# 距離制限された彩色問題の近似

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**概念**

$(h,k)$ 彩色問題は，隣接する 2 点の色の差は $h$ 以上で，距離が 2 である 2 点の色の差は $k$ 以上であるように無向グラフの点を非負整数で彩色する問題である．これは $L(h,k)$ ラベル付け問題としても知られている．小文では，一般のグラフと 2 部グラフに対して，この問題の最良の近似を与える．

# Approximating the $L(h,k)$-labelling problem

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**Abstract**

The $(h,k)$-coloring problem, better known as the $L(h,k)$-labelling problem, is that of vertex coloring an undirected graph with non-negative integers so that adjacent vertices receive colors that differ by at least $h$ and vertices of distance 2 receive colors that differ by at least $k$. We give tight approximations for this problem on general graphs and on bipartite graphs.

## 1 Introduction

The $(h,k)$-coloring problem, better known as the $L(h,k)$-labelling problem, is that of vertex coloring an undirected graph $G$ with non-negative integers so that adjacent vertices receive colors that differ by at least $h$ and vertices of distance 2 receive colors that differ by at least $k$. This problem was introduced by Griggs and Yeh [6] (in the case $h = 2$ and $k = 1$) to model a frequency assignment problem, where wireless transmitter/receivers must be assigned frequencies without causing interference.

A large body of research has developed through the years on $(h,k)$-coloring problems. Most of that effort has been on two cases. The $(1,1)$-coloring problem is known as the *distance-2 coloring problem*, which again is closely related to coloring the square of a graph [2]. The $(2,1)$-coloring problem is also known under the names $\lambda$-coloring and radio-coloring. The recent and online survey [3] gives a thorough treatment of exact solutions and bounds known for numerous classes of graphs.

## 1.1 Our contributions

We consider arbitrary $(h, k)$-coloring problems, where $h$ and $k$ can also be functions of $n$, the number of vertices in the graph.

We show that a greedy First-Fit algorithm attains a performance ratio of $O(\min(\Delta, \sqrt{n} + h/k))$, where $\Delta$ is the maximum degree of the graph, and that that is tight for all values of these parameters, even in the case of bipartite graphs. The First-Fit algorithm is online and can be made distributed, thus the bounds are also competitive ratios.

We show that this is close to best possible, as for any integral $h, k$, it is NP-hard to approximate the $(h, k)$-coloring problem within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$. We also show that the problem is hard to approximate within a factor of $\Omega(h/n^{\epsilon})$, for $h$ as large as $n$. On the positive side, it is never harder to approximate than the ordinary ($(1, 0)$-) coloring problem, hence an upper bound of $O(n(\log\log n)^2/\log^3 n)$ holds.

We obtain tight results for the special case of bipartite graphs, for all $(h, k)$-coloring problems. We give a simple algorithm that attains a performance ratio of $O(\sqrt{n})$, while the $n^{1/2-\epsilon}$-hardness result applies here also. Notice that our constructions for First-Fit show that its performance ratio for bipartite graphs can be as large as $\Omega(n)$.

## 1.2 Previous results

A large body of research exists on $(h, k)$-coloring problems. This includes both exact and inexact bounds on special classes of graphs, and NP-hardness proofs; see the survey of Calamoneri [3]. The constructive upper bounds for coloring special classes of graphs can be viewed as approximation algorithms, while relative approximation results appear to be rare.

Few approximation results exist on general graphs, or on classes without constant factor upper bounds. Mostly, these are restricted to the $(1, 1)$-coloring problem, which is equivalent to the *distance-2 coloring problem* (and closely related to coloring the square of a graph [2]). McCormick [8] showed that a greedy algorithm attains a $O(\sqrt{n})$-approximation (see also [1]). Agnarsson, Greenlaw, and Halldórsson [1] showed that the problem is hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$. This hardness holds also in the case of bipartite graphs and split graphs.

Recently, Calamoneri and Vocca [4] gave a $h\sqrt{n}(1 + o(n))$-approximation of $(h, k)$-approximation problems with $h > k$. They also gave approximations of bipartite graphs, that were asymptotically $\min(h, 2k)\sqrt{n}$ and $4/3\Delta^2$ factors.

# 2 Results

The *span* of a $(h, k)$-coloring $\psi$ is the value of the largest color assigned minus one, $\max_{v \in V(G)} \psi(v) - 1$. [2] Let $\lambda_{h,k}(G)$ denote the minimum span of a $(h, k)$-coloring of a graph $G$. The performance ratio $\rho_A$ of an $(h, k)$-coloring algorithm $A$ is the maximum ratio between the maximum and minimum spans, i.e.,

$$\rho_A = \rho_A(n) = \max_{G, |V(G)|=n} \frac{A(G)}{\lambda_{h,k}(G)}.$$

## 2.1 Basic properties

We first simplify the problems by showing that it suffices to consider only restricted subset of possible colorings, and that we can omit the factor $k$ with only a small loss of performance.

It is well known that by uniformly increasing the gap between the vertices, one obtains a proper coloring with larger separations.

**Observation 2.1** *Consider a $(h, k)$-coloring $\psi$ with span $\lambda$. Then, $\psi'(v) = \psi(v) * t$ is a $(h \cdot t, k \cdot t)$-coloring with span $\lambda \cdot t$.*

The converse holds also when the two separation constraints have a common divisor.

**Lemma 2.2** *Consider a $(h \cdot t, k \cdot t)$-coloring $\psi$ with span $\lambda$. Then,*

$$\psi'(v) = \left\lfloor \frac{\psi(v)}{t} \right\rfloor$$

*is a valid $(h, k)$-coloring with span $\lfloor \lambda / t \rfloor$.*

*Proof.* Suppose that the claim is false, and let $u, v$ be a pair of vertices whose colors $\psi'(u)$, $\psi'(v)$ falsify the claim. Then, either $u$ and $v$ are adjacent and $|\psi'(u) - \psi'(v)| \leq h - 1$, or $u$ and $v$ share a common neighbor and $|\psi'(u) - \psi'(v)| < k$. Consider the former case; the other is identical and will be omitted. Let $\psi(u) = t \cdot \psi'(u) + r_v$ and $\psi(v) = t \cdot \psi'(v) + r_u$, for some $0 \leq r_v, r_u < t$. Then, $|\psi(u) - \psi(v)| = |t \cdot (\psi'(u) - \psi'(v)) + (r_u - r_v)| \leq t \cdot |\psi'(u) - \psi'(v)| + |r_u - r_v| < t(h-1) + t = th$. Then, $u$ and $v$ are not properly $(ht, kt)$-colored. This is a contradiction, hence the claim. $\square$

When there is no common divisor, one can create one by rounding up the values with a small increase in the span.

**Lemma 2.3** *Consider a $(h, k)$-coloring $\psi$ with span $\lambda$, and an integer $t \leq k$. Then,*

$$\psi'(v) = \psi(v) + \lfloor \psi(v)/t \rfloor$$

*is a valid $(\lceil h/t \rceil t, k)$-coloring with span $(1 + 1/\lceil h/t \rceil)\lambda$.*

[2]Note that the span is frequently defined to be simply the largest color used. Our definition matched the size of the color palette used, including the "holes".

*Proof.* □

**Corollary 2.4** *A $(h, k)$-coloring with span $\lambda$ can be turned into a $(\lceil h/k \rceil, 1)$-coloring with span at most $2\lambda/k$, for any $h, k$.*

*Proof.* Use the two lemmas, with $t = k$ in the second lemma. □

## 2.2 Analysis of First-Fit

The First-Fit algorithm is one of the simplest coloring strategies. Processing the vertices in an arbitrary order, each vertex is assigned the smallest color compatible with its neighborhood. For the $(h, k)$-coloring problem, that means satisfying the distance constraints to the previously colored neighbors as well as previously colored vertices of distance two.

First-Fit is an online algorithm, thus the upper bounds proven also give upper bounds on competitive ratio of online coloring algorithms. It can also be component of a distributed strategy, when complemented by a synchronization primitive, such as only vertices that form a distance-2 independent set are colored at the same time.

Let $d_2(v)$ be the number of neighbors of distance-2 from a vertex $v$ and $\Delta_2 = \max_v d_2(v) \leq \Delta(\Delta - 1)$ be the maximum of these values of vertices in the graph.

**Lemma 2.5** *The span of a First-Fit $(h, k)$-coloring of a graph $G$ is at most*

$$FF(G) \leq \max_{v \in V}[d_2(v) \cdot (2k - 1) + d(v) \cdot (2h - 1)] \leq \Delta_2 \cdot (2k - 1) + \Delta \cdot (2h - 1) + 1.$$

*Further, $FF(G) \leq n \cdot h$.*

*Proof.* Each of the $\Delta$ neighbors of $v$ can cause $2h - 1$ colors to be unavailable for $v$ to use: $h - 1$ above, $h - 1$ colors below, and then the color of the neighbor. Similarly for the $d_2(v)$ distance-2 neighbors of $v$. Finally, there is the color used by $v$. □

**Lemma 2.6** *For any graph $G$, the minimum span of a $(h, k)$-coloring of $G$ is bounded below by*

$$\lambda_{h,k}(G) \geq (\Delta - 1) \cdot k + h + 1.$$

*Proof.* Each of the $\Delta$ neighbors of a maximum degree vertex $v$, as well as $v$ itsefl, must be mutually $k$ colors apart, using at least $\Delta k + 1$ colors. The separation from $v$ to its nearest colored neighbor must be an additional $h - k$. □

**Theorem 2.7** *The performance ratio of First-Fit, denoted as $\rho_{FF}$, is at most $O(\min(\Delta, h/k + \sqrt{n}))$. Furthermore, this is tight within a constant factor, for any combination of the parameters, even in the case of bipartite graphs.*

*Proof.* By Corollary 2.4, we may assume without loss of generality that $k = 1$. Let $G$ be a graph with $n$ vertices and maximum degree $\Delta$, $FF(G)$ be the span of a First-Fit $(h, 1)$-coloring of $G$, and $\lambda_{h,1}(G)$ be the minimum span. Let $\rho_{FF} = \max_G \frac{FF(G)}{\lambda_{h,1}(G)}$.

By Lemmas 2.5 and 2.6, we have that $FF(G) \leq \min(n, (\Delta-1)\Delta) + \Delta \cdot (2h-1) + 1$ and $\chi_{h,1}(G) \geq \Delta + h$. Now, $(\Delta-1)\Delta/\Delta = \Delta - 1$ and $\Delta(2h-1)/h \leq 2\Delta$, so $FF(G)/\lambda_{h,1}(G) \leq 2\Delta$. Also, if $\Delta > h + \sqrt{n}$, we have that

$$\frac{FF(G)}{\lambda_{h,1}(G)} \leq \frac{n + \Delta(2h-1)}{\Delta} \leq \sqrt{n} + (2h-1).$$

To see that these bounds are tight, consider the bipartite graph $B_{m,m}$ which consists of a complete bipartite graph $K_{m,m}$ from which a perfect matching has been removed. In a worst case First-Fit coloring, the vertices are assigned the colors $0, 1, h, h+1, \ldots, (m-1)h, (m-1)h+1$, while an optimal coloring uses colors $0, 1, \ldots, m-1, m-1+h, m-1+h+1, \ldots, 2(m-1)+h$. The ratio between the two spans is at least $\min(h/2, m-1)$. By letting $m$ range from $\sqrt{n}$ to $n/2$ and adding edges and degree-1 vertices to allow $\Delta$ to range from $m$ to $n-m$, we obtain a tight bound for the second part of the claim. $\square$

## 2.3 General graphs

Consider now the case when $h$ is huge in comparison with $\Delta$. E.g., when $h \geq n^2$.

**Claim 2.8** *Consider the $(h,1)$-coloring problem for a graph $G$ where $h > \Delta_2(G) + \Delta(G)$. Then, the $(h,1)$-coloring and the ordinary coloring problems are equivalent for $G$, within a constant factor.*

*Proof.* Consider an ordinary vertex coloring $\psi$ of $G$ that uses $\chi$ colors. Form distance-2 coloring $\phi$ of $G$ using at most $\Delta_2 + \Delta$ colors. Form the $(h,1)$-coloring $\psi'$ of $G$ by

$$\psi'(v) = 2h \cdot \psi(v) + \phi(v).$$

Then, vertices $u$ and $v$ assigned different color by $\psi$ are separated by at least $2h - |\phi(u) - \phi(v)| > h$ colors, and vertices of distance-2 are assigned different color based on the $\phi$ value. The span of this coloring is at most $(2h+1)\chi$.

On the other hand, an $(h,1)$-coloring $\psi'$ of $G$ of span $\lambda$ can be turned into an ordinary coloring by

$$\psi(v) = \lfloor 2\psi'(v)/h \rfloor$$

with span at most $\lambda/(h/2)$. $\square$

For general graphs, we can obtain improved hardness results when $h$ is large.

**Theorem 2.9** *The $(h,k)$-coloring problem is NP-hard to approximate within a factor of $n^{1/2-\epsilon}$, for any $\epsilon > 0$ and $h$ in the range $[n^{1/2-\epsilon}, n]$. This holds even in the case of bipartite graphs.*

We use a hardness construction from [1]. Given a graph $G$ on $N$ vertices, we construct a bipartite graph $H$ on $N^2 + N$ vertices that satisfies

1. $\alpha(H^2) \leq \alpha(G)$, and
2. $\chi(H^2) \leq N \cdot \chi(G) + h$.

*Proof.* The hardness construction of Feige and Kilian [5] for graph coloring shows that for any $\epsilon > 0$, it is NP-hard to distinguish between graph instances $G$ where a) $\alpha(G) \leq N^\epsilon$, and b) $\chi(G) \leq N^\epsilon$. When a) holds, then $\alpha(H^2) \leq N^\epsilon$ and $\chi(H^2) \geq (N^2 + N)/\alpha(H^2) \geq N^{2-\epsilon}$, thus the span of a $(h, 1)$-coloring is at least that much. On the other hand, if b) holds, then $\chi(H^2) \leq N^{1+\epsilon} + h$. Thus, we obtain a gap of $\min(N^{1-2\epsilon}, N^{2-\epsilon}/h)$ which is $n^{1/2-\epsilon}$ for $h \leq \sqrt{n}$. For $h \geq \sqrt{n}$, the gap is smaller, but a comparable upper bound of $(n + h)/h \leq 2n/h$ follows also easily. □

**Theorem 2.10** *The $(h, k)$-coloring problem is NP-hard to approximate within a factor of $h/n^\epsilon$, for any $\epsilon > 0$.*

*Proof.* Recall the Feige-Kilian gap. We show that on the same graph, we obtain a gap of nearly $\Omega(h)$.

When $\alpha(G) \leq n^\epsilon$, we have that any set of $h$ adjacent colors of a $(h, 1)$-coloring can contain at most $\alpha(G)$-vertices. Thus the span of the coloring is at least $nh/\alpha(G) \geq n^{1-\epsilon}h$.

When $\chi(G) \leq n^\epsilon$, we can construct a $(h, 1)$-coloring by coloring the vertices in order of their ordinary color, using a new color for each vertex, but adding a separation of $h$ when a new color class is considered. This gives a span of at most $n + (\chi(G) - 1)h \leq n + n^\epsilon h$. The gap between the two ratios is therefore at least $\min(n^{1-2\epsilon}, h/n^\epsilon)$. □

**Theorem 2.11** *The $(h, k)$-coloring problem can be approximated within a ratio no worse than the graph coloring problem (the $(1, 0)$-coloring problem). In particular, it is $O(n(\log \log n)^2/\log^3 n)$-approximable.*

*Proof.* We may assume that $h \geq \sqrt{n}$ and that $k = 1$. Let $G$ be a graph, and let $\phi$ be its graph coloring using $\gamma$. Given a graph coloring $\phi$ of a graph $G$, we form an $(h, 1)$-coloring of $G$ by $\psi_{h,1}(v_i) = \phi(v) \cdot h + i$, for vertices $v_1, v_2, \ldots, v_n$. It is easy to observe that this forms a valid $(h, 1)$-coloring and that its span is $h\chi(G) + n$ colors.

Since vertices colored with any of $h$ adjacent colors in a $(h, 1)$-coloring must be independent, we have that

$$\chi(G) \leq \lambda_{h,1}(G)/h.$$

Thus, our performance ratio is bounded by

$$\frac{|\psi(G)|}{\lambda_{h,1}(G)} \leq \frac{h|\phi(G)| + n}{h\chi(G)} \leq \rho_\phi + \sqrt{n}.$$

Since it is known that the graph coloring problem is hard to approximate within $n^{1-\epsilon}$, for any $\epsilon > 0$ [5], $\rho_\phi > \sqrt{n}$. Hence, the performance ratio for $(h, k)$-coloring is asymptotically no worse than for ordinary graph coloring.

The best performance ratio known for graph coloring is $O(n(\log \log n)^2/\log^3 n)$ [7]. □

## 2.4 Bipartite graphs

For the class of bipartite graphs, we can give essentially tight bounds on the approximability of the $(h, k)$-coloring problem.

**Theorem 2.12** *For any $h, k$, possibly functions of $n$, the $(h, k)$-coloring problem can be approximated within a factor of $O(\sqrt{n})$ on bipartite graphs.*

*Proof.* By Corollary 2.4, we may assume without loss of generality that $k = 1$. Given a bipartite graph $G = (U, V, E)$, we color $U$ and $V$ separately, and separate the color sets by a distance of $h$. The coloring of each set corresponds to a distance-2 coloring of the induced subgraphs, which requires at most $\Delta_2 + 1$ colors. In total, the algorithm uses at most $2\Delta_2 + 2 + h$ colors, and trivially also at most $n$ colors. Compared with the easy lower bound of $\Delta + h$, this gives a performance ratio of at most $\min(2\Delta, \sqrt{n})$. □

Recall that the hardness proof of Theorem 2.9 applied also to bipartite graphs. Thus, we have an essentially tight bound within lower order factors on the approximability.

## 2.5 Inductive graphs

Consider now the case of graphs with small *inductiveness* or *degeneracy*. The inductiveness of a graph $G$, denoted $D(G)$, is given by $D(G) = \max_{H \subseteq G} \delta(H) = \max_{H \subseteq G} \min_{v \in H} d_H(v)$, where the maximum is taken over all subgraphs of $G$. It can also be defined in terms of an ordering of the vertices; it implies that there exists a numbering of the vertices of $G$ such that each vertex has at most $D(G)$ higher-numbered neighbors. The *minimum-degree greedy* coloring algorithm iteratively selects a vertex $v$ of minimum degree ($\leq D(G)$), inductively colors the rest of the graph, and finally colors $v$ with the smallest color compatible with its colored neighbors.

**Lemma 2.13** *The minimum-degree greedy coloring algorithm attains a performance ratio of at most $2D(G)$.*

*Proof.* By an analysis similar to Lemma 2.5, one can see that the Greedy algorithm has span of at most

$$D(G) \cdot (2h - 1) + D(G) \cdot (\Delta - 1) \cdot (2k - 1) + 1.$$

Comparing this with the bound of Lemma 2.6 yields the result. □

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