On the Orthogonal Drawing of Series-Parallel Graphs

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Abstract. We show in this paper that every series-parallel graph with maximum degree at most 4 has a 1-bend 2-D orthogonal drawing. We also show that every series-parallel graph with maximum degree at most 6 has a 2-bend 3-D orthogonal drawing.

1 Introduction

We consider the problem of generating orthogonal drawings of series-parallel graphs in the plane and space. The problem has obvious applications in the design of 2-D and 3-D VLSI circuits and optoelectronic integrated systems.

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of vertices of G. G is called a k-graph if $\Delta(G) \leq k$.

It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph G is called a 3-D drawing of G. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a 2-D drawing of G.

A 2-D orthogonal drawing of a planar graph G is a 2-D drawing of G such that each edge is drawn by a sequence of contiguous horizontal and vertical line segments. A 3-D orthogonal drawing of a graph G is a 3-D drawing of G such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph G has a 2-D[3-D] orthogonal drawing only if $\Delta(G) \leq 4[\Delta(G) \leq 6]$. An orthogonal drawing with no more than b bends per edge is called a b-bend orthogonal drawing.

Biedl and Kant [2], and Liu, Morgana, and Simeone [7] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing with the only exception of the octahedron, which has a 3-bend 2-D orthogonal drawing. Moreover, Kant [6] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of K_4. Nomura, Tayu, and Ueno [8] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia proved that it is NP-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing [5]. Battista, Liotta, and Vargiu showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs [1]. We show in Section 3 the following theorem.

Theorem 1. *Every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.* □

Eades, Symvonis, and Whitesides [4], and Papakostas and Tollis [9] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Moreover, Wood showed that every 5-graph has a 2-bend 3-D orthogonal drawing [11]. Nomura, Tayu, and Ueno [8] showed that every outerplanar 6-graph has a 0-bend 3-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Eades, Stirk, and Whitesides proved that it is NP-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing [3]. We show in Section 4 the following theorem.

Theorem 2. *Every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing.* □

2 Preliminaries

A *series-parallel graph* is defined recursively as follows:

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the terminals.
(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:
   (i) *Series composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.
   (ii) *Parallel composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into a vertex t. Vertices s and t are the terminals of G.

A series-parallel graph G is naturally associated with a binary tree $T(G)$, which is called a *decomposition tree* of G. The nodes of $T(G)$ are of three types, S-nodes, P-nodes, and Q-nodes. $T(G)$ is defined recursively as follows:

(1) If G is a single edge, then $T(G)$ consists of a single Q-node.
(2-i) If G is obtained from series-parallel graphs G_1 and G_2 by the series composition, then the root of $T(G)$ is a S-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.
(2-ii) If G is obtained from series-parallel graphs G_1 and G_2 by the parallel composition, then the root of $T(G)$ is a P-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

Notice that the leaves of $T(G)$ are the Q-nodes, and an internal node of $T(G)$ is either an S-node or P-node. Notice also that every P-node has at most one Q-node as a child, since G is a simple graph. If G has n vertices then $T(G)$ has $O(n)$ nodes, and $T(G)$ can be constructed in $O(n)$ time [10].

3 Proof of Theorem 1 (Sketch)

Let G be a series-parallel 4-graph with terminals s and t. We generate for G several 1-bend 2-D orthogonal drawings of distinct types depending on $d_G(s)$ and $d_G(t)$. The number of distinct types $\nu(d_G(s), d_G(t))$ is no more than 4 for every pair of $d_G(s)$ and $d_G(t)$. We denote by $\tau(d_G(s), d_G(t), i)$ a type of drawing for G, where $0 \le i \le \nu(d_G(s), d_G(t))$. Fig. 1 shows the types of 1-bend 2-D orthogonal drawings of G, where terminals are indicated by circles. We denote by $\Gamma_i(G)$ a 1-bend 2-D orthogonal drawing of type $\tau(d_G(s), d_G(t), i)$ for G. The drawings $\Gamma_i(G)$ are generated by Algorithm 1 below.

Algorithm 1 (Outline)
Input: a series-parallel 4-graph G with terminals s and t.
Output: 1-bend 2-D orthogonal drawings $\Gamma_i(G)$, $0 \le i \le \nu(d_G(s), d_G(t))$.
Step 0 Compute $T(G)$.
Step 1 If G consists of a single edge, let $\Gamma_0(G)$ be a drawing of type $\tau(1, 1, 0)$ and $\Gamma_1(G)$ be a drawing of type $\tau(1, 1, 1)$ for G.
Step 2 If G is the series composition of G_1 and G_2, drawings $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ are first recursively generated for $0 \le j \le \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $0 \le k \le \nu(d_{G_2}(s_2), d_{G_2}(t_2))$. Then for each i, $1 \le i \le \nu(d_G(s), s_G(t))$, generate $\Gamma_i(G)$ by combining appropriate $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ as shown in Table 1.

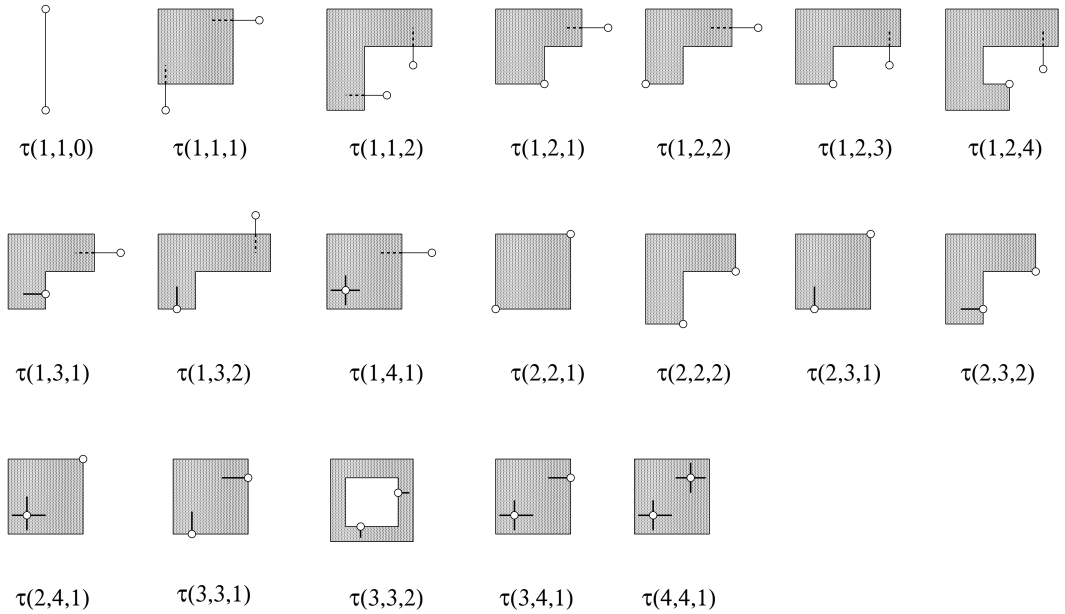

Fig. 1. Types of 1-bend 2-D orthogonal drawings, where $\tau(i,j,k) = \tau(j,i,k)$.

Step 3 If G is the parallel composition of G_1 and G_2, drawings $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ are first recursively generated for $1 \le j \le \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $1 \le k \le \nu(d_{G_2}(s_2), d_{G_2}(t_2))$. Then for each i, $1 \le i \le \nu(d_G(s), s_G(t))$, generate $\Gamma_i(G)$ by combining appropriate $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ as shown in Table 2. □

The correctness of the algorithm is guaranteed by the following lemma.

Lemma 1. *If G contains more than one edge, then for any $\tau(d_G(s), d_G(t), i)$, $1 \le i \le \nu(d_G(s), d_G(t))$, there always exist a drawing $\Gamma_j(G_1)$, $0 \le j \le \nu(d_{G_1}(s_1), d_{G_1}(t_1))$, and a drawing $\Gamma_k(G_2)$, $0 \le k \le \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that we can generate $\Gamma_i(G)$ by combining $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ with the only exception of $\tau(3,3,2)$ for G with edge (s,t).* □

The proof of the lemma is obvious from the tables 1 and 2 below, which show types of such $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ for each type of $\Gamma_i(G)$, where $\tau(i,j,k)$ is indicated by (i,j,k) in the tables. It is tedious but easy to check the tables.

4 Proof of Theorem 2 (Sketch)

Let G be a series-parallel 6-graph with terminals s and t. We use a vector $R(G) \in \{+1,-1\}^3$ to represent relative positions of terminals in the space. For vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, define that $\boldsymbol{a} * \boldsymbol{b} = (a_1b_1, a_2b_2, a_3b_3)$. Let $\mathcal{D}^+ = \{X, Y, Z\}$, $\mathcal{D}^- = \{-X, -Y, -Z\}$, $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, and let $D_G(s)$ and $D_G(t)$ be subsets of $\mathcal{D}$ satisfying the following conditions:

1. $|D_G(s)| = d_G(s)$ and $|D_G(t)| = d_G(t)$.
2. There exist $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \ne -B$.

The conditions above implies that the elements of $D_G(s)$ and $D_G(t)$ can be ordered $A_1, A_2, \ldots, A_{d_G(s)}$ and $B_1, B_2, \ldots B_{d_G(t)}$, respectively, such that $A_i \ne -B_i$ for each i, $1 \le i \le \min\{d_G(s), d_G(t)\}$. We denote by $[D_G(s)]$ and $[D_G(t)]$ such sequesces of elements. $D_G(s)$ and $D_G(t)$ are said to be inner-directed if there exist $A \in D_G(s)$ and $B \in D_G(t)$ satisfying the following conditions:

1. $A \in \mathcal{D}^-$ and $B \in \mathcal{D}^+$
2. $A \neq -B$
3. If $D_G(s) - \{A\} \neq \phi$ and $D_G(t) - \{B\} \neq \phi$ then there exist $A' \in D_G(s) - \{A\}$ and $B' \in D_G(t) - \{B\}$ such that $A' \neq -B'$.

A 2-bend 3-D orthogonal drawing $\Gamma(G)$ of G is generated by Algorithm 2 in section 4.1.

4.1 Algorithm 2: 3D-DRAW$(G, D_G(s), D_G(t), R(G))$ (Outline)

3D-DRAW$(G, D_G(s), D_G(t), R(G))$
Input: a series-parallel 6-graph G with terminal s and t, $D_G(s), D_G(t)$, and $R(G)$
Output: 2-bend 3-D orthogonal drawing $\Gamma(G)$
begin
  Compute $T(G)$
  if G consists of a single edge **then** draw $\Gamma(G)$ depending on $D_G(s)$, $D_G(t)$, and $R(G)$
  else
    if G is the series composition of G_1 and G_2
      SER-DECOM$(G, G_1, G_2, D_G(s), D_G(t), R(G))$ (in Section 4.1.1)
    end if
    if G is the parallel composition of G_1 and G_2,
      PAR-DECOM$(G, G_1, G_2, D_G(s), D_G(t), R(G))$ (in Section 4.1.2)
    end if
    $\Gamma(G_1)$ =3D-DRAW$(G_1, D_{G_1}(s_1), D_{G_1}(t_1), R(G_1))$
    $\Gamma(G_2)$ =3D-DRAW$(G_2, D_{G_2}(s_2), D_{G_2}(t_2), R(G_2))$
    if G is the seires composition of G_1 and G_2,
      SER-COM$(\Gamma(G_1), \Gamma(G_2))$ (in Section 4.1.3)
    end if
    if G is the parallel composition of G_1 and G_2,
      PAR-COM$(\Gamma(G_1), \Gamma(G_2))$ (in Section 4.1.4)
    end if
  end if
end

4.1.1 SER-DECOM$(G, G_1, G_2, D_G(s), D_G(t), R(G))$

Input: $G, G_1, G_2, D_G(s), D_G(t), R(G)$
Output: $D_{G_1}(s_1), D_{G_1}(t_1), D_{G_2}(s_2), D_{G_2}(t_2), R(G_1), R(G_2)$
Step 1 Define that $(X_G, Y_G, Z_G) = (X, Y, Z) * R(G)$, $\mathcal{D}_G^+ = \{X_G, Y_G, Z_G\}$, and $\mathcal{D}_G^- = \{-X_G, -Y_G, -Z_G\}$.
Step 2 If $D_G(s)$ and $D_G(t)$ are inner-directed, then select $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \in \mathcal{D}_G^-$ and $B \in \mathcal{D}_G^+$. Else select $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \neq -B$.
Step 3 Output $D_{G_1}(s_1)$, $D_{G_1}(t_1)$, $D_{G_2}(s_2)$, $D_{G_2}(t_2)$, $R(G_1)$, and $R(G_2)$ depending on A and B as follows:
  Case 1 $A \in \mathcal{D}_G^-, B \in \mathcal{D}_G^+$:
    Case 1-1 $B \in \{X_G, Z_G\}$: Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-Y_G\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-Y\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_{G_2}(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = R(G)$.

Case 1-2 $B = Y_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-X_G\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-X\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = R(G)$.

Case 2 $A \in \mathcal{D}_G^+, B \in \mathcal{D}_G^-$:

Case 2-1 $A = X_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{A\}$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-A\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-A\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = (+1, -1, -1) * R(G)$.

Case 2-2 $A = Y_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 2-1. Let $R(G_1) = (+1, +1, -1) * R(G)$ and $R(G_2) = (-1, -1, +1) * R(G)$.

Case 2-3 $A = Z_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 2-1. Let $R(G_1) = (+1, -1, +1) * R(G)$ and $R(G_2) = (-1, +1, -1) * R(G)$.

Case 3 $A \in \mathcal{D}_G^-, B \in \mathcal{D}_G^-$:

Case 3-1 $A = B = -Z_G$: Let $D_{G_2}(t_2) = D_G(t)$. If $D_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S| = D_{G_2}(s_2)$ and $S \subseteq \mathcal{D}_G^- - \{B\}$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - \{X_G\}$. Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S'| = D_{G_1}(t_1)$, $\{X\} \subseteq S' \subseteq \mathcal{D}_G^+$, and $D_{G_2}(s_2) \cap S = \emptyset$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = R(G)$ and $R(G_2) = (+1, +1, -1) * R(G)$.

Case 3-2 $A = B = -Y_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 3-1. Let $R(G_1) = R(G)$ and $R(G_2) = (+1, -1, +1) * R(G)$.

Case 3-3 $A = B = -X_G$: Let $D_{G_2}(t_2) = D_G(t)$. If $D_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S| = D_{G_2}(s_2)$ and $S \subseteq \mathcal{D}_G^- - \{B\}$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - \{Z_G\}$. Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S'| = D_{G_1}(t_1)$, $\{Z_G\} \subseteq S' \subseteq \mathcal{D}_G^+$, and $D_{G_2}(s_2) \cap S = \emptyset$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = R(G)$ and $R(G_2) = (-1, +1, +1) * R(G)$.

Case 3-4 $A \neq B$: Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $D_{G_1}(t_1) \geq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ - \{-A\} \subseteq S \subseteq \mathcal{D} - \{-A\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-A\} \subseteq S' \subseteq \mathcal{D}_G^- - \{A\} + \{-A\}$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_2}(s_2) \geq 4$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- + \{-A\} \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = (-1, -1, -1) * R(G)$.

Case 4 $A \in \mathcal{D}_G^+, B \in \mathcal{D}_G^+$:

Case 4-1 $A = B = Z_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{A\}$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-Y\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-Y\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = R(G)$.

Case 4-2 $A = B = Y_G$: Let $D_{G_1}(s\_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{A\}$. If $D_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-Z_G\}$. If $D_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $\{-Z_G\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If

$D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (+1, -1, +1) * R(G)$ and $R(G_2) = R(G)$.

Case 4-3 $A = B = X_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 4-1. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = R(G)$.

Case 4-4 $A \neq B$: If $D_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$, $S' \subseteq \mathcal{D}_G^- - \{-B\}$. If $D_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = D_{G_2}(s_2)$ and $\mathcal{D}_G^- - \{-B\} + \{B\} \subseteq S' \subseteq \mathcal{D} - \{-B\}$. Let $D_G(t_2) = D_G(t)$. Let $D_{G_1}(s_1) = D_G(s)$. If $D_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\{-B\} \subseteq S \subseteq \mathcal{D}_G^+ - \{B\} + \{-B\}$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $D_{G_1}(t_1) \geq 4$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ + \{-B\} \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = (-1, -1, -1) * R(G)$ and $R(G_2) = R(G)$.

4.1.2 PAR-DECOM($G, G_1, G_2, D_G(s), D_G(t), R(G)$)

Input: $G, G_1, G_2, D_G(s), D_G(t), R(G)$

Output: $D_{G_1}(s_1), D_{G_1}(t_1), D_{G_2}(s_2), D_{G_2}(t_2), R(G_1), R(G_2)$

Step 1 Define that $(X_G, Y_G, Z_G) = (X, Y, Z)*R(G)$, $\mathcal{D}_G^+ = \{X_G, Y_G, Z_G\}$, and $\mathcal{D}_G^- = \{-X_G, -Y_G, -Z_G\}$.

Step 2 Construct $[D_G(s)] = (A_1, A_2, \ldots, A_{D_G(s)})$ and $[D_G(t)] = (B_1, B_2, \ldots, B_{D_G(t)})$ such that $A_i \neq -B_i$, $1 \leq i \leq \min\{d_G(s), d_G(t)\}$. If $D_G(s)$ and $D_G(t)$ are inner-directed, we assume without loss of generality that $A_1 \in \mathcal{D}_G^-$ and $B_1 \in \mathcal{D}_G^+$.

Step 3 Output $D_{G_1}(s_1)$, $D_{G_1}(t_1)$, $D_{G_2}(s_2)$, $D_{G_2}(t_2)$, $R(G_1)$, and $R(G_2)$ depending on $d_{G_1}(s_1)$ and $d_{G_1}(t_1)$ as follows:

Case 1 $k_1 = d_{G_1}(s_1) \leq d_{G_1}(t_1)$:

Case 1-1 $e = (s,t) \in G_1 : D_{G_1}(s_1) = \{A_1, A_2, \ldots, A_{k_1}\}$, $D_{G_1}(t_1) = \{B_1, B_2, \ldots, B_{k_1}, B_{k_1+D_{G_2}(t_2)+1}, \ldots, B_{D_G(t)}\}$, $D_{G_2}(s_2) = \{A_{k_1+1}, A_{k_1+2}, \ldots, A_{D_G(s)}\}$, $D_{G_2}(t_2) = \{B_{k_1+1}, B_{k_1+2}, \ldots, B_{k_1+D_{G_2}(t_2)}\}$, $R(G_1) = R(G_2) = R(G)$.

Case 1-2 $e = (s,t) \in G_2 : D_{G_1}(s_1) = \{A_2, A_3, \ldots, A_{k_1+1}\}$, $D_{G_1}(t_1) = \{B_2, B_3, \ldots, B_{k_1+1}, B_{k_1+D_{G_2}(t_2)+1}, \ldots, B_{D_G(t)}\}$, $D_{G_2}(s_2) = \{A_1, A_{k_1+2}, A_{k_1+3}, \ldots, A_{D_G(s)}\}$, $D_{G_2}(t_2) = \{B_1, B_{k_1+2}, B_{k_1+3}, \ldots, B_{k_1+D_{G_2}(t_2)}\}$, and $R(G_1) = R(G_2) = R(G)$.

Case 2 $d_{G_1}(s_1) \geq d_{G_1}(t_1) = k_1$:

Case 2-1 $e = (s,t) \in G_1 : D_{G_1}(s_1) = \{A_1, A_2, \ldots, A_{k_1}, A_{k_1+D_{G_2}(s_2)+1}, \ldots, A_{D_G(s)}\}$, $D_{G_1}(t_1) = \{B_1, B_2, \ldots, B_{k_1}\}$, $D_{G_2}(s_2) = \{A_{k_1+1}, A_{k_1+2}, \ldots, A_{k_1+D_{G_2}(s_2)}\}$, $D_{G_2}(t_2) = \{B_{k_1+1}, B_{k_1+2}, \ldots, B_{D_G(t)}\}$, and $R(G_1) = R(G_2) = R(G)$.

Case 2-2 $e = (s,t) \in G_2 : D_{G_1}(s_1) = \{A_2, A_3, \ldots, A_{k_1+1}, A_{k_1+D_{G_2}(s_2)+1}, \ldots, A_{D_G(s)}\}$, $D_{G_1}(t_1) = \{B_2, B_3, \ldots, B_{k_1+1}\}$, $D_{G_2}(s_2) = \{A_1, A_{k_1+2}, A_{k_1+3}, \ldots, A_{k_1+D_{G_2}(s_2)}\}$, $D_{G_2}(t_2) = \{B_1, B_{k_1+2}, B_{k_1+3}, \ldots, B_{D_G(t)}\}$, and $R(G_1) = R(G_2) = R(G)$.

4.1.3 SER-COM($\Gamma(G_1), \Gamma(G_2)$)

Input: $\Gamma(G_1), \Gamma(G_2)$

Output: $\Gamma(G)$

Step 1 Translate $\Gamma(G_1)$ and $\Gamma(G_2)$ so that t_1 and s_2 can be identified.

Step 2 Generate $\Gamma'(G)$ by identifying t_1 with s_2.

Step 3 Generate $\Gamma(G)$ by modifying $\Gamma'(G)$ so that there are no overlaps.

4.1.4 PAR-COM($\Gamma(G_1), \Gamma(G_2)$)

Input: $\Gamma(G_1), \Gamma(G_2)$
Output: $\Gamma(G)$
Step 1 Modify and translate $\Gamma(G_1)$ and $\Gamma(G_2)$ so that the terminals can be identified.
Step 2 Generate $\Gamma'(G)$ by identifying s_1 with s_2, and t_1 with t_2.
Step 3 Generate $\Gamma(G)$ by modifying $\Gamma'(G)$ so that there are no overlaps.

4.2 Analysis of Algorithm 2

Omitted.

5 Concluding Remarks

It should be noted that $K_{2,3}$, which is a series-parallel 3-graph, has no 0-bend 2-D orthogonal drawing. It is an interesting open problem to decide if every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

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$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$
(1, 1, 1)	(1, 1, 1)	(1, 1, 1)		(1, 2, 1)	(2, 2, 2)		(2, 2, 1)	(1, 3, 2)
	(1, 1, 1)	(2, 1, 2)		(1, 3, 1)	(1, 2, 1)		(2, 2, 1)	(2, 3, 1)
	(1, 1, 1)	(3, 1, 2)	(1, 3, 1)	(1, 1, 1)	(1, 3, 1)		(2, 3, 1)	(1, 3, 2)
	(1, 2, 2)	(1, 1, 1)		(1, 1, 1)	(2, 3, 1)	(2, 3, 2)	(2, 1, 1)	(1, 3, 1)
	(1, 2, 1)	(2, 1, 2)		(1, 1, 1)	(3, 3, 1)		(2, 1, 2)	(2, 3, 2)
	(1, 3, 2)	(1, 1, 1)		(1, 2, 2)	(1, 3, 1)		(2, 1, 1)	(3, 3, 1)
(1, 1, 2)	(1, 1, 1)	(1, 1, 1)		(1, 2, 1)	(2, 3, 1)		(2, 2, 2)	(1, 3, 2)
	(1, 1, 1)	(2, 1, 1)		(1, 3, 1)	(1, 3, 2)		(2, 2, 1)	(2, 3, 2)
	(1, 1, 1)	(3, 1, 1)	(1, 3, 2)	(1, 1, 1)	(1, 3, 1)		(2, 3, 2)	(1, 3, 2)
	(1, 2, 1)	(1, 1, 1)		(1, 1, 1)	(2, 3, 1)	(2, 4, 1)	(2, 1, 2)	(1, 4, 1)
	(1, 2, 1)	(2, 1, 1)		(1, 1, 1)	(3, 3, 1)		(2, 1, 2)	(2, 4, 1)
	(1, 3, 1)	(1, 1, 1)		(1, 2, 2)	(1, 3, 2)		(2, 1, 2)	(3, 4, 1)
(1, 2, 1)	(1, 1, 1)	(1, 2, 1)		(1, 2, 2)	(2, 3, 1)		(2, 2, 1)	(1, 4, 1)
	(1, 1, 1)	(2, 2, 1)		(1, 3, 2)	(1, 3, 2)		(2, 2, 1)	(2, 4, 1)
	(1, 1, 1)	(3, 2, 1)	(1, 4, 1)	(1, 1, 1)	(1, 4, 1)		(2, 3, 1)	(1, 4, 1)
	(1, 2, 1)	(1, 2, 2)		(1, 1, 1)	(2, 4, 1)	(3, 3, 1)	(3, 1, 2)	(1, 3, 2)
	(1, 2, 1)	(2, 2, 1)		(1, 1, 1)	(3, 4, 1)		(3, 1, 2)	(2, 3, 1)
	(1, 3, 1)	(1, 2, 2)		(1, 2, 1)	(1, 4, 1)		(3, 1, 2)	(3, 3, 1)
(1, 2, 2)	(1, 1, 1)	(1, 2, 2)		(1, 2, 1)	(2, 4, 1)		(3, 2, 1)	(1, 3, 2)
	(1, 1, 1)	(2, 2, 1)		(1, 3, 1)	(1, 4, 1)		(3, 2, 1)	(2, 3, 1)
	(1, 1, 1)	(3, 2, 1)	(2, 2, 1)	(2, 1, 2)	(1, 2, 2)		(3, 3, 1)	(1, 3, 2)
	(1, 2, 2)	(1, 2, 2)		(2, 1, 2)	(2, 2, 1)	(3, 3, 2)	(3, 1, 1)	(1, 3, 1)
	(1, 2, 2)	(2, 2, 1)		(2, 1, 2)	(3, 2, 1)		(3, 1, 1)	(2, 3, 1)
	(1, 3, 2)	(1, 2, 2)		(2, 2, 1)	(1, 2, 2)		(3, 1, 2)	(3, 3, 2)
(1, 2, 3)	(1, 1, 1)	(1, 2, 2)		(2, 2, 1)	(2, 2, 1)		(3, 2, 1)	(1, 3, 1)
	(1, 1, 1)	(2, 2, 2)		(2, 3, 1)	(1, 2, 2)		(3, 2, 2)	(2, 3, 2)
	(1, 1, 2)	(3, 2, 1)	(2, 2, 2)	(2, 1, 1)	(1, 2, 1)		(3, 3, 2)	(1, 3, 2)
	(1, 1, 0)	(3, 2, 2)		(2, 1, 1)	(2, 2, 1)	(3, 4, 1)	(3, 1, 1)	(1, 4, 1)
	(1, 2, 1)	(1, 2, 2)		(2, 1, 1)	(3, 2, 1)		(3, 1, 1)	(2, 4, 1)
	(1, 2, 2)	(2, 2, 2)		(2, 2, 1)	(1, 2, 1)		(3, 1, 2)	(3, 4, 1)
	(1, 3, 1)	(1, 2, 2)		(2, 2, 1)	(2, 2, 2)		(3, 2, 1)	(1, 4, 1)
(1, 2, 4)	(1, 1, 1)	(1, 2, 1)		(2, 3, 1)	(1, 2, 1)		(3, 2, 1)	(2, 4, 1)
	(1, 1, 1)	(2, 2, 2)	(2, 3, 1)	(2, 1, 2)	(1, 3, 2)		(3, 3, 1)	(1, 4, 1)
	(1, 1, 1)	(3, 2, 2)		(2, 1, 2)	(2, 3, 1)			
	(1, 2, 1)	(1, 2, 1)		(2, 1, 2)	(3, 3, 1)			

Table 1. Series composition.

$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$
(2, 2, 1)	(1, 1, 1)	(1, 1, 1)		(1, 2, 2)	(2, 1, 3)	(4, 4, 1)	(1, 1, 1)	(3, 3, 2)
(2, 2, 2)	(1, 1, 1)	(1, 1, 2)	(3, 3, 2)	(1, 1, 2)	(2, 2, 2)		(1, 1, 2)	(3, 3, 1)
(2, 3, 1)	(1, 1, 1)	(1, 2, 1)		(1, 2, 1)	(2, 1, 4)		(1, 2, 1)	(3, 2, 2)
(2, 3, 2)	(1, 1, 1)	(1, 2, 4)	(3, 4, 1)	(1, 1, 1)	(2, 3, 2)		(1, 3, 1)	(3, 1, 1)
(2, 4, 1)	(1, 1, 1)	(1, 3, 1)		(1, 2, 1)	(2, 2, 2)		(2, 2, 2)	(2, 2, 2)
(3, 3, 1)	(1, 1, 1)	(2, 2, 2)		(1, 3, 1)	(2, 1, 1)			

Table 2. Parallel composition.