# 完全グラフの均衡的 $(C_4, C_4, C_6)$-Trefoil 分解アルゴリズム

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**アブストラクト**

グラフ理論において、グラフの分解問題は主要な研究テーマである。$C_4$、$C_6$ をそれぞれ 4 点、6 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル $C_4$、$C_4$、$C_6$ からなるグラフを $(C_4, C_4, C_6)$-trefoil という。本研究では、完全グラフ $K_n$ を $(C_4, C_4, C_6)$-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

**キーワード:** 均衡的 $(C_4, C_4, C_6)$-trefoil 分解; 完全グラフ; グラフ理論

# Balanced $(C_4, C_4, C_6)$-Trefoil Decomposition Algorithm of Complete Graphs

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**Abstract**

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced $(C_4, C_4, C_6)$-trefoil decomposition algorithm of the complete graph $K_n$.

**Keywords:** Balanced $(C_4, C_4, C_6)$-trefoil decomposition; Complete graph; Graph theory

## 1. Introduction

Let $K_n$ denote *the complete graph* of $n$ vertices. Let $C_4$ and $C_6$ be *the* 4-*cycle* and *the* 6-*cycle*, respectively. *The* $(C_4, C_4, C_6)$-*trefoil* is a graph of 3 edge-disjoint cycles $C_4$, $C_4$ and $C_6$ with a common vertex and the common vertex is called *the center of the* $(C_4, C_4, C_6)$-*trefoil*.
When $K_n$ is decomposed into edge-disjoint sum of $(C_4, C_4, C_6)$-trefoils, we say that $K_n$ *has a* $(C_4, C_4, C_6)$-*trefoil decomposition*. Moreover, when every vertex of $K_n$ appears in the same number of $(C_4, C_4, C_6)$-trefoils, we say that $K_n$ *has a balanced* $(C_4, C_4, C_6)$-*trefoil decomposition* and

this number is called *the replication number*.
It is a well-known result that $K_n$ has a $C_3$ decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that $K_n$ has a $(C_3, C_3)$-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_n$ is $n \equiv 1$ (mod 28).

## 2. Balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_n$

We use the following notation for a $(C_4, C_4, C_6)$-trefoil.

**Notation.** We denote a $(C_4, C_4, C_6)$-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_1 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7), (v_1, v_8, v_9, v_{10}, v_{11}, v_{12})\}$.

We have the following theorem.

**Theorem.** $K_n$ has a balanced $(C_4, C_4, C_6)$-trefoil decomposition if and only if $n \equiv 1$ (mod 28).

**Proof. (Necessity)** Suppose that $K_n$ has a balanced $(C_4, C_4, C_6)$-trefoil decomposition. Let $b$ be the number of $(C_4, C_4, C_6)$-trefoils and $r$ be the replication number. Then $b = n(n-1)/28$ and $r = 3(n-1)/7$. Among $r$ $(C_4, C_4, C_6)$-trefoils having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $(C_4, C_4, C_6)$-trefoils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/28$ and $r_2 = 11(n-1)/28$. Therefore, $n \equiv 1$ (mod 28) is necessary.
**(Sufficiency)** Put $n = 28t + 1$. Construct $tn$ $(C_4, C_4, C_6)$-trefoils as follows:
$B_i^{(1)} = \{(i, i+2t+1, i+22t+2, i+4t+1), (i, i+3t+1, i+24t+2, i+5t+1), (i, i+1, i+10t+2, i+24t+3, i+12t+2, t+1)\}$
$B_i^{(2)} = \{(i, i+2t+2, i+22t+4, i+4t+2), (i, i+3t+2, i+24t+4, i+5t+2), (i, i+2, i+10t+4, i+24t+6, i+12t+4, t+2)\}$
$B_i^{(3)} = \{(i, i+2t+3, i+22t+6, i+4t+3), (i, i+3t+3, i+24t+6, i+5t+3), (i, i+3, i+10t+6, i+24t+9, i+12t+6, t+3)\}$
...
$B_i^{(t)} = \{(i, i+3t, i+24t, i+5t), (i, i+4t, i+26t, i+6t), (i, i+t, i+12t, i+27t, i+14t, 2t)\}$
$(i = 1, 2, ..., n)$,
where the additions $i + x$ are taken modulo $n$ with residues $1, 2, ..., n$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_n$.

**Note.** We consider the vertex set $V$ of $K_n$ as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo $n$ with residues $1, 2, ..., n$.

**Example 1. A balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{29}$.**
Construct 29 $(C_4, C_4, C_6)$-trefoils as follows:
$B_1 = \{(1, 4, 25, 6), (1, 5, 27, 7), (1, 2, 13, 28, 15, 3)\}$
$B_2 = \{(2, 5, 26, 7), (2, 6, 28, 8), (2, 3, 14, 29, 16, 4)\}$
$B_3 = \{(3, 6, 27, 8), (3, 7, 29, 9), (3, 4, 15, 1, 17, 5)\}$
$B_4 = \{(4, 7, 28, 9), (4, 8, 1, 10), (4, 5, 16, 2, 18, 6)\}$
$B_5 = \{(5, 8, 29, 10), (5, 9, 2, 11), (5, 6, 17, 3, 19, 7)\}$
$B_6 = \{(6, 9, 1, 11), (6, 10, 3, 12), (6, 7, 18, 4, 20, 8)\}$
$B_7 = \{(7, 10, 2, 12), (7, 11, 4, 13), (7, 8, 19, 5, 21, 9)\}$

$B_8 = \{(8, 11, 3, 13), (8, 12, 5, 14), (8, 9, 20, 6, 22, 10)\}$
$B_9 = \{(9, 12, 4, 14), (9, 13, 6, 15), (9, 10, 21, 7, 23, 11)\}$
$B_{10} = \{(10, 13, 5, 15), (10, 14, 7, 16), (10, 11, 22, 8, 24, 12)\}$
$B_{11} = \{(11, 14, 6, 16), (11, 15, 8, 17), (11, 12, 23, 9, 25, 13)\}$
$B_{12} = \{(12, 15, 7, 17), (12, 16, 9, 18), (12, 13, 24, 10, 26, 14)\}$
$B_{13} = \{(13, 16, 8, 18), (13, 17, 10, 19), (13, 14, 25, 11, 27, 15)\}$
$B_{14} = \{(14, 17, 9, 19), (14, 18, 11, 20), (14, 15, 26, 12, 28, 16)\}$
$B_{15} = \{(15, 18, 10, 20), (15, 19, 12, 21), (15, 16, 27, 13, 29, 17)\}$
$B_{16} = \{(16, 19, 11, 21), (16, 20, 13, 22), (16, 17, 28, 14, 1, 18)\}$
$B_{17} = \{(17, 20, 12, 22), (17, 21, 14, 23), (17, 18, 29, 15, 2, 19)\}$
$B_{18} = \{(18, 21, 13, 23), (18, 22, 15, 24), (18, 19, 1, 16, 3, 20)\}$
$B_{19} = \{(19, 22, 14, 24), (19, 23, 16, 25), (19, 20, 2, 17, 4, 21)\}$
$B_{20} = \{(20, 23, 15, 25), (20, 24, 17, 26), (20, 21, 3, 18, 5, 22)\}$
$B_{21} = \{(21, 24, 16, 26), (21, 25, 18, 27), (21, 22, 4, 19, 6, 23)\}$
$B_{22} = \{(22, 25, 17, 27), (22, 26, 19, 28), (22, 23, 5, 20, 7, 24)\}$
$B_{23} = \{(23, 26, 18, 28), (23, 27, 20, 29), (23, 34, 6, 21, 8, 25)\}$
$B_{24} = \{(24, 27, 19, 29), (24, 28, 21, 1), (24, 25, 7, 22, 9, 26)\}$
$B_{25} = \{(25, 28, 20, 1), (25, 29, 22, 2), (25, 26, 8, 23, 10, 27)\}$
$B_{26} = \{(26, 29, 21, 2), (26, 1, 23, 3), (26, 27, 9, 24, 11, 28)\}$
$B_{27} = \{(27, 1, 22, 3), (27, 2, 24, 4), (27, 28, 10, 25, 12, 29)\}$
$B_{28} = \{(28, 2, 23, 4), (28, 3, 25, 5), (28, 29, 11, 26, 13, 1)\}$
$B_{29} = \{(29, 3, 24, 5), (29, 4, 26, 6), (29, 1, 12, 27, 14, 2)\}$.
This decomposition can be written as follows:
$B_i = \{(i, i+3, i+24, i+5), (i, i+4, i+26, i+6), (i, i+1, i+12, i+27, i+14, i+2)\}$ $(i = 1, 2, ..., 29)$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{29}$.

**Example 2. A balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{57}$.**
Construct 114 $(C_4, C_4, C_6)$-trefoils as follows:
$B_i^{(1)} = \{(i, i+5, i+46, i+9), (i, i+7, i+50, i+11), (i, i+1, i+22, i+51, i+26, i+3)\}$
$B_i^{(2)} = \{(i, i+6, i+48, i+10), (i, i+8, i+52, i+12), (i, i+2, i+24, i+54, i+28, i+4)\}$
$(i = 1, 2, ..., 57)$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{57}$.

**Example 3. A balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{85}$.**
Construct 255 $(C_4, C_4, C_6)$-trefoils as follows:
$B_i^{(1)} = \{(i, i+7, i+68, i+13), (i, i+10, i+74, i+16), (i, i+1, i+32, i+75, i+38, i+4)\}$
$B_i^{(2)} = \{(i, i+8, i+70, i+14), (i, i+11, i+76, i+17), (i, i+2, i+34, i+78, i+40, i+5)\}$
$B_i^{(3)} = \{(i, i+9, i+72, i+15), (i, i+12, i+78, i+18), (i, i+3, i+36, i+81, i+42, i+6)\}$
$(i = 1, 2, ..., 85)$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{85}$.

**Example 4. A balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{113}$.**
Construct 452 $(C_4, C_4, C_6)$-trefoils as follows:
$B_i^{(1)} = \{(i, i+9, i+90, i+17), (i, i+13, i+98, i+21), (i, i+1, i+42, i+99, i+50, i+5)\}$
$B_i^{(2)} = \{(i, i+10, i+92, i+18), (i, i+14, i+100, i+22), (i, i+2, i+44, i+102, i+52, i+6)\}$
$B_i^{(3)} = \{(i, i+11, i+94, i+19), (i, i+15, i+102, i+23), (i, i+3, i+46, i+105, i+54, i+7)\}$
$B_i^{(4)} = \{(i, i+12, i+96, i+20), (i, i+16, i+104, i+24), (i, i+4, i+48, i+108, i+56, i+8)\}$
$(i = 1, 2, ..., 113)$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{113}$.

**Example 5. A balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{141}$.**
Construct 705 $(C_4, C_4, C_6)$-trefoils as follows:
$B_i^{(1)} = \{(i, i+11, i+112, i+21), (i, i+16, i+122, i+26), (i, i+1, i+52, i+123, i+62, i+6)\}$
$B_i^{(2)} = \{(i, i+12, i+114, i+22), (i, i+17, i+124, i+27), (i, i+2, i+54, i+126, i+64, i+7)\}$
$B_i^{(3)} = \{(i, i+13, i+116, i+23), (i, i+18, i+126, i+28), (i, i+3, i+56, i+129, i+66, i+8)\}$
$B_i^{(4)} = \{(i, i+14, i+118, i+24), (i, i+19, i+128, i+29), (i, i+4, i+58, i+132, i+68, i+9)\}$
$B_i^{(5)} = \{(i, i+15, i+120, i+25), (i, i+20, i+130, i+30), (i, i+5, i+60, i+135, i+70, i+10)\}$
$(i = 1, 2, ..., 141)$.
Then they comprise a balanced $(C_4, C_4, C_6)$-trefoil decomposition of $K_{141}$.

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