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グラフに関する丸め問題：外平面グラフの場合

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概要　頂点に実数値の重みを持つグラフ G が与えられた時、これらの重みを整数値に丸めることを考える。このとき、全ての最短路の上での重み和が入力の実数重みの和の丸めになっているとき、この丸めを大域丸め（global rounding) と言う。任意の n 頂点グラフに対し、大域丸めは $n+1$ 個以下しかないと予想されている。本論文では外平面グラフに対してこの予想を示し、かつ、全ての大域丸めを列挙する多項式時間アルゴリズムを与える。

Rounding Problem on Graphs: Case of Outerplanar Graphs

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Abstract. Given a connected weighted graph $G = (V, E)$ and a given real-valued assignment $\mathbf{a}$ on V satisfying $0 \leq \mathbf{a}(v) \leq 1$, a global rounding α with respect to G is a binary assignment satisfying that $|\sum_{v \in P} \mathbf{a}(v) - \alpha(v)| < 1$ for every shortest path P in G. Asano et al [1] conjectured that there are at most $|V| + 1$ global roundings for G. We prove that the conjecture holds if G is an outerplanar graph.

1　Introduction

Given a real number a, an integer k is a *rounding* of a if the difference between a and k is strictly less than 1, or equivalently, if k is the floor $\lfloor a \rfloor$ or the ceiling $\lceil a \rceil$ of a. We extend this usual notion of rounding into that of *global rounding* on hypergraphs as follows.

Let $\mathcal{H} = (V, \mathcal{F})$, where $\mathcal{F} \subset 2^V$, be a hypergraph on a set V of n nodes. Given a real valued function (often called an *input assignment*) $\mathbf{a}$ on V, we say that an integer valued function α on V is a *global rounding* of $\mathbf{a}$ with respect to $\mathcal{H}$, if $w_F(\alpha)$ is a rounding of $w_F(\mathbf{a})$ for each $F \in \mathcal{F}$, where $w_F(f)$ denotes $\sum_{v \in F} f(v)$. We assume that the hypergraph contains all the singleton sets as its edges; thus, $\alpha(v)$ is a rounding of $\mathbf{a}(v)$ for each v, and we can restrict our attention to the case where the ranges of $\mathbf{a}$ and α are $[0, 1]$ and $\{0, 1\}$ respectively.

This notion of global roundings on hypergraphs is closely related to that of *linear* or *inhomogeneous discrepancy* of hypergraphs[9, 5]. Given $\mathbf{a}$ and $\mathbf{b} \in [0, 1]^V$, define the *discrepancy* $D_{\mathcal{H}}(\mathbf{a}, \mathbf{b})$ between them by $D_{\mathcal{H}}(\mathbf{a}, \mathbf{b}) = \max_{F \in \mathcal{F}} |w_F(\mathbf{a}) - w_F(\mathbf{b})|$. The supremum $\sup_{\mathbf{a} \in [0,1]^V} \min_{\alpha \in \{0,1\}^V} D_{\mathcal{H}}(\mathbf{a}, \alpha)$ is called the linear (or inhomogeneous) discrepancy of $\mathcal{H}$, and it is a quality measure of approximability of a real vector with an integral vector to satisfy a constraint given by a linear system corresponding to $\mathcal{H}$.

Thus, the set of global roundings of $\mathbf{a}$ is the set on integral points in the open unit ball around $\mathbf{a}$ by using the discrepancy D_H as the distance function. It is known that the open ball always contains an integral point for any “input” $\mathbf{a}$ if and only if the hypergraph is unimodular (see [5, 6]). The fact is utilized in digital halftoning applications [2, 3].

We give in this paper a class of hypergraphs for which all the global roundings of a given input can be efficiently enumerated. For the purpose, we first consider the number of global roundings, since enumeration is expensive if the output size is large. Given $\mathbf{a} \in [0, 1]^V$, we are interested in the number $\nu(\mathcal{H}, \mathbf{a})$ of all global roundings of given $\mathbf{a}$ on $\mathcal{H}$ and its maximum value $\nu(\mathcal{H}) = \max_{\mathbf{a} \in [0,1]^V} \nu(\mathcal{H}, \mathbf{a})$ over all possible inputs $\mathbf{a}$. In other words, $\nu(\mathcal{H})$ is the maximum number of integral points in a unit ball with respect to $D_{\mathcal{H}}$.

This direction of research is initiated by Sadakane *et al.*[10] where the authors discovered a somewhat surprising fact that $\nu(\mathcal{I}_n) \le n+1$ where $\mathcal{I}_n$ is a hypergraph on $V = \{1,2,..,n\}$ with edge set $\{[i,j]; 1 \le i \le j \le n\}$ consisting of all subintervals of V; moreover, they give an efficient algorithm to enumerate all the global roundings of a given input on $\mathcal{I}_n$. On the other hand, $\nu(\mathcal{H}) \ge n+1$ for any hypergraph $\mathcal{H}$: if we let $\mathbf{a}(v) = \epsilon$ for every v, where $\epsilon < 1/n$, then any binary assignment on V that assigns 1 to at most one vertex is a global rounding of $\mathcal{H}$, and hence $\nu(\mathcal{H}) \ge n+1$. Given this discovery, it is natural to ask for which class of hypergraphs this property $\nu(\mathcal{H}) = n+1$ holds.

Given a connected G in which edges are possibly weighted by a positive value, we define a *shortest-path hypergraph* $\mathcal{H}_G$ generated by G as follows: a set F of vertices of G is an edge of $\mathcal{H}_G$ if and only if F is the set of vertices of some shortest path in G with respect to the given edge weights. We permit more than one shortest path between a pair of nodes if they have the same length. $\mathcal{H}_G$ is non-unimodular if G is not a path. Asano *et al.* [1] proposed the following conjecture:

Conjecture 1.1 ([1]) $\nu(\mathcal{H}_G) = n+1$ *for any connected graph G with n nodes.*

Sadakane *et al.*'s result implies that the conjecture holds for a path, and Asano *et al.* [1] proved it for special graphs including trees and cycles. Indeed, if we consider the hypergraph corresponding to the set of all (simple) paths in G, instead of shortest paths, it is easy to see that it has at most $n+1$ global roundings.

A set A of binary functions on V is called $\mathcal{H}$-*compatible* if, for each pair α and β in A, $|w_F(\alpha) - w_F(\beta)| \le 1$ holds for every hyperedge F of $\mathcal{H}$. Let $\mu(\mathcal{H})$ be the maximum cardinality of an $\mathcal{H}$-compatible set.

Intuitively, a compatible set forms a cluster with the unit diameter, while global roundings are in the interior of the unit ball around $\mathbf{a}$. Since the distance between two integral points in the unit ball must be at most 1 if we consider $D_{\mathcal{H}}$ as the distance function, $\mu(\mathcal{H}_G) \ge \nu(\mathcal{H}_G)$.

In particular, $\{(0000), (1000), (0100), (0010), (0001)\}$ is a compatible set for any hypergraph on four vertices, and it is the neighborhood of origin with respect to the Hamming distance. Indeed, if all doubletons are hyperedges (e.g., $\mathcal{H} = \mathcal{H}_{K_n}$ where K_n is the unweighted complete graph), a compatible set must be a subset of such a neighborhood of a binary vector. However, an $\mathcal{I}_6$ compatible set $\{(101010), (010101), (110101), (011010), (101101), (010110), (101011)\}$ has a different structure. In this paper, we prove the following:

Theorem 1.2 $\mu(\mathcal{H}_G) = n+1$ *holds for the shortest-path hypergraph $\mathcal{H}_G$, if G is an outerplanar graph.*

Thus, we have that Conjecture1.1 holds for an outerplanar graph. We then investigate the structure of global roundings, and give an algorithm to enumerate all the global roundings of an outerplanar graph G for an input assignment $\mathbf{a}$ in polynomial time.

The algorithm has a potential application to digital halftoning. One method [11] to solve the digital halftoning problem is to fill the grid by a space filling curve such as Hilbert curve, and consider a global rounding along the path, where the curve is regarded as a path (with the pixels as the vertices) and thus the hypergraph $\mathcal{I}_n$ or $\mathcal{I}_{k,n}$ is considered. However, it often happens that a pair of adjacent pixels in the grid is very far from each other on the curve. In order to resolve it, we can add some short-cut edges to make the path into an outerplanar graph, and compute its global rounding.

2 Preliminaries

We start with the following easy observations:

Lemma 2.1 *For hypergraphs $\mathcal{H} = (V, \mathcal{F})$ and $\mathcal{H}' = (V, \mathcal{F}')$ such that $\mathcal{F} \subset \mathcal{F}'$, $\mu(\mathcal{H}) \ge \mu(\mathcal{H}')$.*

For a binary assignment α on V and a subset X of V, $\alpha|_X$ denotes the restriction of α on X. Let $V = X \cup Y$ be a partition of V into nonintersecting subsets X and Y of vertices. For binary assignments α on X and β on Y, $\alpha \oplus \beta$ is a binary assignment on V obtained by concatenating α and β: That is, $\alpha \oplus \beta(v) = \alpha(v)$ if $v \in X$, otherwise it is $\beta(v)$.

The following lemma is a key lemma:

Lemma 2.2 *Let $G = (V, E)$ be a connected graph, and let $V = X \cup Y$ be a partition of V. Let α_1 and α_2 be different assignments on X and let β_1 and β_2 be different assignments on Y. Then, the set $\{ \alpha_1 \oplus \beta_1, \alpha_1 \oplus \beta_2, \alpha_2 \oplus \beta_1, \alpha_2 \oplus \beta_2 \}$ cannot be $\mathcal{H}_G$-compatible.*

Proof: Consider $x \in X$ satisfying $\alpha_1(x) \neq \alpha_2(x)$ and $y \in Y$ satisfying $\beta_1(y) \neq \beta_2(y)$. We choose such x and y with the minimum shortest path length. Thus, on each internal node of a shortest path $\mathbf{P}$ from x to y, all four assignments take the same value. Without loss of generality, we assume $\alpha_1(x) = \beta_1(y) = 0$ and $\alpha_2(x) = \beta_2(y) = 1$. Then, $w_{\mathbf{P}}(\alpha_2 \oplus \beta_2) = w_{\mathbf{P}}(\alpha_1 \oplus \beta_1) + 2$, and hence violate the compatibility. □

2.1 Bridging Two Graphs

An edge e in a connected graph G is called a *bridge* if the graph is separated into two connected components by removing e from G.

Proposition 2.3 *Suppose that a graph G has a bridge e separating $G - \{e\}$ into two connected components G_1 and G_2. Then, $\mu(G) \leq \mu(G_1) + \mu(G_2) - 1$.*

Proof: Consider an $\mathcal{H}_G$-compatible set A. Let $A_i = \{\alpha|_{V_i} : \alpha \in A\}$, where V_i are vertex sets of G_i for $i = 1, 2$. It is clear that A_i is a $\mathcal{H}_{G_i}$-compatible set for each $i = 1, 2$. We construct a bipartite graph M whose vertex set corresponds to A_1 and A_2, where an edge is given between two roundings $\beta \in A_1$ and $\gamma \in A_2$ if and only if $\beta \oplus \gamma \in A$. We claim that the M is a forest. From this claim, it is straightforward to see that $\mu(G) \leq \mu(G_1) + \mu(G_2) - 1$.

In order to prove the claim, consider the endpoint v_1 of the bridge e in G_1. We construct a shortest-path tree T from v_1 in G_1, and give the breadth-first ordering $v_1, v_2, \ldots, v_t$ of vertices of G_1 along this tree. Let $U^j = \{v_1, v_2, \ldots, v_j\}$, and let A_1^j be $A_1|_{U^j}$. We consider a bipartite graph M_j whose vertex set corresponds to A_1^j and A_2, where an edge is given between two roundings $\beta^j \in A_1^j$ and $\gamma \in A_2$ if and only if there exists $\beta \in A_1$ such that $\beta^j = \beta|_{U^j}$ and $\beta \oplus \gamma \in A$. It suffices to show that M_j is a forest for every i, since $M_t = M$. The graph M_0 is defined to be a star graph connecting all the nodes corresponding to assignments in A_2 to a node (representing the empty assignment). We can construct M_j from M_{j-1} by splitting each node $x(\alpha)$ corresponding to an assignment in $\alpha \in A_1^j$ into two nodes $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$, one assigns 0 and the other assigns 1 to v_j. The neighbors of x is connected to $x(\alpha \oplus 0)$ and/or $x(\alpha \oplus 1)$ by definition. We can prove that at most one neighbor of x can be connected to both of $x(\alpha \oplus 0)$ and $x(\alpha \oplus 1)$. This can be proved analogously to the proof of Lemma 2.2, since for each $u \in V_2$, at least one shortest path between u and v_j is a path in $T \cup G_2$. If M_{j-1} is a forest, we can see that such a splitting operation keeps the graph to be a forest, and accordingly, M_j is a forest. Thus, we can prove the claim by induction. □

A graph G is *series connection* of two graphs G_1 and G_2 if $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$ (implying that they share no edge), where v is called the *separator*.

Proposition 2.4 *Suppose that a graph G is a series connection of two connected graphs G_1 and G_2. Then, $\mu(G) \leq \mu(G_1) + \mu(G_2) - 2$.*

Proof: We consider a $\mathcal{H}_G$-compatible set A. For each of $i = 1, 2$, every shortest path within G_i is a shortest path in G, and hence the restriction A_i of A to G_i is an $\mathcal{H}_{G_i}$-compatible set. Let x be the vertex shared by G_1 and G_2. Let A^0 and A^1 be the subset of A where the values at x are 0 and 1, respectively. We apply the argument of the proof of Proposition 2.3 to each of A^0 and A^1. Thus, we have $|A^j| \leq |A_1^j| + |A_2^j| - 1$ for each of $j = 0, 1$. Thus, $|A| \leq |A_1| + |A_2| - 2$. □

2.2 The Structure of a Compatible Set for a Cycle

Let C_n be a directed cycle on n vertices $V = \{1, 2, \ldots, n\}$ with edge set $\{e_1, \ldots, e_n\}$ where $e_i = (i, i+1)$, $1 \leq i \leq n$. The arithmetic on vertices are cyclic, i.e., $n + 1 = 1$.

For an assignment α, we define $w(\alpha) = w_V(\alpha) = \sum_{v \in C_n} \alpha(v)$ to be the weight of α over all vertices in C_n.

Lemma 2.5 *Let α and β be $\mathcal{H}_{C_n}$-compatible assignments on C_n. Then, $w(\alpha)$ and $w(\beta)$ differ by at most 1.*

Lemma 2.6 *Suppose $w(\alpha) = w(\beta)$ for assignments α and β. Then, if α and β are $\mathcal{H}_{C_n}$-compatible they are compatible on* every *path of C_n.*

The following result is given by Asano et al. [1].

Theorem 2.7 $\mu(\mathcal{H}_{C_n}) = n + 1$.

We sharpen the result slightly. Let A be an $\mathcal{H}_{C_n}$-compatible set. Let w be the minimum of $w_V(\alpha)$ for $\alpha \in A$, where V is the vertex set of C_n. Thus, because of Lemma 2.5, either $w(\alpha) = w$ or $w(\alpha) = w + 1$ for each $\alpha \in A$. Let $A_0 = \{\alpha \in A | w(\alpha) = w\}$ and $A_1 = \{\alpha \in A | w(\alpha) = w + 1\}$.

An ordered pair of edges (e_i, e_j) of C_n is called a *binding* pair if the path $\mathbf{P}$ between the end vertex v_{i+1} of e_i and the starting vertex v_j of e_j has the properties that (1) $w_{\mathbf{P}}(\alpha)$ has a same value for all $\alpha \in A_0$ and (2) $w_{\mathbf{P}}(\alpha)$ has a same value for all $\alpha \in A_1$. We can easily see that (e_j, e_i) is binding if (e_i, e_j) is binding, and (e_i, e_k) is binding if both (e_i, e_j) and (e_j, e_k) are binding ;thus, the set of binding pairs gives an equivalence relation on the edge set E of C_n. Let $r(A)$ be the number of equivalence classes of the above relation in E.

Lemma 2.8 $|A| \leq r(A) + 1$.

Proof: This lemma is given by modifying the argument of Asano *et al.* [1]. We omit details. □

We investigate basic structure of an $\mathcal{H}_{C_n}$- compatible set. Let $V_k = \{v_1, v_2, \ldots, v_k\}$, and let $A(V_k)$ be the set of prefixes of A on V_k (i.e., restrictions of roundings to V_k). Similarly, we define $A_0(V_k)$ and $A_1(V_k)$ to be the set of prefixes of A_0 and A_1 on V_k. We set $V_0 = \emptyset$, and $A(V_0) = \{\emptyset\}$; thus, $|A(V_0)| = 1$. Note that a prefix in $A(V_k)$ need not be a global rounding of the spanning subgraph G_k of V_k in the cycle C_n, since the shortest path in G_k between a pair of vertices may be different from that in G between the same pair. Also, a global rounding of G_k is not always in $A(V_k)$.

A prefix $\alpha \in A(V_k)$ is called *double* if $\alpha \in A_0(V_k) \cap A_1(V_k)$. It is called *large* and *small* if $\alpha \in A_1(V_k) \setminus A_0(V_k)$ and $\alpha \in A_0(V_k) \setminus A_1(V_k)$, respectively.

We form a tree T of depth n each of whose node $v(\alpha)$ correspond to a prefix α of a global rounding: Precisely speaking, its root corresponds to the unique element $\emptyset$ in $A(V_0)$, and a depth k node corresponds to an element in $A(V_k)$. A node $v(\alpha)$ corresponding to $\alpha \in A(V_k)$ is a son of $v(\beta)$ ($\beta \in A(V_{k-1})$) if β is the prefix of α of length $k-1$. Clearly, T is a binary tree.

If $v(\alpha)$ is a branching node in T, we call α a branching prefix; In other words, α is a branching prefix if and only if both $\alpha \oplus 0$ and $\alpha \oplus 1$ are prefixes of global roundings. If one branch is large and the other is small, we say that the branching node (and prefix) *split*. If one of the branches is double, we say the branching prefix *multiple*. Other branching prefixes are called *normal*.

By definition, T has $|A(V_n)| \leq \mu(\mathcal{H}_{C_n}) = n + 1$ leaves, and hence it has at most n branching nodes.

Thus, there are at most n branching prefixes for the $\mathcal{H}_{C_n}$ compatible set A.

3 Outerplanar graph

A graph G is an outerplanar graph if and only if it has a planar embedding where all of its vertices lie on the boundary of its outer face. Since series connection has been already considered, we can assume that G is 2-connected. Thus, every edge is either on the cycle C bounding the outer face or a chord of the cycle.

We can assume that every edge e is the shortest path between its endpoints in G; otherwise, we can simply remove it from our consideration. Furthermore, we can assume that e is the unique shortest path between its endpoints. Indeed, if there is another shortest path in G, adding e makes the condition of the global rounding more strict, and hence does not increase the number of global roundings.

Suppose we are given an outerplanar graph G and consider an $\mathcal{H}_G$-compatible set Γ. A face cycle X of G consisting of a part of C and a chord edge e is called an *ear*. Let Y be the graph removing all vertices and edges of X from G except $e = (x, y)$ and its endpoint. Thus, $V(X) \cap V(Y) = \{x, y\}$. Clearly, Y is an outerplanar graph.

Let $n = |V(X)|$. It suffices to prove that $\mu(G) \leq \mu(Y) + n - 2$, since by induction we can show that $\mu(G) \leq |V(G)| + 1$ from that.

For improving readability, we first give a weaker result that $\mu(G) \leq \mu(Y) + 2(n-2)$, from which we can obtain $\mu(G) \leq 2|V(G)| + 1$.

Lemma 3.1 *Given $\gamma \in \Gamma$, consider its restricted assignments γ_X and γ_Y to X and Y, respectively. Then, $A = \{\gamma_X | \gamma \in \Gamma\}$ and $B = \{\gamma_Y | \gamma \in \Gamma\}$ are $\mathcal{H}_X$-compatible and $\mathcal{H}_Y$-compatible sets, respectively.*

Proof: For any two vertices u and v in Y, the shortest path $\mathbf{p}$ between u and v in G must be in Y, since otherwise $\mathbf{p}$ contains a path (which is not e) between x and y in X, and we can reduce the length by replacing it with e. Thus, B is a compatible set. Similarly, we can prove that A is a compatible set. □

If α and β are binary assignments on X and Y respectively such that α and β have the same value at each of x and y, they define a binary assignment on G, denoted by $\alpha \odot \beta$. The previous lemma implies that an element in Γ is always written as $\alpha \odot \beta$ for $\alpha \in A$ and $\beta \in B$.

We consider prefixes of elements of A if we set $V_i = \{v_1, v_2, \ldots, v_i\}$, where $v_1 = x$ and $v_2 = y$. We often call them A-prefixes. We also consider $\alpha \odot \beta$ for an A-prefix α a member β of B.

Let $\Gamma_0 = \{\alpha \odot \beta \in \Gamma | \alpha \in A_0\}$ and $\Gamma_1 = \{\alpha \odot \beta \in \Gamma | \alpha \in A_1\}$, where A_0 and A_1 are the sets defined in the previous section (considering X as a cycle). An assignment $\alpha \odot \beta$ on $V_i \cup V(Y)$ is called a Γ-prefix if it is a restriction of a global rounding of G on $V_i \cup V(Y)$. G_i is the induced subgraph of G by $V_i \cup V(Y)$. A Γ-prefix is called *double*, *large*, or *small* analogously to an A-prefix.

A Γ-prefix is called a *branching* Γ-prefix if both $(\alpha \oplus 0) \odot \beta$ and $(\alpha \oplus 1) \odot \beta$ are Γ-prefixes. Analogously to A-prefixes, we define *split*, *multiple*, and *normal* branching Γ-prefixes.

Lemma 3.2 *Given a branching A-prefix α of length $k \geq 2$, there is at most one $\beta \in B$ such that $\alpha \odot \beta$ is a normal (or multiple) branching Γ-prefix.*

Proof: It suffices to consider normal branching Γ-prefixes, since multiple branching Γ-prefixes are easier to handle. Suppose that both β and β' give normal branching Γ-prefixes combined with α, and let q be one of nearest nodes from v_{k+1} in Y such that $\beta(q) \neq \beta(q')$. Let $\delta_1 = (\alpha \oplus 0) \odot \beta$, $\delta_2 = (\alpha \oplus 1) \odot \beta$, $\delta_3 = (\alpha \oplus 0) \odot \beta'$, and $\delta_4 = (\alpha \oplus 1) \odot \beta'$. Let $\gamma_i \in \Gamma$ has δ_i as its prefix ($i = 1, 2, 3, 4$). Without loss of generality, we assume that δ_1 and δ_2 are small. If δ_3 and δ_4 are large, comparing δ_2 and δ_3, the path $v_{k+2}, v_{k+3}, \ldots, v_n$ cannot be a shortest path. Thus, the shortest path between q and v_{k+1} must be in G_i, and we derive contradiction from the argument given in the proof of Lemma 2.2.

We thus can assume that δ_3 and δ_4 are small. By symmetry, we can assume that $\beta(q) = 0$ and $\beta(q') = 1$. If the shortest path $\mathbf{P}$ between v_{k+1} and q contains $v_2, \ldots, v_{k+1}$, we can see that $\gamma_4(\mathbf{P}) - \gamma_1(\mathbf{P}) = 2$ to have a contradiction. For the other case, we consider the shortest path $\mathbf{P'}$ from v_{k+2} to q, and can see that $\gamma_3(\mathbf{P'}) - \gamma_2(\mathbf{P'}) = 2$. □

Lemma 3.3 *Given a branching A-prefix α of length $k \geq 2$, there is at most one β such that $\alpha \odot \beta$ is a split (or multiple) branching Γ-prefix.*

Proof: Suppose that there are two split branching Γ-prefixes $\alpha \odot \beta$ and $\alpha \odot \beta'$. Consider the situation on v_{k+1}. Let q be one of nearest nodes in Y from v_{k+1} on which β and β' take different values from each other (say, $\beta(q) = 0$ and $\beta'(q) = 1$) Let $\mathbf{P}_1 = v_2, v_3, \ldots, v_k$ and $\mathbf{P}_2 = v_{k+2}, v_{k+2}, \ldots, v_n, v_1$. We define δ_1, δ_2, δ_3, and δ_4, and γ_1, γ_2, γ_3, and γ_4 as defined in the previous lemma.

If δ_1 is large and δ_2 is small, we can see that the shortest path between v_1 and v_{k+2} must be $v_1, v_2, \ldots v_{k+2}$, since the weights of γ_1 and γ_2 differs by 2 on the other path in the cycle. Thus, the shortest path $\mathbf{P}$ between v_{k+1} and q must contain $\mathbf{P}_1$, and the weights of $\gamma_1(\mathbf{P})$ and $\gamma_4(\mathbf{P})$ on $\mathbf{P}$ differ by 2 from each other (they only differ from each other on the both ends of $\mathbf{P}$).

Thus, we assume that δ_1 and δ_3 are small and δ_2 and δ_4 are large. Then, by definition, γ_i $(i = 1, 2, 3, 4)$ are exactly same to each other on $\mathbf{P}_1$. Thus, it can be easily seen that γ_i takes the same sum on the path $\mathbf{P}_2$. $\mathbf{P}$ must contain either $\mathbf{P}_1$ or $\mathbf{P}_2$, and it is routine to see that $\gamma_4(\mathbf{P}) - \gamma_1(\mathbf{P}) = 2$ to have contradiction. □

Corollary 3.4 $\mu(G) \leq \mu(Y) + 2(n-2)$

Proof: For each branching A-prefix, we have shown that there are at most two (one normal and one split) branching Γ-prefixes. Consider the tree T giving the prefix tree of A. We only need to consider branching nodes below the second level.

T has $|A| - 1$ branching nodes. If both the root and one of the nodes in depth 1 are branching nodes, starting from v_3 (recall that $x = v_1$ and $y = v_2$ are shared by Y), there are at most $n-2$ branching A-prefixes below the second level.

If there is only one branching node in levels 0 and 1 in T, we can see that one of (e_0, e_1), (e_1, e_2) and (e_0, e_2) is a binding pair, where e_0 is the edge between v_n and v_1; thus, $r(A) \leq n-1$. Therefore, we have $|A| \leq n$ from Lemma 2.8. If there is no branching node in levels 0 and 1, both (e_0, e_1) and (e_1, e_2) are binding pairs, and hence we have $r(A) \leq n-2$ and $|A| \leq n-1$. Thus for each case, there are at most $n-2$ branching A-prefixes below the second level.

Thus we have at most $2(n-2)$ branching Γ-prefixes.

□

Now, consider the situation that α is an A-prefix of length k and there are $\beta \neq \beta'$ such that $\alpha \odot \beta$ is a normal Γ-branching and simultaneously that $\alpha \odot \beta'$ is a split Γ-branching. Without loss of generality, we can assume that both $(\alpha \oplus 0) \odot \beta$, $(\alpha \oplus 1) \odot \beta$ are small. We can also assume that $(\alpha \oplus 0) \odot \beta'$ is small and $(\alpha \oplus 1) \odot \beta'$ is large, since it is easy to show that the other case cannot happen. Let γ and γ' are members of Γ obtained by extending $(\alpha \oplus 1) \odot \beta$ and $(\alpha \oplus 1) \odot \beta'$. Let K be the largest index such that $\gamma(v_K) = \gamma'(v_K)$. Since γ is small and γ' is large, $K \neq n$. Let $\tilde{\alpha}$ be the corresponding A-prefix of length K. Then, $\tilde{\alpha}$ gives a split A-branching in T.

Lemma 3.5 *In the above situation, there is no β'' such that $\tilde{\alpha} \odot \beta''$ is a split branching Γ-prefix.*

Proof: If $(\tilde{\alpha} \oplus 0) \odot \beta''$ is large and $(\tilde{\alpha} \oplus 1) \odot \beta''$ is small, we can easily derive contradiction. Thus, we assume that $(\tilde{\alpha} \oplus 0) \odot \beta''$ is small and $(\tilde{\alpha} \oplus 1) \odot \beta''$ is large. Let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are corresponding elements in Γ.

If $\beta'' = \beta'$, $(\alpha \oplus 1) \odot \beta'$ is a double prefix. Thus, $\alpha \oplus \beta'$ is a multiple branching Γ- prefix, contradicting the assumption.

Therefore, we assume that $\beta'' \neq \beta'$. Let q be one of the nearest vertex from v_{K+1} satisfying that $\beta''(q) \neq \beta'(q)$. If the shortest path $\mathbf{P}$ between v_{K+1} and q goes through v_K, we can easily have contradiction. Thus, we assume that $\mathbf{P}$ goes through v_{K+2}. If $\beta''(q) = 0$ and $\beta'(q) = 1$, $\gamma'(\mathbf{P}) = \tilde{\gamma}_0(\mathbf{P}) + 2$. Thus, $\beta''(q) = 1$ and $\beta'(q) = 0$. If $\beta(q) = 0$, we can derive contradiction since $\gamma(\mathbf{P}) = \tilde{\gamma}_1(\mathbf{P}) - 2$. Thus, $\beta(q) = 1$.

Now, consider the shortest path $\mathbf{Q}$ from v_{k+1} to q. If this path goes through v_{k+2}, we can easily have contradiction by comparing $\tilde{\gamma}_1$ and the small Γ-prefix $(\alpha \oplus 0) \odot \beta'$ on the path. Thus, we assume that $\mathbf{Q}$ goes through v_k. Consider the nearest node r to v_{k+1} on $\mathbf{Q}$ satisfying $\beta(r) \neq \beta'(r)$. Then, we have contradiction to the fact that both $\alpha \odot \beta$ and $\alpha \odot \beta'$ are branching Γ-prefixes, since they are same on $\mathbf{Q}$ except both end vertices, and we have both $0, 0$ and $1, 1$ for the combination of the assignment on the end vertices.

□

Theorem 3.6 $\mu(G) \leq \mu(Y) + n - 2$.

Proof: An A-prefix α is extended to a split branching Γ-prefix $\alpha \odot \beta$ only if α gives a split or multiple branching node in the prefix tree T of A. On the other hand, α is extended to a normal/multiple branching Γ-prefix only if α gives a non-split (i.e. normal or multiple) branching node in T.

By definition, a multiple branching node in T must have a split branching node as its descendent. Consider any path $\mathbf{P}$ from a leaf to the root in T. The previous lemma means that among all α corresponding to the nodes of the path $\mathbf{P}$ at most one α corresponds to a split branching Γ-prefix. Thus, the number of split branching Γ-prefixes is bounded by the number of split branching nodes of T.

On the other hand, the number of normal or multiple branching Γ-prefixes is bounded by the number of non-split branching nodes. Thus, the total number of branching Γ-prefixes is bounded by the number of branching nodes of T. Thus, we obtain the theorem. □

Thus, we conclude that $\mu(G) \leq |V(G)| + 1$ if G is an outerplanar graph.

3.1 Enumeration Algorithm

Since the number of global roundings of an outerplanar graph G is bounded by $n+1$, we have hope to enumerate all of them in polynomial time. Indeed, the proof in the previous section leads us to such an algorithm.

Theorem 3.7 *The set Γ of all global roundings of an input assignment $\mathbf{a}$ for an outerplanar graph G can be computed in $O(n^3)$ time.*

Proof:

Let $|X| = n_0$ and $|Y| = n_1 = n - n_0 + 2$. Given a Γ-prefix $\alpha \odot \beta$ on $V_i \cup V(Y)$, we want to check its extensions $(\alpha \oplus 0) \odot \beta$ and $(\alpha \oplus 1) \odot \beta$ whether they are extendable to members of Γ or not.

Unfortunately, it is expensive to check the extendibility exactly, since there are exponential number of possible extensions. Instead, we check whether they satisfy the global rounding conditions for the shortest paths between pairs of nodes in $V_i \cup V(Y)$ for each case where it is small or large (i.e., the node sum on X is w or $w+1$). Note that the shortest paths may go through vertices in $V \setminus (V_i \cup V(Y))$ A prefix is called a *weak* Γ*-prefix* if it satisfies this check. From our argument in the previous section, the number of weak Γ prefixes on $V_i \cup V(Y)$ is at most $|Y| + 2(n_0 - 2)$ for each i, and a weak Γ-prefix of $V_{n-1} \cup V(Y)$ is a global rounding of G by definition.

The check is done as follows. We first compute the shortest path tree T_v from $v = v_{i+1}$ in G, and then check for each extension of weak Γ-prefix using paths in the shortest path tree. The sum of entries on a path of T_v can be queried in $O(1)$ time after $O(n)$ time preprocessing. Thus, the set of global roundings of G can be computed in $O(n^2 n_0)$ time from that of Y. Therefore, it can be computed in $O(n^3)$ time (without giving the global roundings of Y). $\square$

4 Concluding remarks

As we mentioned in the introduction, our enumeration algorithm for the outerplanar graph has a potential application to digital halftoning. We want to implement our algorithm to see whether the method is effective or not; however, we have the following two drawbacks: (1) it may happen that no global rounding exists (2) the high time complexity prevent us to execute the algorithm on a digital image (for example if $n = 1024 \times 1024$). (1) can be avoided by restricting the length of shortest paths and make a graph giving the global roundings following the idea for generating global roundings of $\mathcal{I}_{k,n}$ given in [10, 11]. (2) is serious, and it will be nice if we can reduce the time complexity.

For a general graph, we do not even know whether $\nu(\mathcal{H}_G)$ is polynomially bounded by the number of vertices. It is plausible that the number of roundings can become large if the entries have some middle values (around 0.5). However, for a special input $\mathbf{a}$ consisting of entries with a same value $0.5+\epsilon$, we can show that the number of global roundings of $\mathbf{a}$ is bounded by $\max\{|V|, |E|\} + 1$ if each edge of $G = (V, E)$ has a unit length [8].

Another interesting question is how small hypergraph attains $\mu(\mathcal{H}) = n + 1$. We only know a naive bound that $\mathcal{H}$ must have $\Omega(\frac{n}{\log n})$ hyperedges, although we suspect that $n(n-1)/2$ is the true answer.

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