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グラフの平面凸描画の枝長と線形カットサイズと交差操作の関係

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あらまし　$G = (N, c)$ を節点集合 $N = \{0, 1, \ldots, n-1\}$ と実数の枝の重み関数 $c : V \times V \to \boldsymbol{R}$ によって定義されるグラフ（ネットワーク）とする。以下の３つの尺度を考える。(1) 凸多角形上にグラフを記述したときの枝長の総和、(2) 線形カットの大きさ、(3) 枝の交差処理による帰着可能性。これらの３つの尺度に関連した半順序関係を導入し、それらが同値であることを証明する。

キーワード　半順序, グラフ描画, 直線描画, 線形カット, 環状配置

Relation among Edge Length of Convex Planar Drawings, Size of Linear Cuts, and Cross-Operations on Graphs

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Abstract　Let $G = (N, c)$ be a graph with a vertex set $N = \{0, 1, \ldots, n-1\}$ and a real edge weight function $c : V \times V \to \boldsymbol{R}$. Three measures for comparing two graphs are considered: (1) the sum of edge length when the graph is drawn on a convex polygon, (2) sizes of linear-cuts, and (3) reducibility by using cross-operations. Three partial orders, corresponding the measures respectively, are also introduced. This paper shows that these three partial orders are equivalent. Moreover, it presents a polynomial time algorithm for determining $G \preceq G'$ for given G and G', where, $\preceq$ is the partial order.

Key words: partial order, graph drawing, straight-line, linear-cut, cyclic layout

1 Introduction

Let G be a graph with a vertex set $N = \{0, 1, \ldots, n-1\}$. Each pair of vertices (i, j) of a graph has a weight (= number of parallel edges) $c(i, j) \in \boldsymbol{R}$ ($\boldsymbol{R}$ is the set of real numbers). $c(i, j)$ may be written as $c(i, j; G)$ if the graph should be expressed explicitly. We can define a graph as $G = (N, c)$, where $c : N \times N \to \boldsymbol{R}$. Note that a weight may be zero, negative, or irrational in this paper. If all weights are restricted to nonnegative integers, graphs are called *multigraph*s. If all weights are restricted to $\{0, 1\}$, graphs are called *simple graph*s. Each vertex is labeled by an integer in N and each edge has a real weight, so that we may call such graphs *labeled weighted graph*s. Graphs appeared in this paper are labeled weighted graphs if otherwise stated. In this paper, selfloops, $c(i, i)$, are meaningless. For this reason, if $c(i, j; G) = c(i, j; G')$ for all $i \neq j$, then we say $G = G'$. A singleton set $\{i\}$ may be simply written as i.

For $A, B \subseteq N$,

$$c(A, B; G) := \sum_{i \in A,\ j \in B} c(i, j; G).$$

$c(A, N - A; G)$ may be written as $c(A; G)$. $c(A, B; G)$ and $c(A; G)$ may be expressed as $c(A, B)$ and $c(A)$, respectively, if G is clear. Note that $c(i, G)$ means a degree of $i \in N$.

We adopt the cyclic order for treating integers (vertices) in N. Thus for $i, j \in N$,

$$N[i, j] \;:=\; \begin{cases} \{i, i+1, \ldots, j\}, & \text{if } i \leq j, \\ \{i, i+1, \ldots, n-1, 0, 1, \ldots, j\}, & \\ & \text{if } i > j. \end{cases}$$

Moreover, $i \leq j \leq k$ means $j \in N[i, k]$, $i \leq j \leq k \leq h$ means $i \leq j \leq k$ and $k \leq h \leq i$, and $i \pm j$ is $i' \in N$ such that $i' \equiv i \pm j \pmod{n}$.

Three partial orders are defined as follows.

1. Let $x_0, x_1, ..., x_{n-1}$ be vertices of a convex n-gon P in the plane (each internal angle may be equal to π), where, x_0x_1, x_1x_2, $\ldots$, $x_{n-2}x_{n-1}$, and $x_{n-1}x_0$ are edges of the n-gon. Denote the length of the line segment x_ix_j by $d_P(i, j)$. Define a length of G with respect to P as

$$S_P(G) := \sum_{i, j \in V} c(i, j; G) \cdot d_P(i, j).$$

$S_P(G)$ can be regarded as the sum of edge length of a graph G drawn in the plane such that each vertex i of G is equal to a corresponding vertex x_i of P and each edge of G is given by a straight line segment(e.g., Figure 1).

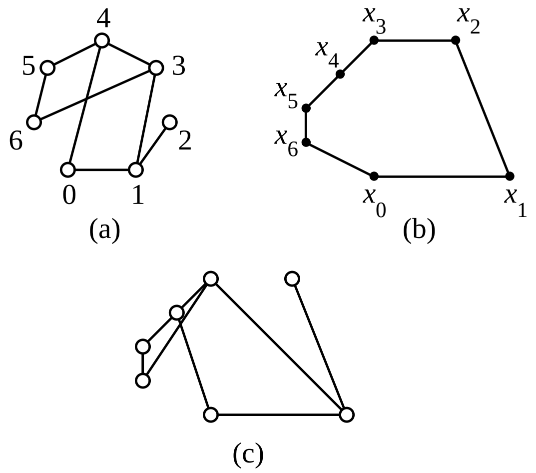

Figure 1: (a) graph G, (b) convex polygon P, (c) G drawn on P.

If $S_P(G) \leq S_P(G')$ for any convex polygon P, then $G \preceq_l G'$ ("l" means length). If $G \preceq_l G'$ and $G \neq G'$, then $G \prec_l G'$.

2. $N[i, j]$ is called a *linear-cut* if $N[i, j] \neq N$ ($N[i, j] \neq \emptyset$ is clear from the definition). The size of a linear-cut $N[i, j]$ is defined as $c(N[i, j]; G)$. If $c(N[i, j]; G) \leq c(N[i, j]; G')$ for all linear-cuts $N[i, j]$, then $G \preceq_c G'$ ("c" means cuts). Skiena [9] showed that if $c(N[i, j]; G) = c(N[i, j]; G')$ for all linear-cuts $N[i, j]$, then $G = G'$. (Although he treated only multigraphs, his proof can be used for general real-weighted graphs.) It directly follows that if $G \preceq_c G'$ and $G' \preceq_c G$, then $G = G'$. Thus we can say $G \prec_c G'$ if $G \preceq_c G'$ and $G \neq G'$.

3. We define a *cross-operation* $X(i,j,k,h;\Delta)$, for $i \le j \le k \le h$ and a positive real value $\Delta > 0$, as removing Δ from $c(i,j)$ and $c(k,h)$, and adding Δ to $c(i,k)$ and $c(j,h)$. Figure 2 illustrates a cross-operation $X(i,j,k,h;1)$. Note that more than one vertices in i,j,k, and h may be equal. For example, $X(i,i,k,k;\Delta)$ means only adding 2Δ to $c(i,k)$. (Remember that all selfloops are meaningless in this paper, thus removing Δ from $c(i,i)$ and $c(k,k)$ can be ignored.) If graph G' can be obtained from graph G by applying a sequence of cross-operations, then we express as $G \preceq_o G'$ ("o" means operations). If $G \preceq_o G'$ and $G \neq G'$, then $G \prec_o G'$.

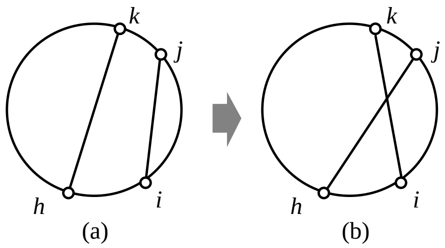

Figure 2: Cross-operation $X(i,j,k,h;1)$.

These three measures, edge-length, cut-size, and reducibility by the operation, are important alone, and have been considered independently. However, this paper shows that they are equivalent. It establishes the next:

Theorem 1 *Three partial orders $\preceq_l$, $\preceq_o$, and $\preceq_c$ are equivalent, i.e.,*

$$G \preceq_l G' \Leftrightarrow G \preceq_c G' \Leftrightarrow G \preceq_o G'$$

for any pair of labeled weighted graphs G and G' □

Sum of edge lengths is one of the crucial criteria on graph drawing. Graph drawing has recently become a very important research area [1, 7].

Some properties on linear-cuts have been found in advance. Mäkinen [6] shows the problem of finding a permutation $\pi = \langle p_0, p_1, \ldots, p_{n-1}\rangle$ of the vertices $\langle 0, 1, \ldots, n-1\rangle$ of a given multigraph G such that $\max_{i,j \in N} c(N[i,j])$ is minimum is NP-hard, and presents a heuristic algorithm. Schröder, et.al. [8] shows some lower bounds of the maximum size of linear-cuts for cylindrical mesh graphs. Skiena [9] considers a problem of reconstructing a graph from information of linear-cut sizes only, and shows that $\binom{n}{2}$ linear-cuts are necessary and sufficient for the reconstruction.

Hakimi [2] considered cross-operations and the reverse of cross-operations and called them *elementary d-invariant transformation*s ("*d*" means dimensions). He showed that every pair of multigraphs G and G' such that $c(i,G) = c(i,G')$ for all $i \in N$ can be transformed from one to another by using a finite sequence of elementary d-invariant transformations.

The author have presented the following Theorem 2 [4]. Here, a graph $G_p := (N, c_p)$ is defined as

$$c_p(i,j) := \begin{cases} 1, & \text{if } j = i + p \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2 [4] *(1) $G_p \prec_l G_{p+1}$ for $p = 0, 1, \ldots, \lfloor n/2 \rfloor - 1$.*
(2) For any 2-regular multigraph $G (\neq G_{\lfloor n/2 \rfloor})$, $G \prec_l G_{\lfloor n/2 \rfloor}$.
(3) If $G (\neq G_1)$ is a 2-regular multigraph such that $c(N[i,j]; G) > 0$ for any linear-cut $N[i,j]$, $G_1 \prec_l G$. □

Theorem 2 (1) was firstly conjectured by Jorge Urrutia in the open problem session of Japan Conference on Discrete and Computational Geometry 1998 (JCDCG'98). For an example of this theorem, see Figure 3. Theorem 2 (1) claims that $S_P(G_1) \le S_P(G_2) \le S_P(G_3)$ for any convex polygon P. (Note that in reference [4], the more general property "$S_P(G_q)$ is a strictly increasing and strictly concave function for any convex polygon P if $1 \le q \le \lfloor n/2 \rfloor - 1$" was shown.)

Theorem 1 is a wide generalization of Theorem 2, i.e., the former gives another proof for the latter. We show it by using an example, $G_q \prec_c G_{q+1}$ is clear for any $q =$

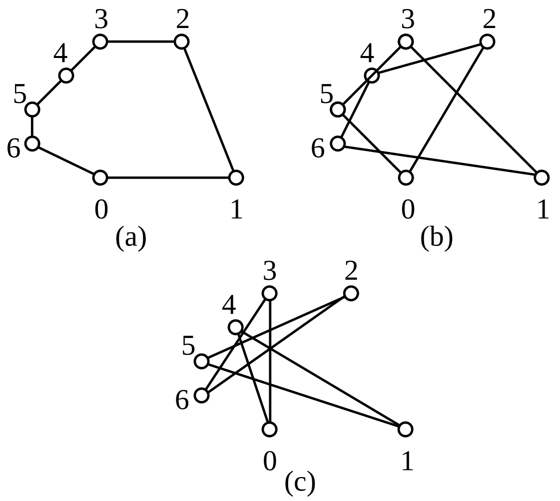

Figure 3: (a) G_1, (b) G_2, and (c) G_3 drawn on a convex polygon P.

$0, 1, \ldots, \lfloor n/2 \rfloor - 1$. Moreover, G_2 can be obtained from G_1 by applying a sequence of cross-operations $\langle (n-1, 0, 1, 2; 1)$, $(0, 1, 3, 4; 1)$, $(2, 3, 4, 1; 1)$, $(0, 3, 4, 5; 1)$, $(0, 4, 5, 6; 1)$, $\ldots$, $(0, n-3, n-2, n-1; 1) \rangle$ (Figure 4), i.e., $G_1 \prec_o G_2$.

We present a proof of Theorem 1 in the next section. Preliminary results of this paper were presented in the Japan Conference on Discrete and Computational Geometry (JCDCG2000) [5]. In the theorem presented in [5], edge weights were restricted to nonnegative integers and only graphs with the same number of edges could be compared.

2 Proof

Define $G_\emptyset = (N, c_\emptyset)$ as $c_\emptyset(i, j) = 0$ for all $i, j \in N$. Note that $c(N[i, j]; G_\emptyset) = 0$ for any linear-cut $N[i, j]$, and $S_P(G_\emptyset) = 0$ for any polygon P. For any pair of $G = (N, c)$ and $G' = (N, c')$, we define $G - G' = (N, c'')$ as $c''(i, j) := c(i, j) - c'(i, j)$ for every $i, j \in N$. $G \preceq G'$ ($\preceq$ is any one of $\preceq_l$, $\preceq_c$, and $\preceq_o$) is equivalent to $G - G' \preceq G_\emptyset$. Therefore, it is enough to consider $G' = G_\emptyset$ for proving Theorem 1, as a result of this fact, the proof of Theorem 1 consists of three parts:

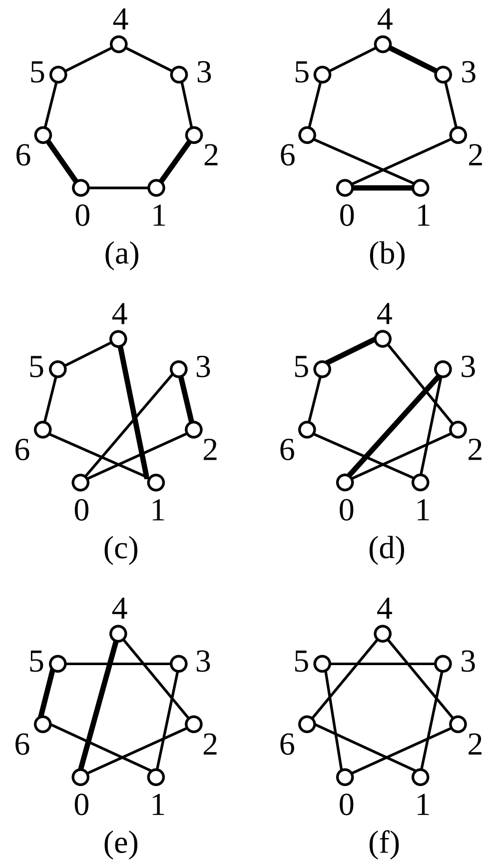

Figure 4: A sequence of cross-operations modifying G_1 into G_2.

(1) $G \preceq_o G_\emptyset \Rightarrow G \preceq_l G_\emptyset$, (Lemma 1)

(2) $G \preceq_l G_\emptyset \Rightarrow G \preceq_c G_\emptyset$, (Lemma 2) and

(3) $G \preceq_c G_\emptyset \Rightarrow G \preceq_o G_\emptyset$. (Lemma 3)

Lemma 1 *If $G \preceq_o G_\emptyset$, then $G \preceq_l G_\emptyset$.*

Proof: It is clear from the triangle inequality. □

Lemma 2 *If $G \preceq_l G_\emptyset$, then $G \preceq_c G_\emptyset$.*

Proof: Suppose that $G \preceq_c G_\emptyset$ does not hold, i.e., there are $i, j \in N$ such that $c(N[i, j]; G) > 0$. We construct a polygon P satisfying $S_P(G) > 0$ as follows. $X = \{x_k \mid k \in N[i, j]\}$ and $Y = \{x_k \mid k \in N - N[i, j]\}$. Let $p, r > 0$ be real numbers. Put all vertices $x_i \in X$ in a circle whose center is $(0, 0)$ and radius is r. Put all vertices $x_i \in Y$ in a circle whose center is $(p, 0)$ and radius is r. We can locate all vertices satisfying the above conditions and convexity for any r and p. By letting p be far larger than r, $S_P(G) > 0$. □

For proving the remaining part, $G \preceq_c G_\emptyset \Rightarrow G \preceq_o G_\emptyset$, we need to introduce some notations as follows. For integers $i, j \in N$,

$$N(i,j) := N[i,j] - \{i,j\}.$$

The following proposition is well-known. Since the proof is easy, it is omitted.

Proposition 1 *Let $A, B, C, D \subset N$ be four mutually disjoint subsets such that $A \cup B \cup C \cup D = N$, then*

$$c(A \cup B) + c(A \cup D) = c(A) + c(C) + 2c(B, D).$$

□

Now, we can prove the next:

Lemma 3 *If $G \preceq_c G_\emptyset$, then $G \preceq_o G_\emptyset$.*

Proof: Assume that $G \preceq_c G_\emptyset$, i.e.,

$$c(N[i,j]; G) \leq 0 \text{ for all } i, j \in N. \quad (1)$$

Let k be the largest integer such that $c(N[i,j]) = 0$ for all $(i,j) \in \{(i,j) \mid i,j \in N, |N[i,j]| \leq k\}$. If $k \geq \lceil n/2 \rceil$, $G = G_\emptyset$. Hence, we assume $k < \lceil n/2 \rceil$. Then their exists (i_0, j_0) such that $|N[i_0, j_0]| = k + 1$ and

$$c(N[i_0, j_0]) < 0. \quad (2)$$

By considering Proposition 1 with $A = N(i_0, j_0)$, $B = \{j_0\}$, $C = N(j_0, i_0)$, and $D = \{i_0\}$, we obtain

$$\begin{aligned} & c(i_0, j_0) \\ = \ & \frac{1}{2}\{c(N[i_0+1, j_0]) + c(N[i_0, j_0 - 1]) \\ & \quad - c(N(i_0, j_0)) - c(N(j_0, i_0))\} \\ > \ & 0, \end{aligned}$$

since $c(N[i_0 + 1, j_0]) = c(N[i_0, j_0 - 1]) = c(N(i_0, j_0)) = 0$ and $c(N(j_0, i_0)) = c(N[i_0, j_0]) < 0$.

If there is a pair i' and j' satisfying the following (a)–(c) (Figure 5 (a)):

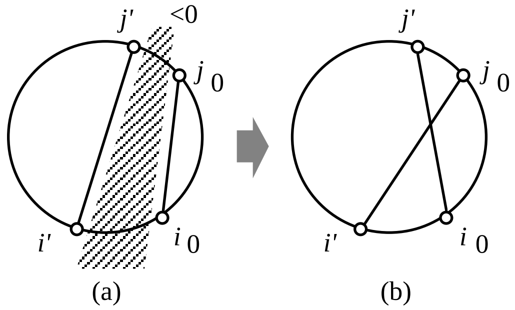

Figure 5: Cross-operation $X(i_0, j_0, j', i'; \Delta)$

(a) $j_0 < j' \leq i' < i_0$,

(b) $c(i', j') > 0$ or $i' = j'$, and

(c) $c(N[i,j]) < 0$ for all $i' < i \leq i_0$ and $j_0 \leq j < j'$,

then we can apply a cross-operation $X(i_0, j_0, j', i' : \Delta)$ to G without violating the relation $G \preceq_c G_\emptyset$ (Figure 5), where

$$\Delta = \min\{c(i_0, j_0),\ c(i', j'),\ \min_{i' < i \leq i_0,\ j_0 \leq j < j'} \frac{-c(N[i,j])}{2}\}. \quad (3)$$

Therefore we try to find such i' and j' as follows.

Let j_1 ($j_0 < j_1 < i_0$) be a vertex such that $c(N[i_0, j]) < 0$ for all $j_0 \leq j < j_1$ and

$$c(N[i_0, j_1]) = 0. \quad (4)$$

If there is no such j_1, then we find a desired pair (i', j') by letting $i' := j' := i_0 - 1$ (note (1) and (2)). Thus we assume such j_1 exists. Let $i_1 := i_0 - 1$. Assume that there exists $j' \in N[j_0 + 1, j_1]$ such that $c(i_1, j') > 0$. Then $i' := i_1$ and j' satisfy (a)–(c). Therefore, we assume there is no such j', i.e., $c(i_1, j) \leq 0$ for all $j \in N[j_0 + 1, j_1]$. It follows that

$$c(i_1, N[j_0 + 1, j_1]) \leq 0. \quad (5)$$

Consider Proposition 1 with $A = N[i_0, j_0]$, $B = N[j_0 + 1, j_1]$, $C = (j_1, i_1)$, and $D = \{i_1\}$ (Figure 6). Since (2), (4), (5), and

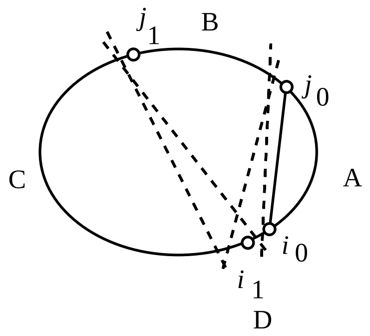

Figure 6: Applying Proposition 1

$c(N[i_1, j_1]) \leq 0$(because (1)), we obtain

$$\begin{aligned} & c(N[i_1, j_0]) \\ = \ & -c(N[i_0, j_1]) + c(N[i_0, j_0]) \\ & +c(N[i_1, j_1]) + 2c(i_1, N[j_0+1, j_1]) \\ < \ & 0. \end{aligned}$$

Let $i'_1 := i''_1 := i_1$, and $i_1 := i_1 - 1$ (Figure 7, which is illustrated generally. $i'_1 = i''_1$ here). Note that

$$c(N[i, j]) < 0 \quad \text{for all } i''_1 < i \leq i_0,\ j_0 \leq j < j_1, \tag{6}$$

$$c(N[i''_1, j_0]) < 0, \tag{7}$$

$$c(N[i''_1, i'_1], N[j_0+1, j_1]) \leq 0. \tag{8}$$

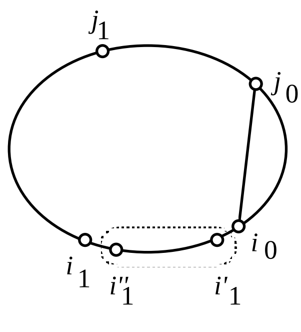

Figure 7: i_1, i'_1, and i''_1

Let j'_1 be the closest vertex to j_1 such that $j_0 < j'_1 \leq j_1$, $c(N[i''_1, j]) < 0$ for all $j_0 \leq j < j'_1$, and $c(i_1, j) \leq 0$ for all $j_0 < j < j'_1$. If $j_1 \neq j'_1$, rename $j_1 := j'_1$. The old j_1 is called j_1^{old} for distinction. Note that (6)–(8) also hold for the new j_1. From the definition of j_1, we get

$$\begin{aligned} c(N[i''_1, j]) &< 0 \text{ for all } j_0 \leq j < j_1, \\ c(i_1, j) &\leq 0 \text{ for all } j_0 < j < j_1. \end{aligned} \tag{9}$$

If $j_1 = i_1$, then $i' := j' := i_1$ satisfies (a)–(c). Thus we assume $j_1 \neq i_1$. If $c(i_1, j_1) > 0$, then $i' := i_1$ and $j' := j_1$ satisfy (a)–(c). Then we assume that

$$c(i_1, j_1) \leq 0. \tag{10}$$

By considering (9) and (10),

$$c(i_1, N[j_0+1, j_1]) \leq 0. \tag{11}$$

We will show $c(N[i_0, j_1]) = 0$. If $j_1 = j_1^{old}$, it is clear from (4) (note that j_1 in (4) is j_1^{old}, here). Then assume that $j_1 \neq j_1^{old}$, i.e., $j_0 < j_1 < j_1^{old}$. If $c(N[i_0, j_1]) < 0$, then j_1 should be chosen closer than the present j_1 since the definition of j_1 and (10), contradiction. Thus we get

$$c(N[i_0, j_1]) = 0. \tag{12}$$

We will show $c(N[i_1, j_0]) < 0$. For this purpose, we make the following assumption and lead contradiction:

Assumption 1: $c(N[i_1, j_0]) = 0$.

From Proposition 1 with $A := N[i''_1, j_0]$, $B = N[j_0+1, j_1]$, $C = N(j_1, i_1)$, and $D = \{i_1\}$ (Figure 8 (a)), we obtain

$$\begin{aligned} & c(N[i''_1, j_1]) + c(N[i_1, j_0]) \\ = \ & c(N[i''_1, j_0]) + c(N[i_1, j_1]) \\ & +2c(i_1, N[j_0+1, j_1]). \end{aligned}$$

Thus, by considering Assumption 1, (11), and $c(N[i_1, j_1]) \leq 0$ (because (1)),

$$\begin{aligned} & c(N[i''_1, j_0]) - c(N[i''_1, j_1]) \\ = \ & c(N[i_1, j_0]) - c(N[i_1, j_1]) \\ & -2c(i_1, N[j_0+1, j_1]) \\ \geq \ & 0. \end{aligned}$$

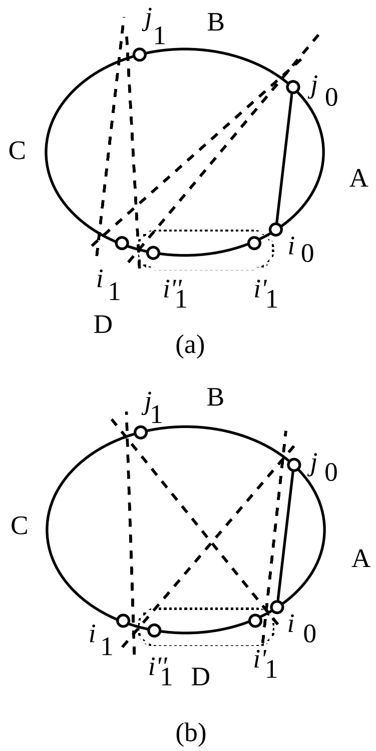

Figure 8: Applying Proposition 1

By considering Proposition 1 with $A = N[i_0, j_0]$, $B = N[j_0+1, j_1]$, $C = N[j_1+1, i_1]$, and $D = N[i_1'', i_1']$ (Figure 8 (b)),

$$\begin{aligned} & c(N[i_1'', j_0]) - c(N[i_1'', j_1]) \\ = \; & -c(N[i_0, j_1]) + c(N[i_0, j_0]) \\ & +2c(N[i_1'', i_1'], N[j_0+1, j_1]). \end{aligned}$$

From (2), (8), and (12), we obtain

$$c(N[i_1'', j_0]) - c(N[i_1'', j_1]) < 0,$$

contradicting (13). Therefore, Assumption 1 is denied, i.e.,

$$c(N[i_1, j_0]) < 0. \qquad (13)$$

Here, let $i_1'' := i_1$ and $i_1 := i_1 - 1$ (i_1' is not changed), then (6)–(8) also hold. Thus the preceding discussion (from (6) to (13)) can be also applied. However, $N[i_1'', i_1']$ becomes larger in the new iteration. Thus, such procedure must be stopped at most $|N(j_0, i_0)|$ iterations. Therefore, we must finally find i' and j' satisfying (a)–(c).

By setting the value of Δ as (3), we can apply $X(i_0, j_0, j', i'; \Delta)$ to G without violating the relation $G \preceq_c G_\emptyset$.

Now, we have found a cross operation that makes G be closer to $G_\emptyset$. By applying the preceding discussion iteratively, we can find a sequence of cross-operations that makes G be closer to $G_\emptyset$. For completing the proof, we must show that the length of the sequence is finite. It is shown as follows.

Let G' be a graph obtained by applying $X(i_0, j_0, j', i'; \Delta)$ to G. There are three cases: (I) $\Delta = c(i_0, j_0; G)$, (II) $\Delta = \min_{i'<i\le i_0,\ j_0\le j<j'}(-c(N[i,j];G))/2$, and (III) $\Delta = c(i', j'; G)$. We consider each case as follows.

(I) $\Delta = c(i_0, j_0; G)$. $c(i_0, j_0; G')$ becomes zero. Then by applying Proposition 1 with $A = N(i_0, j_0)$, $B = \{j_0\}$, $C = N(j_0, i_0)$, and $D = \{i_0\}$, we obtain $c(N[i_0, j_0]; G') = 0$. Thus, the number of zero-linear-cuts of G' is greater than the one of G.

(II) $\Delta = \min_{i'<i\le i_0,\ j_0\le j<j'}(-c(N[i,j];G))/2$. Let i'' and j'' be vertices satisfying $i' < i'' \le i_0$, $j_0 \le j'' < j'$, and $\Delta = -c(N[i'', j'']; G)/2$. $c(N[i'', j'']; G')$ becomes zero. Thus, the number of zero-linear-cuts of G' is greater than the one of G.

(III) $\Delta = c(i', j'; G)$. $c(i', j'; G')$ becomes zero. It is enough to assume $c(i_0, j_0; G') > 0$, because if $c(i_0, j_0; G') = 0$, then case (I) can be applied. We can find new i' and j' satisfying (a)–(c). The number of pairs (i'', j'') in G' such that $j_0 < j'' < i'' < i_0$ and $c(i'', j'') > 0$ is smaller than the one in G, so that (III) occurs successively at most $\binom{|N(j_0,i_0)|}{2} < n^2$ times.

From (I)–(III), the number of zero-linear-cuts increases during at most n^2 cross-operations. The number of linear-cuts is $\binom{n}{2} < n^2$. It follows that the length of the sequence of cross-operations is less than n^4. By using the sequence, G is modified to $G_\emptyset$, i.e., $G \preceq_o G_\emptyset$. □

Proof of Theorem 1: Follows immediately from Lemmas 1, 2, and 3. □

3 Concluding Remarks

This paper shows that three partial-orders $\preceq_l$, $\preceq_o$, and $\preceq_c$ are equivalent. For investigating $G \preceq_c G'$, only linear-cuts are tested, thus it can be determined in polynomial time. Therefore, we can solve a problem of determining whether or not $S_P(G) \le S_P(G')$ for any convex polygon P for given two labeled weighted graphs G and G' in polynomial time. Moreover, if $G \preceq_c G'$, we can find a sequence of cross-operations for modifying G to G' by using the discussion of the proof of Lemma 3 in polynomial time.

In this paper, Euclidean distance is used. However, for any distance (for example, L_k distance) in which the triangle inequality holds, the same results can be obtained.

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