完全グラフの均衡的 (C_4, C_8, C_8)-Trefoil 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_8 をそれぞれ 4 点、8 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_8、C_8 からなるグラフを (C_4, C_8, C_8)-trefoil という。本研究では、完全グラフ K_n を (C_4, C_8, C_8)-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 (C_4, C_8, C_8)-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_8, C_8)-Trefoil Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_8, C_8)-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_8, C_8)-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_8 be *the 4-cycle* and *the 8-cycle*, respectively. *The* (C_4, C_8, C_8)*-trefoil* is a graph of 3 edge-disjoint cycles C_4, C_8 and C_8 with a common vertex and the common vertex is called *the center of the* (C_4, C_8, C_8)*-trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_8, C_8)-trefoils, we say that K_n *has a* (C_4, C_8, C_8)*-trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_8, C_8)-trefoils, we say that K_n *has a balanced* (C_4, C_8, C_8)*-trefoil decomposition* and this

number is called *the replication number.*

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system.*
In this sense, our balanced (C_4, C_8, C_8)-trefoil decomposition of K_n is to be known as *a balanced (C_4, C_8, C_8)-trefoil system.*

2. Balanced (C_4, C_8, C_8)-trefoil decomposition of K_n

We use the following notation for a (C_4, C_8, C_8)-trefoil.

Notation. We denote a (C_4, C_8, C_8)-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1$, $v_1 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_1$, $v_1 - v_{12} - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}), (v_1, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_8, C_8)-trefoil decomposition if and only if $n \equiv 1$ (mod 40).

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_8, C_8)-trefoil decomposition. Let b be the number of (C_4, C_8, C_8)-trefoils and r be the replication number. Then $b = n(n-1)/40$ and $r = 18(n-1)/40$. Among r (C_4, C_8, C_8)-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_8, C_8)-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/40$ and $r_2 = 17(n-1)/40$. Therefore, $n \equiv 1$ (mod 40) is necessary.

(Sufficiency) Put $n = 40t + 1$. Construct tn (C_4, C_8, C_8)-trefoils as follows:
$B_i^{(1)} = \{(i, i+16t+1, i+37t+2, i+17t+1), (i, i+4t+1, i+18t+2, i+18t+3, i+28t+4, i+20t+3, i+8t+2, i+6t+1), (i, i+4t+2, i+18t+4, i+18t+6, i+28t+8, i+20t+6, i+8t+4, i+6t+2)\}$
$B_i^{(2)} = \{(i, i+16t+2, i+37t+4, i+17t+2), (i, i+4t+3, i+18t+6, i+18t+9, i+28t+12, i+20t+9, i+8t+6, i+6t+3), (i, i+4t+4, i+18t+8, i+18t+12, i+28t+16, i+20t+12, i+8t+8, i+6t+4)\}$
...
$B_i^{(t)} = \{(i, i+17t, i+39t, i+18t), (i, i+6t-1, i+22t-2, i+24t-3, i+36t-4, i+26t-3, i+12t-2, i+8t-1), (i, i+6t, i+22t, i+24t, i+36t, i+26t, i+12t, i+8t)\}$
$(i = 1, 2, ..., n)$,
where the additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a balanced (C_4, C_8, C_8)-trefoil decomposition of K_n.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Example 1. A balanced (C_4, C_8, C_8)-trefoil decomposition of K_{41}.
$B_1 = \{(1, 18, 40, 19), (1, 6, 21, 22, 33, 24, 11, 8), (1, 7, 23, 25, 37, 27, 13, 9)\}$
$B_2 = \{(2, 19, 41, 20), (2, 7, 22, 23, 34, 25, 12, 9), (2, 8, 24, 26, 38, 28, 14, 10)\}$
$B_3 = \{(3, 20, 1, 21), (3, 8, 23, 24, 35, 26, 13, 10), (3, 9, 25, 27, 39, 29, 15, 11)\}$
$B_4 = \{(4, 21, 2, 22), (4, 9, 24, 25, 36, 27, 14, 11), (4, 10, 26, 28, 40, 30, 16, 12)\}$
$B_5 = \{(5, 22, 3, 23), (5, 10, 25, 26, 37, 28, 15, 12), (5, 11, 27, 29, 41, 31, 17, 13)\}$
$B_6 = \{(6, 23, 4, 24), (6, 11, 26, 27, 38, 29, 16, 13), (6, 12, 28, 30, 1, 32, 18, 14)\}$

$B_7 = \{(7,24,5,25),(7,12,27,28,39,30,17,14),(7,13,29,31,2,33,19,15)\}$
$B_8 = \{(8,25,6,26),(8,13,28,29,40,31,18,15),(8,14,30,32,3,34,20,16)\}$
$B_9 = \{(9,26,7,27),(9,14,29,30,41,32,19,16),(9,15,31,33,4,35,21,17)\}$
$B_{10} = \{(10,27,8,28),(10,15,30,31,1,33,20,17),(10,16,32,34,5,36,22,18)\}$
$B_{11} = \{(11,28,9,29),(11,16,31,32,2,34,21,18),(11,17,33,35,6,37,23,19)\}$
$B_{12} = \{(12,29,10,30),(12,17,32,33,3,35,22,19),(12,18,34,36,7,38,24,20)\}$
$B_{13} = \{(13,30,11,31),(13,18,33,34,4,36,23,20),(13,19,35,37,8,39,25,21)\}$
$B_{14} = \{(14,31,12,32),(14,19,34,35,5,37,24,21),(14,20,36,38,9,40,26,22)\}$
$B_{15} = \{(15,32,13,33),(15,20,35,36,6,38,25,22),(15,21,37,39,10,41,27,23)\}$
$B_{16} = \{(16,33,14,34),(16,21,36,37,7,39,26,23),(16,22,38,40,11,1,28,24)\}$
$B_{17} = \{(17,34,15,35),(17,22,37,38,8,40,27,24),(17,23,39,41,12,2,29,25)\}$
$B_{18} = \{(18,35,16,36),(18,23,38,39,9,41,28,25),(18,24,40,1,13,3,30,26)\}$
$B_{19} = \{(19,36,17,37),(19,24,39,40,10,1,29,26),(19,25,41,2,14,4,31,27)\}$
$B_{20} = \{(20,37,18,38),(20,25,40,41,11,2,30,27),(20,26,1,3,15,5,32,28)\}$
$B_{21} = \{(21,38,19,39),(21,26,41,1,12,3,31,28),(21,27,2,4,16,6,33,29)\}$
$B_{22} = \{(22,39,20,40),(22,27,1,2,13,4,32,29),(22,28,3,5,17,7,34,30)\}$
$B_{23} = \{(23,40,21,41),(23,28,2,3,14,5,33,30),(23,29,4,6,18,8,35,31)\}$
$B_{24} = \{(24,41,22,1),(24,29,3,4,15,6,34,31),(24,30,5,7,19,9,36,32)\}$
$B_{25} = \{(25,1,23,2),(25,30,4,5,16,7,35,32),(25,31,6,8,20,10,37,33)\}$
$B_{26} = \{(26,2,24,3),(26,31,5,6,17,8,36,33),(26,32,7,9,21,11,38,34)\}$
$B_{27} = \{(27,3,25,4),(27,32,6,7,18,9,37,34),(27,33,8,10,22,12,39,35)\}$
$B_{28} = \{(28,4,26,5),(28,33,7,8,19,10,38,35),(28,34,9,11,23,13,40,36)\}$
$B_{29} = \{(29,5,27,6),(29,34,8,9,20,11,39,36),(29,35,10,12,24,14,41,37)\}$
$B_{30} = \{(30,6,28,7),(30,35,9,10,21,12,40,37),(30,36,11,13,25,15,1,38)\}$
$B_{31} = \{(31,7,29,8),(31,36,10,11,22,13,41,38),(31,37,12,14,26,16,2,39)\}$
$B_{32} = \{(32,8,30,9),(32,37,11,12,23,14,1,39),(32,38,13,15,27,17,3,40)\}$
$B_{33} = \{(33,9,31,10),(33,38,12,13,24,15,2,40),(33,39,14,16,28,18,4,41)\}$
$B_{34} = \{(34,10,32,11),(34,39,13,14,25,16,3,41),(34,40,15,17,29,19,5,1)\}$
$B_{35} = \{(35,11,33,12),(35,40,14,15,26,17,4,1),(35,41,16,18,30,20,6,2)\}$
$B_{36} = \{(36,12,34,13),(36,41,15,16,27,18,5,2),(36,1,17,19,31,21,7,3)\}$
$B_{37} = \{(37,13,35,14),(37,1,16,17,28,19,6,3),(37,2,18,20,32,22,8,4)\}$
$B_{38} = \{(38,14,36,15),(38,2,17,18,29,20,7,4),(38,3,19,21,33,23,9,5)\}$
$B_{39} = \{(39,15,37,16),(39,3,18,19,30,21,8,5),(39,4,20,22,34,24,10,6)\}$
$B_{40} = \{(40,16,38,17),(40,4,19,20,31,22,9,6),(40,5,21,23,35,25,11,7)\}$
$B_{41} = \{(41,17,39,18),(41,5,20,21,32,23,10,7),(41,6,22,24,36,26,12,8)\}$.

This decomposition can be written as follows:
$B_i = \{(i,i+17,i+39,i+18),(i,i+5,i+20,i+21,i+32,i+23,i+10,i+7),(i,i+6,i+22,i+24,i+36,i+26,i+12,i+8)\}$
$(i = 1,2,...,41)$.

Example 2. A balanced (C_4,C_8,C_8)-trefoil decomposition of K_{81}.
$B_i^{(1)} = \{(i,i+33,i+76,i+35),(i,i+9,i+38,i+39,i+60,i+43,i+18,i+13),(i,i+10,i+40,i+42,i+64,i+46,i+20,i+14)\}$
$B_i^{(2)} = \{(i,i+34,i+78,i+36),(i,i+11,i+42,i+45,i+68,i+49,i+22,i+15),(i,i+12,i+44,i+48,i+72,i+52,i+24,i+16)\}$
$(i = 1,2,...,81)$.

Example 3. A balanced (C_4,C_8,C_8)-trefoil decomposition of K_{121}.
$B_i^{(1)} = \{(i,i+49,i+113,i+52),(i,i+13,i+56,i+57,i+88,i+63,i+26,i+19),(i,i+14,i+$

$58, i+60, i+92, i+66, i+28, i+20)\}$
$B_i^{(2)} = \{(i, i+50, i+115, i+53), (i, i+15, i+60, i+63, i+96, i+69, i+30, i+21), (i, i+16, i+62, i+66, i+100, i+72, i+32, i+22)\}$
$B_i^{(3)} = \{(i, i+51, i+117, i+54), (i, i+17, i+64, i+69, i+104, i+75, i+34, i+23), (i, i+18, i+66, i+72, i+108, i+78, i+36, i+24)\}$
$(i = 1, 2, ..., 121)$.

Example 4. A balanced (C_4, C_8, C_8)-trefoil decomposition of K_{161}.
$B_i^{(1)} = \{(i, i+65, i+150, i+69), (i, i+17, i+74, i+75, i+116, i+83, i+34, i+25), (i, i+18, i+76, i+78, i+120, i+86, i+36, i+26)\}$
$B_i^{(2)} = \{(i, i+66, i+152, i+70), (i, i+19, i+78, i+81, i+124, i+89, i+38, i+27), (i, i+20, i+80, i+84, i+128, i+92, i+40, i+28)\}$
$B_i^{(3)} = \{(i, i+67, i+154, i+71), (i, i+21, i+82, i+87, i+132, i+95, i+42, i+29), (i, i+22, i+84, i+90, i+136, i+98, i+44, i+30)\}$
$B_i^{(4)} = \{(i, i+68, i+156, i+72), (i, i+23, i+86, i+93, i+140, i+101, i+46, i+31), (i, i+24, i+88, i+96, i+144, i+104, i+48, i+32)\}$
$(i = 1, 2, ..., 161)$.

Example 5. A balanced (C_4, C_8, C_8)-trefoil decomposition of K_{201}.
$B_i^{(1)} = \{(i, i+81, i+187, i+86), (i, i+21, i+92, i+93, i+144, i+103, i+42, i+31), (i, i+22, i+94, i+96, i+148, i+106, i+44, i+32)\}$
$B_i^{(2)} = \{(i, i+82, i+189, i+87), (i, i+23, i+96, i+99, i+152, i+109, i+46, i+33), (i, i+24, i+98, i+102, i+156, i+112, i+48, i+34)\}$
$B_i^{(3)} = \{(i, i+83, i+191, i+88), (i, i+25, i+100, i+105, i+160, i+115, i+50, i+35), (i, i+26, i+102, i+108, i+164, i+118, i+52, i+36)\}$
$B_i^{(4)} = \{(i, i+84, i+193, i+89), (i, i+27, i+104, i+111, i+168, i+121, i+54, i+37), (i, i+28, i+106, i+114, i+172, i+124, i+56, i+38)\}$
$B_i^{(5)} = \{(i, i+85, i+195, i+90), (i, i+29, i+108, i+117, i+176, i+127, i+58, i+39), (i, i+30, i+110, i+120, i+180, i+130, i+60, i+40)\}$
$(i = 1, 2, ..., 201)$.

References

[1] C. J. Colbourn and A. Rosa, Triple Systems. Clarendom Press, Oxford (1999).
[2] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* 26 (1988), pp. 91–105.
[3] K. Ushio and H. Fujimoto, Balanced bowtie and trefoil decomposition of complete tripartite multigraphs, *IEICE Trans. Fundamentals*, Vol.E84-A, No.3, pp.839–844, March 2001.
[4] K. Ushio and H. Fujimoto, Balanced foil decomposition of complete graphs, *IEICE Trans. Fundamentals*, Vol.E84-A, No.12, pp.3132–3137, December 2001.
[5] W. D. Wallis, Combinatorial Designs. Marcel Dekker, New York and Basel (1988).