b_1 \mid q_2, a_2, b_2 \rangle \\ &= \left\langle q_1, a_1, b_1 \middle| \left(U_x^\delta\right)^* U_x^\delta \middle| q_2, a_2, b_2 \right\rangle \\ &= \sum_{q_1', c_1, d_1} \sum_{q_2', c_2, d_2} \Big( \delta^*(q_1, \sigma_1, s_1, q_1', c_1, d_1) \delta(q_2, \sigma_2, s_2, q_2', c_2, d_2) \\ &\qquad \langle q_1', a_1 + c_1, b_1 + \mu(d_1) \mid q_2', a_2 + c_2, b_2 + \mu(d_2) \rangle \Big) \end{aligned}$$

Note that two quantum states are the same if and only if all of the states, the counter values, and the head positions coincide with each other. Then we have

$$\begin{aligned} &\langle q_1', a_1 + c_1, b_1 + \mu(d_1) \mid q_2', a_2 + c_2, b_2 + \mu(d_2) \rangle \\ &= \begin{cases} 1 \ (q_1' = q_2', a_1 + c_1 = a_2 + c_2, b_1 + \mu(d_1) = b_2 + \mu(d_2)) \\ 0 \ (\text{otherwise}) \end{cases} \end{aligned} \tag{14}$$

It follows that the inner product is

$$\begin{aligned} &\langle q_1, a_1, b_1 \mid q_2, a_2, b_2 \rangle \\ &= \sum_{\substack{q' \\ a_1 + c_1 = a_2 + c_2 \\ b_1 + \mu(d_1) = b_2 + \mu(d_2)}} \delta^*(q_1, \sigma_1, s_1, q', c_1, d_1) \delta(q_2, \sigma_2, s_2, q', c_2, d_2). \end{aligned}$$

1. In case $a_1 = a_2$ and $b_1 = b_2$, *left hand side* of (14) = 1 if $q_1 = q_2$ and 0 otherwise. Thus $\delta$ satisfies (1).
2. In case $a_1 = a_2$ and $b_1 = b_2 \pm 1$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (2).
3. In case $a_1 = a_2$ and $b_1 = b_2 \pm 2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (3).
4. In case $a_1 = a_2 \pm 1$ and $b_1 = b_2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (4).
5. In case $a_1 = a_2 \pm 1$ and $b_1 = b_2 \pm 1$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (5).
6. In case $a_1 = a_2 \pm 1$ and $b_1 = b_2 \mp 1$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (6).
7. In case $a_1 = a_2 \pm 1$ and $b_1 = b_2 \pm 2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (7).
8. In case $a_1 = a_2 \pm 1$ and $b_1 = b_2 \mp 2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (8).
9. In case $a_1 = a_2 \pm 2$ and $b_1 = b_2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (9).
10. In case $a_1 = a_2 \pm 2$ and $b_1 = b_2 \pm 1$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (10).
11. In case $a_1 = a_2 \pm 2$ and $b_1 = b_2 \mp 1$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (11).
12. In case $a_1 = a_2 \pm 2$ and $b_1 = b_2 \pm 2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (12).
13. In case $a_1 = a_2 \pm 2$ and $b_1 = b_2 \mp 2$, *left hand side* of (14) = 0. Thus $\delta$ satisfies (13).
14. In case $|a_1 - a_2| > 2$ or $|b_1 - b_2| > 2$, *left hand side* of (14) = 0. And *right hand side* of (14) = 0 since two quantum states are always different from each other. Thus, in this case, we need condition for $\delta$.

From these, we conclude that, if $U_x^\delta$ is a unitary operator $\delta$ satisfies the well-formedness conditions.

On the other hand, we assume that $\delta$ satisfies the well-formedness conditions. Then we can easily check $\left(U_x^\delta\right)^* U_x^\delta = I$, that is, $U_x^\delta$ is a unitary operator. □

**Definition 2** *A language $L$ is said to be recognizable by a 2Q1CA with probability $p$, if there exists a 2Q1CA $M$ which accepts any input $x \in L$ with probability at least $p > 1/2$ and rejects any input $x \notin L$ with probability at least $p$.*

We may use the term "accepting probability" for denoting this probability $p$.

## 2.1 Simple 2Q1CAs

### 2.1.1 Counter-simple 2Q1CAs

**Definition 3** *A 2Q1CA $(Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}})$ is said to be counter-simple, if there are unitary operators $\{V_{\sigma,s}\}$ on $l_2(C_n)$ and a counter function $C : Q \times \Gamma \to \{-1, 0, +1\}$ such that for any $q, q' \in Q$, $\sigma \in \Gamma$, $s \in \{0, 1\}$, $c \in \{-1, 0, +1\}$, $d \in \{\leftarrow, \downarrow, \rightarrow\}$,*

$$\delta(q, \sigma, s, q', c, d) = \begin{cases} \langle q'|V_{\sigma,s}|q\rangle & (C(q', \sigma) = c) \\ 0 & (otherwise) \end{cases},$$

*where $\langle q'|V_{\sigma,s}|q\rangle$ is the coefficient of $|q'\rangle$ in $V_{\sigma,s}|q\rangle$.*

In this case, increase or decrease of the counter value is determined by the new state and the symbol it read. Thus the 2Q1CA satisfies the conditions (4), (5), (9) automatically.

### 2.1.2 Head-simple 2Q1CAs

**Definition 4** *A 2Q1CA $(Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}})$ is said to be head-simple, if there are unitary operators $\{V_{\sigma,s}\}$ on $l_2(C_n)$ and a head function $D : Q \to \{\leftarrow, \downarrow, \rightarrow\}$ such that for any $q, q' \in Q$, $\sigma \in \Gamma$, $s \in \{0, 1\}$, $c \in \{-1, 0, +1\}$, $d \in \{\leftarrow, \downarrow, \rightarrow\}$,*

$$\delta(q, \sigma, s, q', c, d) = \begin{cases} \langle q'|V_{\sigma,s}|q\rangle & (D(q') = d) \\ 0 & (otherwise) \end{cases},$$

*where $\langle q'|V_{\sigma,s}|q\rangle$ is the coefficient of $|q'\rangle$ in $V_{\sigma,s}|q\rangle$.*

In this case, the tape head move is determined by the new state. Thus the 2Q1CA satisfies the conditions (2), (3), (5), (6), (7), (8), (10), (11), (12), (13) automatically.

### 2.1.3 Simple 2Q1CAs

**Definition 5** *A 2Q1CA $(Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}})$ is said to be simple, if there are unitary operators $\{V_{\sigma,s}\}$ on $l_2(C_n)$, a counter function $C : Q \times \Gamma \to \{-1, 0, +1\}$ and a head function $D : Q \to \{\leftarrow, \downarrow, \rightarrow\}$ such that for any $q, q' \in Q$, $\sigma \in \Gamma$, $s \in \{0, 1\}$, $c \in \{-1, 0, +1\}$, $d \in \{\leftarrow, \downarrow, \rightarrow\}$,*

$$\begin{aligned} &\delta(q, \sigma, s, q', c, d) \\ &= \begin{cases} \langle q'|V_{\sigma,s}|q\rangle & (C(q', \sigma) = c, D(q') = d) \\ 0 & (otherwise) \end{cases}, \end{aligned}$$

*where $\langle q'|V_{\sigma,s}|q\rangle$ is the coefficient of $|q'\rangle$ in $V_{\sigma,s}|q\rangle$.*

In this case, increase or decrease of the counter value is determined by the new state and the symbol it read, and the head move is determined by the new state. Thus the 2Q1CA satisfies the conditions (2)–(13) automatically.

**Lemma 2** *A simple 2Q1CA satisfies the well-formedness conditions if there are unitary operators $\{V_{\sigma,s}\}$ such that for any $\sigma \in \Gamma$ and $s \in \{0, 1\}$,*

$$\sum_{q'} \langle q'|V_{\sigma,s}|q_1\rangle^* \langle q'|V_{\sigma,s}|q_2\rangle = \begin{cases} 1 & (q_1 = q_2) \\ 0 & (q_1 \neq q_2) \end{cases}, \quad (15)$$

*Proof.* By the definition of simple 2Q1CAs, the well-formedness conditions except for (1) are satisfied automatically.

Now, we let

$$\begin{aligned} &\delta(q_1, \sigma, s, q', c, d) \\ &= \begin{cases} \langle q'|V_{\sigma,s}|q_1\rangle & c = C(q', \sigma), d = D(q') \\ 0 & \text{(otherwise)} \end{cases}, \\ &\delta(q_2, \sigma, s, q', c, d) \\ &= \begin{cases} \langle q'|V_{\sigma,s}|q_2\rangle & c = C(q', \sigma), d = D(q') \\ 0 & \text{(otherwise)} \end{cases}, \end{aligned}$$

then it is trivial that a simple 2Q1CA satisfies (1) if it satisfies (15). □

# 3 Recognizability

## 3.1 2Q1CA for $\{a^n b^{n^2} \mid n \geq 1\}$

**Proposition 1** *Let $L_{\mathrm{square}}$ be $\{a^n b^{n^2} \mid n \geq 1\}$. For an arbitrary fixed positive integer $N \geq 2$, there exists a 2Q1CA $M_{\mathrm{square}}$ which accepts $x \in L_{\mathrm{square}}$ with probability 1 and rejects $x \notin L_{\mathrm{square}}$ with probability $1 - 1/N$. In either case, $M_{\mathrm{square}}$ halts after $O(N|x|)$ steps with certainty.*

*Proof.* Let the state set $Q = \{q_0, q_1, q_2, q_3, q_4, q^i_{5,j_1}, q^i_{6,j_2}, q^i_7 \mid 1 \leq i \leq N, 1 \leq j_1 \leq i, 1 \leq j_2 \leq N - i + 1\}$, $Q_{\mathrm{acc}} = \{q^N_7\}$ and $Q_{\mathrm{rej}} = \{q^j_7 \mid 1 \leq j \leq N - 1\}$. For each $q \in Q$, $\sigma \in \Gamma$, $s \in \{0, 1\}$, we define the transition matrices $\{V_{\sigma,s}\}$, the counter function $C$ and the head function $D$ as follows:

$$
\begin{aligned}
V_{¢,0}|q_0\rangle &= |q_0\rangle, \\
V_{¢,0}|q^i_{5,0}\rangle &= |q^i_{6,N-i+1}\rangle, \\
V_{a,0}|q_0\rangle &= |q_0\rangle, \\
V_{a,0}|q_1\rangle &= |q_2\rangle, \\
V_{a,0}|q_3\rangle &= |q_4\rangle, \\
V_{a,s}|q^i_{5,j+1}\rangle &= |q^i_{5,j}\rangle \quad (j \neq 0, i), \\
V_{a,0}|q^i_{5,i+1}\rangle &= |q^i_{5,i}\rangle, \\
V_{a,1}|q^i_{5,i+1}\rangle &= |q^i_{5,2i}\rangle, \\
V_{a,1}|q^i_{5,1}\rangle &= |q^i_{5,i}\rangle, \\
V_{a,0}|q^i_{6,j+1}\rangle &= |q^i_{6,j}\rangle \quad (1 \leq j \leq N-i), \\
V_{a,0}|q^i_{6,1}\rangle &= |q^i_{6,N-i+1}\rangle, \\
V_{b,0}|q_0\rangle &= |q_1\rangle, \\
V_{b,0}|q_2\rangle &= |q_2\rangle, \\
V_{b,0}|q_3\rangle &= |q_3\rangle, \\
V_{b,0}|q_4\rangle &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|q^i_{5,0}\rangle, \\
V_{b,1}|q^i_{5,i}\rangle &= |q^i_{5,0}\rangle, \\
V_{b,1}|q^i_{5,0}\rangle &= |q^i_{5,2i}\rangle, \\
V_{b,0}|q^i_{6,j+1}\rangle &= |q^i_{6,j}\rangle, \\
V_{b,0}|q^i_{6,1}\rangle &= |q^i_{6,N-i+1}\rangle, \\
V_{\$,0}|q_2\rangle &= |q_3\rangle, \\
V_{\$,0}|q^i_{6,N-i+1}\rangle &= \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{\frac{2\pi i k}{N}\sqrt{-1}}|q^k_7\rangle,
\end{aligned}
$$

$$
\begin{aligned}
C(q^i_{5,2i}, a) &= -1, \\
C(q^i_{5,i}, a) &= +1, \\
C(q^i_{5,2i}, b) &= -1, \\
C(q^i_{5,0}, a) &= +1, \\
C(q, \sigma) &= 0 \quad \text{(otherwise)},
\end{aligned}
$$

$$
\begin{aligned}
D(q_j) &= \rightarrow \quad (j = 0, 2, 4), \\
D(q_j) &= \leftarrow \quad (j = 1, 3), \\
D(q^i_{5,2i}) &= \leftarrow, \\
D(q^i_{5,i}) &= \rightarrow, \\
D(q^i_{6,N-i+1}) &= \rightarrow, \\
D(q) &= \downarrow \quad \text{(otherwise)}.
\end{aligned}
$$

By the construction of $M_{\text{square}}$, we see that the computation consists of three phases. The first phase rejects any input not of the form $a^+b^+$. This phase is straightforward, similar to 2-way reversible finite automata (with no-counter) which recognizes the input of the form $a^+b^+$. If the input is not of the indicated form, the computation terminates. Otherwise, the second phase begins with the state $q_4$ with the tape head reading the left-most $b$.

At the start of the second phase, the computation branches into $N$ paths, indicated by the states $q^1_{5,0}, \ldots, q^N_{5,0}$, each with amplitude $1/\sqrt{N}$. For each of these paths, $M_{\text{square}}$ moves the tape head to left and right deterministically in the following way:

Along the $i$th path, the automaton decreases the counter value by 1 and moves the tape head to left. If the tape head reads the symbol $a$ then it remains stationary for $i$ steps. After that, if the counter value is not 0 then the automaton decreases the counter value by 1 and moves the tape head to left again, else it increases the counter value by 1 and moves the tape head to right until the tape head reads the left-most $b$.

If the tape head reads the symbol ¢, the third phase begins with the state $q^i_{5,0}$. Thus, while $M_{\text{square}}$ is scanning $a$'s in the input, the tape head requires precisely

$$
(i+1)\left(\sum_{c=0}^{m-1}(2c+1)+m\right) = (i+1)\left(m^2+m\right)
$$

steps along the $i$th path, where $m$ is the length of $a$'s.

Along the $i$th path on the third phase, if the tape head reads the symbol $a, b$ then it remains stationary for $N-i+1$ steps and after that moves to right. Upon reading the symbol \$, each computation path again splits according to the Discrete-Fourier Transformation, yielding the single accepting state $q^N_7$ and the other rejecting states $q^i_7 (1 \leq i \leq N-1)$. Thus, while the automaton is scanning $a$'s and $b$'s in the input, the tape head requires precisely $(N-i+1)(m+n)$ steps along the $i$th path, where $n$ is the length of $b$'s. Therefore, it is easy to see that, under the assumption $i \neq i'$, $(i+1)\left(m^2+m\right) + (N-i+1)(m+n) = (i'+1)\left(m^2+m\right) + (N-i'+1)(m+n)$ if and only if $m^2 = n$.

Consider first the case that $m^2 = n$. Since each of the $N$ computation paths reaches the symbol \$ at the same time, we have the superposition

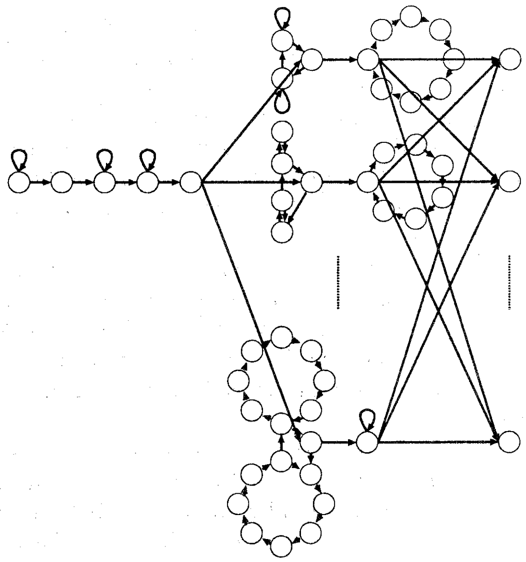

Figure 1: Transition image of $M_{\text{square}}$

immediately after performing the Discrete-Fourier Transformation is

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} e^{\frac{2\pi i k}{N}\sqrt{-1}}|q_7^k\rangle = |q_7^N\rangle.$$

Hence, the accepting state $q_7^N$ is entered with probability 1.

Next suppose that $m^2 \neq n$. In this case, each of $N$ computation paths do not reach the symbol \$ at the same time. Thus, there is no cancellation among the rejection states. For each of $N$ paths, the conditional probability that an observation results in $q_7^N$ at the time is $1/N$. It follows that the total probability that an observation results in $q_7^N$ is also $1/N$. Consequently the input is rejected with probability $1 - 1/N$.

We clearly see that each possible computation path has length $O(N|x|)$, since each path terminates in a halting states with certainty. □

### 3.2 2Q1CA for $\{a^m b^n c^{mn} \mid m, n \geq 1\}$

**Proposition 2** *Let $L_{\text{prod}}$ be $\{a^m b^n c^{mn} \mid m, n \geq 1\}$. For an arbitrary fixed positive integer $N \geq 2$, there exists a 2Q1CA $M_{\text{prod}}$ which accepts $x \in L_{\text{prod}}$ with probability 1 and rejects $x \notin L_{\text{prod}}$ with probability $1 - 1/N$. In either case, $M_{\text{prod}}$ halts after $O(N|x|)$ steps with certainty.*

*Proof.* Let the state set $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6^i, q_7^i, q_{8,j_1}^i, q_9^i, q_{10,j_2}^i, q_{11}^i \mid 1 \leq i \leq N, 1 \leq j_1, \leq i, 1 \leq j_2 \leq N - i + 1\}$, $Q_{\text{acc}} = \{q_{11}^N\}$ and $Q_{\text{rej}} = \{q_{11}^j \mid 1 \leq j \leq N-1\}$. For each $q \in Q$, $\sigma \in \Gamma$, $s \in \{0,1\}$, we define the transition matrices $\{V_{\sigma,s}\}$, the counter function $C$ and the head function $D$ as follows:

$$
\begin{aligned}
V_{\text{¢},0}|q_0\rangle &= |q_0\rangle, \\
V_{\text{¢},0}|q_5\rangle &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|q_6^i\rangle, \\
V_{\text{¢},1}|q_9^i\rangle &= |q_6^i\rangle, \\
V_{a,0}|q_0\rangle &= |q_0\rangle, \\
V_{a,0}|q_1\rangle &= |q_2\rangle, \\
V_{a,0}|q_5\rangle &= |q_5\rangle, \\
V_{a,s}|q_6^i\rangle &= |q_6^i\rangle, \\
V_{a,s}|q_9^i\rangle &= |q_9^i\rangle, \\
V_{a,1}|q_7^i\rangle &= |q_{8,1}^i\rangle, \\
V_{b,0}|q_0\rangle &= |q_1\rangle, \\
V_{b,0}|q_2\rangle &= |q_2\rangle, \\
V_{b,0}|q_3\rangle &= |q_4\rangle, \\
V_{b,0}|q_5\rangle &= |q_5\rangle, \\
V_{b,0}|q_9^i\rangle &= |q_{10,1}^i\rangle, \\
V_{b,1}|q_6^i\rangle &= |q_7^i\rangle, \\
V_{b,1}|q_{8,j}^i\rangle &= |q_{8,j+1}^i\rangle \quad (1 \leq j \leq i-1), \\
V_{b,1}|q_{8,i}^i\rangle &= |q_{8,1}^i\rangle, \\
V_{b,1}|q_9^i\rangle &= |q_9^i\rangle, \\
V_{c,0}|q_2\rangle &= |q_3\rangle, \\
V_{c,0}|q_4\rangle &= |q_4\rangle, \\
V_{c,0}|q_5\rangle &= |q_5\rangle, \\
V_{c,0}|q_{10,j}^i\rangle &= |q_{10,j+1}^i\rangle \quad (1 \leq j \leq N-i), \\
V_{c,0}|q_{10,N-i+1}^i\rangle &= |q_{10,1}^i\rangle, \\
V_{c,1}|q_{8,1}^i\rangle &= |q_9^i\rangle, \\
V_{\$,0}|q_4\rangle &= |q_5\rangle, \\
V_{\$,0}|q_{10,1}^i\rangle &= \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{\frac{2\pi i k}{N}\sqrt{-1}}|q_{11}^k\rangle,
\end{aligned}
$$

$$
\begin{aligned}
C(q_6^i, a) &= +1, \\
C(q_9^i, a) &= -1, \\
C(q_9^i, c) &= -1, \\
C(q, \sigma) &= 0 \quad \text{(otherwise)},
\end{aligned}
$$

$$
\begin{aligned}
D(q_j) &= \rightarrow \quad (j=0,2,4),\\
D(q_j) &= \leftarrow \quad (j=1,3,5),\\
D(q_6^i) &= \rightarrow,\\
D(q_7^i) &= \leftarrow,\\
D(q_{8,1}^i) &= \rightarrow,\\
D(q_9^i) &= \leftarrow,\\
D(q_{10}^i) &= \rightarrow,\\
D(q) &= \downarrow \quad \text{(otherwise)}.
\end{aligned}
$$

By the construction of the automaton, we see that the computation consists of four phases. The first phase rejects any input not of the form $a^+b^+c^+$. This phase is straightforward, similar to 2-way reversible finite automata (with no-counter) which recognizes the input of the form $a^+b^+c^+$. If the input is not of the indicated form, the computation terminates. Otherwise, the second phase begins with the state $q_5$ with the tape head reading the symbol ¢.

At the start of the second phase, the computation branches into $N$ paths, indicated by the states $q_6^1, \ldots, q_6^N$, each with amplitude $1/\sqrt{N}$.

For each of these paths, the automaton moves the tape head to right and increases the counter value by 1 while reading the symbol $a$. Upon reading the symbol $b$, the third phase begins with the state $q_8^i$.

Along the $i$th path on the third phase, if the tape head reads the symbol $b$ then it remains stationary for $i$ steps and after that moves to right. After reading the symbol $c$, if the counter value is not 0 then the automaton moves the tape head to left, with decreasing the counter value by 1 while reading the symbol $a$, until the tape head reaches the symbol ¢, and repeat the second phase. Otherwise, the fourth phase begins with the state $q_{10}^i$. Thus, the tape head requires precisely $m$ steps on the second phase and $in$ steps along the $i$th path on the third phase. It is easy to see that the automaton repeats the second and the third phase $m$ times.

Along the $i$th path on the fourth phase, if the tape head reads the symbol $c$ then it remains stationary for $N-i+1$ steps and after that moves to right. Upon reading the symbol \$, each computation path again splits according to the Discrete-Fourier Transformation, yielding the single accepting state $q_{11}^N$ and the other rejecting states $q_{11}^i (1 \le i \le N-1)$. Thus, the tape head requires precisely $(N-i+1)l$ steps along the $i$ th path,

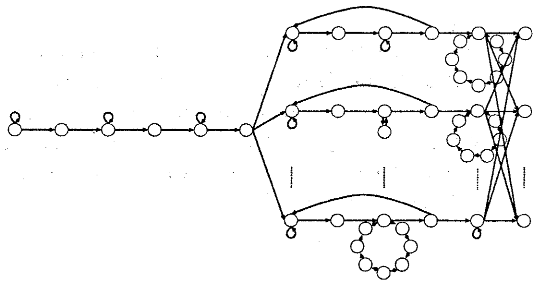

Figure 2: Transition image of $M_{\mathrm{prod}}$

where $l$ is the length of $c$'s.

Therefore, it is easy to see that, under assumption $i \neq i'$, $m^2 + imn + (N-i+1)l = m^2 + i'mn + (N-i'+1)l$ if and only if $l = mn$.

Consider first the case that $l = mn$. Since each of the $N$ computation paths reaches the symbol \$ at the same time, we have the superposition immediately after performing the Discrete-Fourier Transformation is

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} e^{\frac{2\pi ik}{N}\sqrt{-1}} |q_{11}^k\rangle = |q_{11}^N\rangle.$$

Hence, the accepting state $q_{11}^N$ is entered with probability 1.

Next suppose that $l \neq mn$. In this case, each of $N$ computation paths do not reach the symbol \$ at the same time. Thus, there is no cancellation among the rejection states. For each of $N$ paths, the conditional probability that an observation results in $q_{11}^N$ at the time is $1/N$. It follows that the total probability that an observation results in $q_{11}^N$ is also $1/N$. Consequently the input is rejected with probability $1 - 1/N$.

We clearly see that each possible computation path has length $O(N|x|)$, since each path terminates in a halting states with certainty. □

### 3.3 2Q1CA for $\{a_1^{n^1} a_2^{n^2} \ldots a_k^{n^k} \mid n \ge 1\}$

**Proposition 3** *For each fixed $k \ge 1$, there exists a 2Q1CA which recognizes $\{a_1^{n^1} a_2^{n^2} \ldots a_k^{n^k} \mid n \ge 1\}$.*

*Proof.* (Sketch.)

1. $k = 1$
   Trivial.

2. $k = 2$
   In this case, we proved at the Section 3.1.

3. $k \geq 3$
First, the automaton checks whether the input is of the form $a_1^+ a_2^+ \dots a_k^+$. Next, the automaton checks whether $m_1^2 = m_2$ or not, where $m_j (1 \leq j \leq k)$ is the length of $a_j$'s. Then, it checks whether $m_1 m_j = m_{j+1} (2 \leq j \leq k-1)$ or not.

□

## 4 Conclusion

In this paper, we gave the definition of 2Q1CAs and proved that 2Q1CAs can recognize some languages such as $\{a^n b^{n^2} \mid n \geq 1\}$ or $\{a^m b^n c^{mn} \mid m, n \geq 1\}$.

One interesting question is what kind of languages are recognizable by 2Q1CAs? We expect that some non-context-sensitive languages are recognizable but some context-sensitive languages are not.

Another question is concerning to a two-counter case. It is known that two-way deterministic two-counter automata (2D2CAs) can simulate deterministic Turing machines (DTMs) [3]. How about the power of 2Q2CAs?

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