ディジタルハーフトーニングへの応用に向けての魔方陣の一般化 (2)[1]

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要旨 $n \times n$ の行列に整数 $0,\dots,n^2-1$ を配置し，どの行和，列和も等しいものを semimagic square とよばれる．ここでは列和，行和のかわりに $k \times k$ の部分行列に含まれる要素の和を考え，この和が等しい $n \times n$ 行列を *zero $k \times k$-discrepancy matrix* とよぶ．そして，このような行列は k と n がともに偶数であるとき存在し，k と n が互いに素であるとき存在しないことを示す．さらに，$k,m \geq 2$ を整数としたとき $n = k^m$ であるならば zero $k \times k$-discrepancy matrix の存在がいえる．この zero $k \times k$-discrepancy matrix の構成には *constant-gap matrices* を用いる．また，constant-gap matrices の特徴づけを行なう．

A Generalization of Magic Squares with Applications to Digital Halftoning (2)[1]

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Abstract A *semimagic square* of order n is an $n \times n$ matrix containing the integers $0,\dots,n^2-1$ arranged in such a way that each row and column add up to the same value. We generalize this notion to that of a *zero $k \times k$-discrepancy matrix* by replacing the requirement that the sum of each row and each column be the same by that of requiring that the sum of the entries in each $k \times k$ square contiguous submatrix be the same. We show that such matrices exist if k and n are both even, and do not if k and n are are relatively prime. Further, the existence is also guaranteed whenever $n = k^m$, for some integers $k,m \geq 2$. A class of matrices, called *constant-gap matrices* plays an important role. We give a characterization of such matrices.

[1]A preliminary version of this paper appeared in Proceedings of International Symposium on Algorithms and Computation, Hong Kong, December, 2004.

[2]Part of the work on the paper has been carried out when B.A. was visiting JAIST. Work of B.A. on this paper was supported in part by NSF ITR Grant CCR-00-81964.

[3]Work of T.A. was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B).

1 Introduction

A *semimagic square* is an $n \times n$ matrix filled with the numbers $0, \ldots, n^2 - 1$ in such a way that the sum of the numbers in each row and each column are the same. Magic squares and related classes of integer matrices have been studied extensively (for an exhaustive bibliography, see [2] and the references therein).

The notion of a semimagic square is generalized by replacing the requirement that all row and column sums be the same by the analogous requirement for all $k \times k$ contiguous square submatrices; we call such $n \times n$ matrices *zero $k \times k$-discrepancy matrices* of order (k, n). Let $\mathbb{N}(k, n)$ be the set of all such matrices. It is known that $\mathbb{N}(k, n)$ is non-empty if k and n are both even, and empty if they are relatively prime. Further, an explicit construction of a zero $k \times k$-discrepancy matrices of order (k, k^m) for any integers $k, m \geq 2$.

Another property plays an important role in the latter construction of zero $k \times k$-discrepancy matrices. A characterization of matrices with this property is given in this paper.

2 Problem Statement

Generalizing the notion of a semimagic square, we consider an $n \times n$ matrix containing all the integers $0, \ldots, n^2 - 1$ such that the entries contained in every contiguous $k \times k$ submatrix add up to the same value.

More formally, for integers $m, n > 1$, let $\mathbb{Z}(n, m)$ be the class of all $n \times n$ integer matrices with entries from the set $\{0, \ldots, m-1\}$ and let $\mathbb{Z}(n) \subset \mathbb{Z}(n, n^2)$ be the set of those $n \times n$ matrices which contain every value $0, \ldots, n^2 - 1$ exactly once.

A contiguous $k \times k$ submatrix (or *region*, hereafter) $R_{i,j} = R_{i,j}^{(k)}$ with its upper left corner at (i, j) is defined by

$$R_{i,j}^{(k)} = \{(i', j') \mid i' = i, \ldots, i+k-1 \text{ and } j' = j, \ldots, j+k-1\},$$

where indices are calculated modulo n.[4] Given a matrix P and a region $R_{i,j}$ of size k, $P(R_{i,j})$ denotes the sum of the elements of P in locations given by $R_{i,j}$. Analogously, define a $C_{i,j} = C_{i,j}^{(k)}$ to be the $k \times 1$ region of a matrix starting at (i, j) and $P(C_{i,j})$ to be the sum of elements of P in the locations given by $C_{i,j}$. We are interested in *all* $k \times k$ regions in an $n \times n$ matrix:

$$\mathscr{F}_{k,n} = \{R_{i,j}^{(k)} \mid i, j = 0, 1, \ldots, n-1\}.$$

The *$k \times k$-discrepancy* $\mathscr{D}_{k,n}(P)$ of an $n \times n$ matrix P for the family $\mathscr{F}_{k,n}$ is defined as

$$\mathscr{D}_{k,n}(P) = \max_{R \in \mathscr{F}_{k,n}} P(R) - \min_{R' \in \mathscr{F}_{k,n}} P(R').$$

The existence of matrices $P \in \mathbb{Z}(n)$ with $k \times k$-discrepancy $\mathscr{D}_{k,n}(P) = 0$ is known. Let $\mathbb{N}(k, n)$ be the set of all such zero-$k \times k$-discrepancy matrices of order (k, n).

Theorem 1 *The set $\mathbb{N}(k, n)$ of zero-$k \times k$-discrepancy matrices of order (k, n) has the following properties:*

(a) $\mathbb{N}(k, n)$ *is non-empty if k and n are both even.*

(b) $\mathbb{N}(k, n)$ *is empty if k and n are relatively prime.*

(c) $\mathbb{N}(k, n)$ *is empty if k is odd and n is even.*

(d) $\mathbb{N}(k, k^m)$ *is non-empty for any integers k and m, $k \geq 2, m \geq 2$.*

[4] Throughout this paper, index arithmetic is performed modulo matrix dimensions unless otherwise noted.

3 Construction of a $k^m \times k^m$-Matrix of Zero $k \times k$-Discrepancy

In this section we review a scheme for designing a $k^m \times k^m$ matrix from $\mathbb{Z}(k^m)$ for any positive integer m such that its $k \times k$-discrepancy is zero. We first show that there exists a $k^2 \times k^2$ matrix in $\mathbb{Z}(k^2)$ whose $k \times k$ discrepancy is zero, and then extend the result to $k^m \times k^m$ matrices.

Definition 2 *The* simple expansion $\tilde{P}$ *of a $k \times k$ matrix P is the matrix formed by repeating P $k \times k$ times, as follows:*

$$\tilde{P} = \begin{bmatrix} P & P & \cdots & P \\ P & P & \cdots & P \\ & & \vdots & \\ P & P & \cdots & P \end{bmatrix}.$$

Note that the $k \times k$-discrepancy of $\tilde{P}$ is zero, as every $k \times k$ region contains the same set of numbers.

Definition 3 *A* cyclic column shift *of a matrix P is the matrix obtained by shifting each column of P to the right (i.e., shifting the jth column to the $(j+1)$st column) and moving the last column to the first column. A* cyclic row shift *is similarly defined: It means shifting each row of P down to the next lower row (i.e., shifting ith row to the $(i+1)$st row) and moving the bottom row to the top row.*

We denote the matrix obtained by applying cyclic column shift c times and cyclic row shift r times to a $k \times k$ matrix P by $P^{(c,r)}$. That is, element (i,j) in P moves to position $((i+r) \bmod k, (j+c) \bmod k)$ in $P^{(c,r)}$. The *cyclic expansion* $\hat{P} = (\hat{p}_{i,j})$ of a $k \times k$ matrix P is a $k^2 \times k^2$ matrix defined by

$$\hat{P} = \begin{bmatrix} P^{(0,0)} & P^{(0,1)} & \ldots & P^{(0,k-1)} \\ P^{(1,0)} & P^{(1,1)} & \ldots & P^{(1,k-1)} \\ & & \vdots & \\ P^{(k-1,0)} & P^{(k-1,1)} & \ldots & P^{(k-1,k-1)} \end{bmatrix}.$$

An easy calculation shows that, for all i, j, $\hat{p}_{i,j} = p_{i',j'}$, with

$$i' = i + \lfloor j/k \rfloor \text{ and } j' = j + \lfloor i/k \rfloor \pmod{k}. \tag{1}$$

Definition 4 *A* constant-gap matrix $P = (p_{i,j})$ *is one for which*

$$p_{i,j} - p_{i,j'} = p_{i',j} - p_{i',j'} \tag{2}$$

holds for all choices of i, i', j, and j'.

Intuitively, this means that for any two columns j and j' the gap between elements in the same row is independent of the row, hence the "constant gap" name. Since (2) can be rewritten as

$$p_{i,j} - p_{i',j} = p_{i,j'} - p_{i',j'} \text{ or } p_{i,j} + p_{i',j'} = p_{i,j'} + p_{i',j},$$

rows and columns play symmetric roles in the definition. Moreover, a constant-gap matrix has the strong Monge property [1] since the sum of the main diagonal elements is equal to that of the off diagonal elements in any 2×2 submatrix.

Lemma 5 *The constant-gap property is preserved (1) under exchange of any two rows, (2) under exchange of any two columns, and, for square matrices, (3) under mirror reflection across the main diagonal.*

The following lemma is a key to our construction of zero discrepancy matrices.

Lemma 6 *If P is a $k \times k$ constant-gap matrix, the $k \times k$-discrepancy of its cyclic expansion $\hat{P}$ is zero.*

Lemma 7 *Let $P = (p_{ij})$ and $Q = (q_{ij})$ be matrices in $\mathbb{Z}(k)$. Combine $\hat{P}$ and $\tilde{Q}$ into a single matrix in two different ways, namely, put $C^{(1)} = C^{(1)}(P,Q) = (c_{ij}) = \tilde{Q} + k^2\hat{P}$ and $C^{(2)} = C^{(2)}(P,Q) = (c'_{ij}) = \hat{P} + k^2\tilde{Q}$. In other words, $c_{i,j} = \tilde{q}_{i,j} + k^2\hat{p}_{i,j}$ or $c'_{i,j} = \hat{p}_{i,j} + k^2\tilde{q}_{i,j}$, for all i, j. If P has the constant gap property, then*

(a) $C^{(1)}$ and $C^{(2)}$ are in $\mathbb{Z}(k^2)$, and

(b) their $k \times k$-discrepancy is zero.

In addition, $C^{(1)}$ and $C^{(2)}$ are distinct if $P \neq Q$. Thus $|\mathbb{N}(k,k^2)| \geq 2$.

Here is a stronger version of Theorem 1d.

Theorem 8 *$\mathbb{N}(k,k^m) \neq \emptyset$ for any integers $k,m \geq 2$. Moreover, a zero-discrepancy matrix in $\mathbb{N}(k,k^m)$ can be explicitly computed in time linear in its size using $O(mk^2)$ space.*

4 The Class of Constant-Gap Matrices

We have described a scheme for constructing matrices with zero $k \times k$-discrepancy. A key ingredient in the recipe is a constant-gap matrix in $\mathbb{Z}(k)$. It is easily checked that a different choice of such a matrix produces a different zero-discrepancy matrix. Thus a natural question arises: How many different constant-gap matrices of a given size are there? In this section we, in a sense, completely characterize the class of constant-gap matrices in $\mathbb{Z}(n)$. In fact, we discuss a somewhat more general class of matrices. Let $\mathbb{M}(m,n)$ be the set of all integer $m \times n$ matrices with entries $0,\ldots,mn-1$, each used exactly once. A matrix $M = (m_{ij}) \in \mathbb{M}(m,n)$ has *constant-gap property* if, for all i,j,i',j', $m_{ij} + m_{i'j'} = m_{i'j} + m_{ij'}$. It is clear from the definition that this property, as already observed in Lemma 5, is invariant under a number of operations:

Lemma 9 *The constant-gap property is preserved (1) under an arbitrary permutation of rows of a matrix, (2) under an arbitrary permutation of columns of a matrix, and, for square matrices, (3) under mirror reflection through its main diagonal.*

Two constant-gap matrices are *equivalent* if one of them is derived from the other by a sequence of operations listed in the statement of the lemma.[5] We are interested in counting the number of these equivalence classes.

Lemma 10 *Every equivalence class can be represented by a matrix $P = (p_{i,j}) \in \mathbb{M}(m,n)$ in* canonical form, *which satisfies the following additional properties:*

(a) Every row of P is sorted, i.e., $p_{i,0} < p_{i,1} < \cdots < p_{i,n-1}$; for convenience we define $c_j = p_{0,j}$ for $j = 0,\ldots,n-1$.

(b) Every column of P is sorted, i.e., $p_{0,j} < p_{1,j} < \cdots < p_{n-1,j}$; we put $r_j = p_{j,0}$, for $j = 0,\ldots,n-1$.

(c) Generally, $p_{i,j} < p_{i',j'}$ if $i \leq i'$, $j \leq j'$, and $(i,j) \neq (i',j')$.

(d) $p_{0,0} = c_0 = r_0 = 0$, $p_{m,n} = mn-1$.

(e) P is completely specified by (c_j), (r_i): for all i,j, $p_{i,j} = r_i + c_j$.

[5] To avoid cumbersome wording, we will henceforth not discuss the case of square matrices separately. The reasoning there is entirely analogous, with the exception of an occasional invocation of diagonal symmetry to further reduce the number of equivalence classes. We omit further details.

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

(a)

$$\begin{bmatrix}0&1&2&3\\4&5&6&7\\8&9&10&11\\12&13&14&15\end{bmatrix}\quad\begin{bmatrix}0&1&4&5\\2&3&6&7\\8&9&12&13\\10&11&14&15\end{bmatrix}\quad\begin{bmatrix}0&1&8&9\\2&3&10&11\\4&5&12&13\\6&7&14&15\end{bmatrix}$$

(b)

Figure 1: The only equivalence class of constant-gap matrices for $n = 3$ (a), and the three classes for $n = 4$ (b)

Proof: We argue that all the properties can be satisfied without leaving the equivalence class. Columns can be permuted to sort the top row and then rows can be permuted to sort the leftmost column. Because of the constant-gap property, this sorts all rows and columns. Properties (c) and (d) follow from this ordering. The next property follows from the constant-gap condition, namely $p_{i,j} = p_{0,0} + p_{i,j} = p_{i,0} + p_{0,j} = r_i + c_j$. The final property can be ensured for square matrices by taking, if necessary, a reflection through the main diagonal. □

It is not difficult to see that there exists only one equivalence class (up to diagonal reflection) of constant-gap matrices in $\mathbb{Z}(n)$ when $n = 3$; for $n = 4$, there are three equivalence classes, refer to Fig. 1. For $n = 5$ there exists only one equivalence class, whereas there are many when $n = 6$. These observations can be generalized as follows.

The set of all constant-gap matrices in $\mathbb{M}(m,n)$ in canonical form is denoted by $\mathbb{K}(m,n)$. In this section we give a characterization of the sets $\mathbb{K}(m,n)$, for all $m,n > 0$. We begin with some additional terminology and then state our characterization.

We define another operation on matrices. Given matrices $P = (p_{i,j}) \in \mathbb{K}(m,n)$ and $Q = (q_{i,j}) \in \mathbb{K}(m',n')$, their *expansion product* $P \otimes Q$ is the $mm' \times nn'$ matrix $H = (h_{i,j})$ defined by $h_{i,j} = m'n' p_{\lfloor i/m' \rfloor, \lfloor j/n' \rfloor} + q_{i \bmod m', j \bmod n'}$. In addition, define a *simple row* of length k to be the $1 \times k$ matrix filled with consecutive numbers $0, \ldots, k-1$, in this order. Define a *simple column* analogously.

The following facts are easily verified.

Fact 1 *$\mathbb{K}(1,m)$ consists of a single matrix, which is a simple row of length m. An analogous statement holds for $\mathbb{K}(m,1)$.* ***Both row- and column-major order filled members of*** *$\mathbb{M}(m,n)$* ***have constant-gap property.*** *In particular, $|\mathbb{K}(m,n)| \geq 2$, for $m,n > 1$. The two matrices have the same canonical form in $\mathbb{K}(n)$, so $|\mathbb{K}(n)| \geq 1$ for $n \geq 1$.*

Fact 2 *If $P \in \mathbb{K}(m,n)$ and $Q \in \mathbb{K}(m',n')$, then $P \otimes Q \in \mathbb{K}(mm',nn')$.*

Surprisingly, in the sense made more precise by the following theorem, Facts 1 and 2 describe $\mathbb{K}(m,n)$ completely. The remainder of the section is devoted to the proof of this assertion.

Theorem 11 *$\mathbb{K}(m,n)$ can be characterized as follows.*

(a) A matrix $P \in \mathbb{K}(m,n)$ can be written uniquely as

$$P = P_1 \otimes \cdots \otimes P_k,$$

where $P_1, \ldots, P_k$ is an alternating sequence of simple rows and columns, each of length at least two; $k = 0$ if $m = n = 1$.

(b) $\mathbb{K}(m,n) \neq \emptyset$, for $m,n > 0$. $|\mathbb{K}(m,n)| = 1$ if and only if $n = 1$ or $m = 1$ or $n = m$ and n is prime. In this case, $\mathbb{K}(m,n)$ consists just a simple row or column.

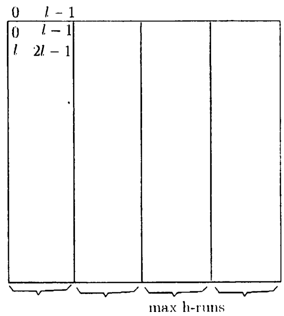

Figure 2: Partition of a matrix by vertical boundaries of h-runs.

A *column difference* $p_{0,j} - p_{0,k} = p_{i,j} - p_{i,k}$, for some $j \neq k$, is the difference between corresponding entries of two columns; it is independent of the row where the difference is taken, by the constant-gap property. Similarly, a *row difference* $p_{i,0} - p_{k,0} = p_{i,j} - p_{k,j}$, for $i \neq k$, is the difference between corresponding entries of two rows.

Lemma 12 *No row difference equals a column difference. In other words, for any* $0 \leq i < k \leq n-1$ *and* $0 \leq j < \ell \leq n-1$, $p_{0,k} - p_{0,i} \neq p_{\ell,0} - p_{j,0}$.

Proof: By the constant-gap property, $p_{0,k} - p_{0,i} = p_{\ell,k} - p_{\ell,i}$ and $p_{\ell,0} - p_{j,0} = p_{\ell,k} - p_{j,k}$. Their equality would imply $p_{\ell,i} = p_{j,k}$ contradicting the assumption that no entry in P is repeated. □

Proof: [Theorem 11] We start with the existence proof for part (a). If $n = 1$ or $m = 1$, we are done, so assume $m,n > 1$. Without loss of generality, assume that $p_{0,1} < p_{1,0}$, so that $p_{0,1} = 1$. By induction, it is sufficient to argue that in this case P can be written as a product of a smaller matrix and a simple row of length at least two.

Let $P = (p_{i,j}) \in \mathbb{K}(m,n)$. A *horizontal run* in P (an *h-run*, for short) is a maximal sequence of consecutive integers appearing in adjacent entries of a row of P; the *length* of an h-run is the number of such integers. Each h-run is associated with an interval defined by its first and last column indices. The *initial h-run* is the one starting at the upper left corner $p_{0,0}$ of the matrix; by our assumption its length ℓ is at least two. The value ℓ must be contained in $p_{1,0}$, for it cannot lie in $p_{0,\ell}$ by maximality of the run and it is the smallest value in the matrix outside of the run. Thus we have the row difference $p_{1,0} - p_{0,0} = \ell$.

Lemma 13 *No h-run is longer than the initial run.*

Proof: If there existed such an h-run, we would have a column difference equal to ℓ appear within the run. However, we already identified a row difference of ℓ in the matrix. This would contradict Lemma 12. □

Recall that in P the differences between consecutive elements in the row are equal to the corresponding differences in the top row, so in terms of presence and length of runs, every row behaves exactly the same. Thus P can be partitioned by vertical lines corresponding to boundaries of h-runs, as shown in Fig. 2.

Lemma 14 *All h-runs have the same length.*

Proof: By Lemma 13, no h-run is longer than the initial h-run of length ℓ. For a contradiction, let j be the first column at which an h-run of length $\ell' < \ell$ starts. Such an h-run in the top row is a sequence $p_{0,j}, p_{0,j}+1, \ldots, p_{0,j}+\ell'-1$. Where in the matrix is the next integer, namely the value $q = p_{0,j}+\ell'$? By definition of an h-run, it cannot be located in the same row. So, it must lie before column j, say in location $p_{k,s\ell}$, for some $s\ell < j$. Since all the h-runs located to the left of column j are of length ℓ, the h-run starting at $p_{k,s\ell}$ must consist of values $q,\ldots,q+\ell-1$. However, since the difference between rows zero and one is ℓ, $p_{1,j}$ must have value $r = p_{0,j}+\ell > q$. Moreover, $r = q-\ell'+\ell < q+\ell-1$, so the value r occurs both at $p_{1,j}$ and in the run starting at value q, which is impossible. □

This property implies the following structure of the matrix P. The number $\ell > 1$ is a divisor of n. The entire matrix consists of h-runs of length ℓ. **The first element of each run is a multiple of ℓ.** If L is a simple row of length ℓ, we can easily verify that indeed $P = P' \otimes \ell$ for the matrix $P' = (p'_{i,j})$ defined by $p'_{i,j} = p_{i,\ell j}/\ell$. We claim that $P' \in \mathbb{K}(m,n/\ell)$. It is easily checked that $P' \in \mathbb{M}(m,n/\ell)$, while properties (a)–(e) follow from the corresponding properties for P.

We now address the question of uniqueness. We have shown that any matrix in $\mathbb{K}(m,n)$ can we written as an expansion product of some number of simple rows and columns. Since the expansion product of two simple rows (resp. columns) is a simple row (resp. column), by consolidating products of consecutive rows (resp. columns) we can express P as a product of alternating rows and columns. This representation is unique, since the type and size of the last factor can be "read" directly from the matrix—the factor is a simple row if $p_{0,1} = 1$ and a simple column if $p_{1,0} = 1$; its length ℓ is the length of the initial run, which is horizontal in the former case and vertical in the latter one.

It remains to note that a matrix from $\mathbb{K}(m,n)$ can always be produced as a product of a simple row of length n and a simple column of length m, thus constant-gap matrices of all orders exist. Reversing the order of multiplication produces a different matrix, provided $m,n > 1$, proving part (b) of the theorem. □

5 Concluding Remarks

We have characterized a family of constant-gap matrices. It leads to an algorithm for enumerating all possible such matrices. **One of our open problems is to find an efficient scheme for enumerating all zero-**discrepancy matrices of order (k,n). The results shown in this paper could be a basis for the direction.

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