完全グラフの均衡的 (C_4, C_6, C_6)-Trefoil 分解アルゴリズム

藤本　英昭　潮　和彦
近畿大学理工学部
電気電子工学科　情報学科
〒577-8502 東大阪市小若江 3-4-1
Tel: +81-6-6721-2332 (ext. 4555(藤本)　4615(潮))
Fax: +81-6-6727-2024(藤本)　+81-6-6730-1320(潮)
E-mail: fujimoto@ele.kindai.ac.jp　ushio@is.kindai.ac.jp

アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_6 をそれぞれ 4 点、6 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_6、C_6 からなるグラフを (C_4, C_6, C_6)-trefoil という。本研究では、完全グラフ K_n を (C_4, C_6, C_6)-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 (C_4, C_6, C_6)-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_6, C_6)-Trefoil Decomposition Algorithm of Complete Graphs

Hideaki Fujimoto　Kazuhiko Ushio
Department of Electric and Electronic Engineering　Department of Informatics
Faculty of Science and Technology
Kinki University
Osaka 577-8502, JAPAN
Tel: +81-6-6721-2332 (ext. 4555(Fujimoto)　4615(Ushio))
Fax: +81-6-6727-2024(Fujimoto)　+81-6-6730-1320(Ushio)
E-mail: fujimoto@ele.kindai.ac.jp　ushio@is.kindai.ac.jp

Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_6, C_6)-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_6, C_6)-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_6 be *the* 4-*cycle* and *the* 6-*cycle*, respectively. *The* (C_4, C_6, C_6)-*trefoil* is a graph of 3 edge-disjoint cycles C_4, C_6 and C_6 with a common vertex and the common vertex is called *the center of the* (C_4, C_6, C_6)-*trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_6, C_6)-trefoils, we say that K_n *has a* (C_4, C_6, C_6)-*trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_6, C_6)-trefoils, we say that K_n *has a balanced* (C_4, C_6, C_6)-*trefoil decomposition* and

this number is called *the replication number*.
It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced (C_4, C_6, C_6)-trefoil decomposition of K_n is $n \equiv 1 \pmod{32}$.

2. Balanced (C_4, C_6, C_6)-trefoil decomposition of K_n

We use the following notation for a (C_4, C_6, C_6)-trefoil.

Notation. We denote a (C_4, C_6, C_6)-trefoil passing through $v_1-v_2-v_3-v_4-v_1-v_5-v_6-v_7-v_8-v_9-v_1-v_{10}-v_{11}-v_{12}-v_{13}-v_{14}-v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9), (v_1, v_{10}, v_{11}, v_{12}, v_{13}, v_{14})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_6, C_6)-trefoil decomposition if and only if $n \equiv 1 \pmod{32}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_6, C_6)-trefoil decomposition. Let b be the number of (C_4, C_6, C_6)-trefoils and r be the replication number. Then $b = n(n-1)/32$ and $r = 7(n-1)/16$. Among r (C_4, C_6, C_6)-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_6, C_6)-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/32$ and $r_2 = 13(n-1)/32$. Therefore, $n \equiv 1 \pmod{32}$ is necessary.
(Sufficiency) Put $n = 32t + 1$. Construct tn (C_4, C_6, C_6)-trefoils as follows:
$B_i^{(1)} = \{(i, i+4t+1, i+29t+2, i+5t+1), (i, i+1, i+8t+2, i+24t+3, i+12t+2, i+2t+1), (i, i+t+1, i+10t+2, i+27t+3, i+14t+2, i+3t+1)\}$
$B_i^{(2)} = \{(i, i+4t+2, i+29t+4, i+5t+2), (i, i+2, i+8t+4, i+24t+6, i+12t+4, i+2t+2), (i, i+t+2, i+10t+4, i+27t+6, i+14t+4, i+3t+2)\}$
$B_i^{(3)} = \{(i, i+4t+3, i+29t+6, i+5t+3), (i, i+3, i+8t+6, i+24t+9, i+12t+6, i+2t+3), (i, i+t+3, i+10t+6, i+27t+9, i+14t+6, i+3t+3)\}$
...
$B_i^{(t)} = \{(i, i+5t, i+31t, i+6t), (i, i+t, i+10t, i+27t, i+14t, i+3t), (i, i+2t, i+12t, i+30t, i+16t, i+4t)\}$ $(i = 1, 2, ..., n)$,
where the additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a balanced (C_4, C_6, C_6)-trefoil decomposition of K_n.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Example 1. A balanced (C_4, C_6, C_6)-trefoil decomposition of K_{33}.
Construct 33 (C_4, C_6, C_6)-trefoils as follows:
$B_1 = \{(1, 6, 32, 7), (1, 2, 11, 28, 15, 4), (1, 3, 13, 31, 17, 5)\}$
$B_2 = \{(2, 7, 33, 8), (2, 3, 12, 29, 16, 5), (2, 4, 14, 32, 18, 6)\}$
$B_3 = \{(3, 8, 1, 9), (3, 4, 13, 30, 17, 6), (3, 5, 15, 33, 19, 7)\}$
$B_4 = \{(4, 9, 2, 10), (4, 5, 14, 31, 18, 7), (4, 6, 16, 1, 20, 8)\}$
$B_5 = \{(5, 10, 3, 11), (5, 6, 15, 32, 19, 8), (5, 7, 17, 2, 21, 9)\}$
$B_6 = \{(6, 11, 4, 12), (6, 7, 16, 33, 20, 9), (6, 8, 18, 3, 22, 10)\}$
$B_7 = \{(7, 12, 5, 13), (7, 8, 17, 1, 21, 10), (7, 9, 19, 4, 23, 11)\}$

$B_8 = \{(8,13,6,14),(8,9,18,2,22,11),(8,10,20,5,24,12)\}$
$B_9 = \{(9,14,7,15),(9,10,19,3,23,12),(9,11,21,6,25,13)\}$
$B_{10} = \{(10,15,8,16),(10,11,20,4,24,13),(10,12,22,7,26,14)\}$
$B_{11} = \{(11,16,9,17),(11,12,21,5,25,14),(11,13,23,8,27,15)\}$
$B_{12} = \{(12,17,10,18),(12,13,22,6,26,15),(12,14,24,9,28,16)\}$
$B_{13} = \{(13,18,11,19),(13,14,23,7,27,16),(13,15,25,10,29,17)\}$
$B_{14} = \{(14,19,12,20),(14,15,24,8,28,17),(14,16,26,11,30,18)\}$
$B_{15} = \{(15,20,13,21),(15,16,25,9,29,18),(15,17,27,12,31,19)\}$
$B_{16} = \{(16,21,14,22),(16,17,26,10,30,19),(16,18,28,13,32,20)\}$
$B_{17} = \{(17,22,15,23),(17,18,27,11,31,20),(17,19,29,14,33,21)\}$
$B_{18} = \{(18,23,16,24),(18,19,28,12,32,21),(18,20,30,15,1,22)\}$
$B_{19} = \{(19,24,17,25),(19,20,29,13,33,22),(19,21,31,16,2,23)\}$
$B_{20} = \{(20,25,18,26),(20,21,30,14,1,23),(20,22,32,17,3,24)\}$
$B_{21} = \{(21,26,19,27),(21,22,31,15,2,24),(21,23,33,18,4,25)\}$
$B_{22} = \{(22,27,20,28),(22,23,32,16,3,25),(22,24,1,19,5,26)\}$
$B_{23} = \{(23,28,21,29),(23,24,33,17,4,26),(23,25,2,20,6,27)\}$
$B_{24} = \{(24,29,22,30),(24,25,1,18,5,27),(24,26,3,21,7,28)\}$
$B_{25} = \{(25,30,23,31),(25,26,2,19,6,28),(25,27,4,22,8,29)\}$
$B_{26} = \{(26,31,24,32),(26,27,3,20,7,29),(26,28,5,23,9,30)\}$
$B_{27} = \{(27,32,25,33),(27,28,4,21,8,30),(27,29,6,24,10,31)\}$
$B_{28} = \{(28,33,26,1),(28,29,5,22,9,31),(28,30,7,25,11,32)\}$
$B_{29} = \{(29,1,27,2),(29,30,6,23,10,32),(29,31,8,26,12,33)\}$
$B_{30} = \{(30,2,28,3),(30,31,7,24,11,33),(30,32,9,27,13,1)\}$
$B_{31} = \{(31,3,29,4),(31,32,8,25,12,1),(31,33,10,28,14,2)\}$
$B_{32} = \{(32,4,30,5),(32,33,9,26,13,2),(32,1,11,29,15,3)\}$
$B_{33} = \{(33,5,31,6),(33,1,10,27,14,3),(33,2,12,30,16,4)\}$.
This decomposition can be written as follows:
$B_i = \{(i,i+5,i+31,i+6),(i,i+1,i+10,i+27,i+14,i+3),(i,i+2,i+12,i+30,i+16,i+4)\}$
$(i = 1,2,...,33)$.
Then they comprise a balanced (C_4,C_6,C_6)-trefoil decomposition of K_{33}.

Example 2. A balanced (C_4,C_6,C_6)-trefoil decomposition of K_{65}.
Construct 130 (C_4,C_6,C_6)-trefoils as follows:
$B_i^{(1)} = \{(i,i+9,i+60,i+11),(i,i+1,i+18,i+51,i+26,i+5),(i,i+3,i+22,i+57,i+30,i+7)\}$
$B_i^{(2)} = \{(i,i+10,i+62,i+12),(i,i+2,i+20,i+54,i+28,i+6),(i,i+4,i+24,i+60,i+32,i+8)\}$
$(i = 1,2,...,65)$.
Then they comprise a balanced (C_4,C_6,C_6)-trefoil decomposition of K_{65}.

Example 3. A balanced (C_4,C_6,C_6)-trefoil decomposition of K_{97}.
Construct 291 (C_4,C_6,C_6)-trefoils as follows:
$B_i^{(1)} = \{(i,i+13,i+89,i+16),(i,i+1,i+26,i+75,i+38,i+7),(i,i+4,i+32,i+84,i+44,i+10)\}$
$B_i^{(2)} = \{(i,i+14,i+91,i+17),(i,i+2,i+28,i+78,i+40,i+8),(i,i+5,i+34,i+87,i+46,i+11)\}$
$B_i^{(3)} = \{(i,i+15,i+93,i+18),(i,i+3,i+30,i+81,i+42,i+9),(i,i+6,i+36,i+90,i+48,i+12)\}$
$(i = 1,2,...,97)$.
Then they comprise a balanced (C_4,C_6,C_6)-trefoil decomposition of K_{97}.

Example 4. A balanced (C_4,C_6,C_6)-trefoil decomposition of K_{129}.
Construct 516 (C_4,C_6,C_6)-trefoils as follows:
$B_i^{(1)} = \{(i,i+17,i+118,i+21),(i,i+1,i+34,i+99,i+50,i+9),(i,i+5,i+42,i+111,i+58,i+13)\}$

$B_i^{(2)} = \{(i, i{+}18, i{+}120, i{+}22), (i, i{+}2, i{+}36, i{+}102, i{+}52, i{+}10), (i, i{+}6, i{+}44, i{+}114, i{+}60, i{+}14)\}$
$B_i^{(3)} = \{(i, i{+}19, i{+}122, i{+}23), (i, i{+}3, i{+}38, i{+}105, i{+}54, i{+}11), (i, i{+}7, i{+}46, i{+}117, i{+}62, i{+}15)\}$
$B_i^{(4)} = \{(i, i{+}20, i{+}124, i{+}24), (i, i{+}4, i{+}40, i{+}108, i{+}56, i{+}12), (i, i{+}8, i{+}48, i{+}120, i{+}64, i{+}16)\}$
$(i = 1, 2, ..., 129)$.
Then they comprise a balanced (C_4, C_6, C_6)-trefoil decomposition of K_{129}.

Example 5. A balanced (C_4, C_6, C_6)-trefoil decomposition of K_{161}.
Construct 805 (C_4, C_6, C_6)-trefoils as follows:
$B_i^{(1)} = \{(i, i{+}21, i{+}147, i{+}26), (i, i{+}1, i{+}42, i{+}123, i{+}62, i{+}11), (i, i{+}6, i{+}52, i{+}138, i{+}72, i{+}16)\}$
$B_i^{(2)} = \{(i, i{+}22, i{+}149, i{+}27), (i, i{+}2, i{+}44, i{+}126, i{+}64, i{+}12), (i, i{+}7, i{+}54, i{+}141, i{+}74, i{+}17)\}$
$B_i^{(3)} = \{(i, i{+}23, i{+}151, i{+}28), (i, i{+}3, i{+}46, i{+}129, i{+}66, i{+}13), (i, i{+}8, i{+}56, i{+}144, i{+}76, i{+}18)\}$
$B_i^{(4)} = \{(i, i{+}24, i{+}153, i{+}29), (i, i{+}4, i{+}48, i{+}132, i{+}68, i{+}14), (i, i{+}9, i{+}58, i{+}147, i{+}78, i{+}19)\}$
$B_i^{(5)} = \{(i, i+25, i+155, i+30), (i, i+5, i+50, i+135, i+70, i+15), (i, i+10, i+60, i+150, i+80, i+20)\}$
$(i = 1, 2, ..., 161)$.
Then they comprise a balanced (C_4, C_6, C_6)-trefoil decomposition of K_{161}.

References

[1] C. J. Colbourn and A. Rosa, Triple Systems. Clarendom Press, Oxford (1999).
[2] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* 26 (1988), pp. 91–105.
[3] K. Ushio and H. Fujimoto, Balanced bowtie and trefoil decomposition of complete tripartite multigraphs, IEICE Trans. Fundamentals, Vol.E84-A, No.3, pp.839–844, March 2001.
[4] K. Ushio and H. Fujimoto, Balanced foil decomposition of complete graphs, IEICE Trans. Fundamentals, Vol.E84-A, No.12, pp.3132–3137, December 2001.
[5] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).